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J-Spectral Factorization

M. Šebek *
Institute of Information Theory and Automation
Czechoslovak Academy of Sciences
18208 Prague 8, Czechoslovakia

H. Kwakernaak
Faculty of Applied Mathematics
University of Twente
7500 AE Enschede, the Netherlands

Abstract

The problem of J-spectral factorization of polynomial matrices is studied. Existence and uniqueness are discussed and the differences from the standard case are pointed out. Two first algorithms are proposed for a numerical solution.

1 Introduction

The problem of spectral factorization of a polynomial matrix appears to be important for practical use in many areas of science and engineering. Namely, the spectral factorization of a polynomial matrix which is positive definite on the imaginary axis, called here *the standard spectral factorization*, has grown into one of the basic tools of modern control theory for its service to linear quadratic ($\mathcal{H}_2$-)optimal control [10,6]. In addition, a new interesting area of application for matrix spectral factorization has arisen recently in $\mathcal{H}_\infty$-optimal control. For this purpose, the task is to compute the spectral factor of a polynomial matrix indefinite on the imaginary axis (the so called *J-spectral factorization*) [7].

Several numerical algorithms to solve the standard factorization have been developed in the last three decades. They are based on computation of zeros and symmetric factor extraction, on Cholesky factorization of a block Toeplitz matrix constructed from matrix coefficients, on Newton-Raphson method and on the solution of algebraic Riccati equation. Among them, chiefly the last two are now very popular. None of the above methods can be implemented directly to solve the J-spectral factorization. In many cases, in fact, the problems connected with their application seem to be principal ones [9].

It is the aim of this paper to look at the problem of J-spectral factorization a little more deeply, to point out the differences from the standard case and, namely, to present two first algorithms for a numerical solution.

The first algorithm incorporates the computation of zeros and a generalized version of the symmetric factor extraction. It features simplicity as well as a nice indication of certain internal singularity which, in fact, reflects the $\mathcal{H}_\infty$-optimality and can be employed to compute directly the reduced optimum compensator [7].

The second one is a new procedure inspired partially by a mathematical work of Jakubovic [5]. It is based on diagonalization, scalar spectral factorization and the backward construction of the matrix factor. It is more complex but it exhibits a little better numerical properties as it avoids the need to compute the zeros. It handles matrices of any signature (including singular ones) and requires no particular form of them to start. It seems to be numerically attractive even for the standard (positive definite) case since, in contrast to all other iterative methods, the iteration is performed just on scalar polynomials so that all typical numerical troubles (appearing with multiple zeros on the stability boundary) are beaten down.

2 Existence and Uniqueness

Let us first recall that an *adjoint* $A^*(s)$ of a real polynomial matrix $A(s)$ is defined by $A^*(s) = A^T(-s)$ and the matrix $A(s)$ is called *para-Hermitian* if $A^*(s) = A(s)$. A general real spectral factorization can be then defined as follows:

Definition 1 *Let be given a para-Hermitian*[1] *real polynomial matrix $A(s)$. Any representation*

$$A(s) = C^*(s)KC(s) \tag{1}$$

where $C(s)$ is a real non-singular square polynomial matrix and K is a symmetric real constant matrix ($K^T = K$) is labelled (right[2]) spectral factorization *of $A(s)$ and any $C(s)$ satisfying (1) is called (right)* spectral factor *of $A(s)$. Moreover, the $C(s)$ is a* Hurwitz spectral factor *of $A(s)$ whenever $C(s)$ is a Hurwitz*[3] *matrix in addition.*

As applications are limited solely to the *Hurwitz* spectral factors, just those will be considered throughout the paper while the adjective "Hurwitz" will be dropped for brevity.

Remark 1 Once the spectral factor exists, the real matrix K can always found in the (block) diagonal form

$$K = \mathrm{diag}\{I_p, -I_n, 0_{o\times o}\},$$

which is the so called *signature matrix*[4]. Without any lost in generality, we shall consider only such K's throughout the paper. The definition above covers a variety of matrix factorization problems. In particular, if $A(s)$ is positive (non-negative) definite on the imaginary axis, we refer to the standard case and always $K = I$ results. On the other hand, if $A(s)$ is indefinite on the imaginary axis, we speak of *J-spectral factorization* and, as usually, we write J $(= \mathrm{diag}\{I_p, -I_n\})$ instead of K.

*The work was done while the first author was with the Systems and Control Group and Mechatronics Research Center of the University of Twente.

[1] No symmetric factorization (1) evidently exists unless $A(s)$ is para-Hermitian. Hence it is no loss in generality to consider just such matrices

[2] The dual *(left or co-) factorization* $A(s) = C(s)KC^*(s)$ can be found simply by applying (1) to $A^T(s)$ and taking the transpose of the result.

[3] In the context of continuous-time systems, which is considered here, a polynomial matrix is called *Hurwitz* if it has no open right half-plain zeros.

[4] Besides, the *signature* of a real symmetric matrix K is a triple (p, n, o) consisting of the number of positive, negative and zero eigenvalues of K, respectively.

The condition for the spectral factorization to exist is as follows:

Theorem 1 *An $(n \times n)$ polynomial matrix $A(\mathsf{s})$ possesses a spectral factorization (1) if for every nonzero invariant polynomial $i_i(A)(\mathsf{s})$ of $A(\mathsf{s})$, $i = 1, \ldots, n$, holds that if it possesses zeros on* Re $s = 0$, *then they must be of even multiplicity.*

PROOF:
See [5,9]. □

To put it in another words, the sufficient condition is violated *only if* some of the invariant factors *is not factorable by itself* because[5] $A(\mathsf{s})$ has a zero on Re $s = 0$ and, moreover, this zero is of an odd multiplicity. So the condition is *almost always satisfied* and hence the spectral factor *generically exists*.

On the other hand, the condition is *not* necessary. When some of the invariant factors is not factorable by itself, the whole matrix need not but *can* be so, all the same:

Example 1 If some of the invariant factors is not factorable by itself, the matrix is not likely to be factorable. So is, for example the matrix

$$A(\mathsf{s}) = \begin{bmatrix} 1+\mathsf{s}^2 & 0 \\ 0 & 1+\mathsf{s}^2 \end{bmatrix} \tag{2}$$

which is already in its Smith form. Despite the fact that its determinant ($\det A(\mathsf{s}) = (1+\mathsf{s}^2)^2$) is factorable, the matrix $A(\mathsf{s})$ itself is evidently not.

Example 2 However, there is still a small gap and the condition is not necessary, indeed: Just consider

$$A(\mathsf{s}) = \begin{bmatrix} 0 & \mathsf{s} \\ -\mathsf{s} & \mathsf{s}^2 \end{bmatrix} \tag{3}$$

Here $i_1(A) = \mathsf{s}$ violates the condition yet $A(\mathsf{s})$ possesses the factor

$$C(\mathsf{s}) = \begin{bmatrix} 1 & 0 \\ -1 & \mathsf{s} \end{bmatrix} \tag{4}$$

all the same. Indeed,

$$\begin{bmatrix} 1 & -1 \\ 0 & -\mathsf{s} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -1 & \mathsf{s} \end{bmatrix} = \begin{bmatrix} 0 & \mathsf{s} \\ -\mathsf{s} & \mathsf{s}^2 \end{bmatrix}$$

Notice that

$$\mathcal{S}(A) = \begin{bmatrix} \mathsf{s} & 0 \\ 0 & \mathsf{s} \end{bmatrix}$$

but

$$\mathcal{S}(C) = \begin{bmatrix} 1 & 0 \\ 0 & \mathsf{s} \end{bmatrix}$$

so that here

$$\mathcal{S}(A) \neq \mathcal{S}(C^*)\mathcal{S}(C).$$

In fact, this situation is not quite clear as yet.

If the Hurwitz spectral factor does exist, it is not unique at all. The class of solutions can be parameterized as follows:

Theorem 2 *Let a polynomial matrix $C(\mathsf{s})$ be a Hurwitz spectral factor of $A(\mathsf{s})$. Then also $B(\mathsf{s})$*

$$B(\mathsf{s}) = T(\mathsf{s})C(\mathsf{s}) \tag{5}$$

is a Hurwitz spectral factor of $A(\mathsf{s})$ for any unimodular matrix $T(\mathsf{s})$ such that

$$T^*(\mathsf{s})KT(\mathsf{s}) = K \tag{6}$$

*(the so called K-*unitary matrix*).*

[5] Due to the para-Hermitian nature of $A(\mathsf{s})$, all the invariant factors are "symmetric": $i_i(A) = \pm i_i(A)$.

There is, of course, a couple of substantial differences between the standard and the J-spectral factorization: For example, all K-unitary matrices appear to be of degree 0 (i.e. real K-orthogonal matrices) whenever K is positive definite. Besides, they are sufficient to provide a particular form of $C(\mathsf{s})$ (having one of the coefficients triangular). In contrast, for an indefinite K, there exist really polynomial K-unitary matrices. Moreover, they need not be able to put $C(\mathsf{s})$ in a particular form.

Example 3 As a counter-example, consider the symmetric constant matrix

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

which does not possess any triangular factorization (1). In fact, one can easy prove that every factor of this matrix must have all its entries non-zero such as, e.g.,

$$\begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Of course, more freedom exists whenever $A(\mathsf{s})$ is a singular matrix. What we have decided here is to put the whole singularity into K and let $C(\mathsf{s})$ be always non-singular. But this is a matter of definition.

3 Symmetric factor extraction

The first algorithm presented is a generalization of the procedure developed by Callier [1] for standard spectral factorization. The method requires the following assumptions:

Assumptions 3

(3.A) $A(\mathsf{s})$ is nonzero on the imaginary axis.

(3.B) $A(\mathsf{s})$ is *diagonally reduced* [1].

Assumption (3.A) implies the existence conditions of Theorem 1. Assumption (3.B) causes no loss of generality because any para-Hermitian matrix $A(\mathsf{s})$ that is not diagonally reduced can easily be made diagonally reduced [1].

The first step of the algorithm is to compute the zeros of $A(\mathsf{s})$, that is, the roots of the polynomial $\det A(\mathsf{s})$. Because $A(\zeta)$ is para-Hermitian, if ζ is a zero, so is $-\zeta$. Likewise, because $A(\zeta)$ has real coefficients, if ζ is a zero, so is its complex conjugate $\bar{\zeta}$. Numerically, the zeros of $A(\mathsf{s})$ may be obtained as the eigenvalues of the (generalized) companion matrix[6] defined by the coefficient matrices of $A(\mathsf{s})$.

The second part of the algorithm is to extract from $A(\mathsf{s})$, sequentially, first- or second-order elementary factors corresponding to each of the zeros. If $T(\mathsf{s})$ is such an elementary factor, we write

$$A = T^* A'' T, \tag{7}$$

where also A'' is polynomial. Depending on whether ζ is real or complex we distinguish three different cases for the elementary factor T.

1. *Case 1.* If the zero ζ is *real* the elementary factor is of the

[6] Compare K. D. Gregson and N. J. Young, "Finite representations of block Hankel operators and balanced realization," *Operator Theory: Advances and Applications*, Vol. 35, pp. 441-480 (1988).

form

$$T(\mathsf{s}) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & -\alpha_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & -\alpha_2 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 1 & -\alpha_{k-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \mathsf{s}-\zeta & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & -\alpha_{k+1} & 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & -\alpha_n & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \tag{8}$$

with $\alpha_1, \alpha_2, \cdots, \alpha_n$ real constants to be determined. The nonunit column of T is the kth column, with k to be determined. The factor T may be seen as a variant of the Hermite standard form of a polynomial matrix factor of degree 1. It has a single real zero ζ. In more compact form we write T as

$$T(\mathsf{s}) = \begin{bmatrix} I & -a_1 & 0 \\ 0 & \mathsf{s}-\zeta & 0 \\ 0 & -a_2 & I \end{bmatrix}. \tag{9}$$

The constant vectors a_1 and a_2 may be found as follows. If ζ is a zero of A, clearly $A(\zeta)$ is singular, so that there exists a real constant *null vector* e corresponding to ζ such that

$$A(\zeta)e = 0. \tag{10}$$

Because $T^*(\zeta)A''(\zeta)T(\zeta)e = 0$ and ζ is a zero of T, we may determine a_1 and a_2 by letting $T(\zeta)e = 0$. Writing out this identity component-by-component it is easily found that a_1 and a_2 follow from

$$a = e/e_k, \tag{11}$$

where e_k is the kth component of e, and the constant vector a is defined by

$$a = \begin{bmatrix} a_1 \\ 1 \\ a_2 \end{bmatrix}. \tag{12}$$

It remains to determine the "remaining factor" A''. To this end, we first extract the factor T "on the right" and write

$$A = A'T, \tag{13}$$

with the square polynomial matrix A' to be determined. Multiplying the equality $A = A'T$ out element-by-element it is easy to see that all entries of A and A' are equal except those in their kth columns. Denoting the kth column of A' as p'_k it follows that

$$A(\mathsf{s})a = p'_k(\mathsf{s})(\mathsf{s}-\zeta). \tag{14}$$

From this, p'_k may easily be computed. The polynomial matrix A'' now may be obtained by the left extraction

$$A' = T^*A'', \tag{15}$$

which we rewrite as the right extraction

$$(A')^* = (A'')^*T. \tag{16}$$

This extraction follows by the same procedure as before. Because A'' is para-Hermitian, it is sufficient to compute the kth diagonal entry p''_{kk} of A''. It may be solved from the equation

$$p'^*_k(\mathsf{s})a = p''_{kk}(\mathsf{s})(s-\zeta). \tag{17}$$

The nondiagonal elements of the kth column of A'' equal the corresponding entries of p'_k, the nondiagonal elements of the kth row of A'' follow by adjugation, while the remaining elements of A'' equal the corresponding elements of A. This defines A''.

2. *Case 2.* If the zero ζ is ***complex*** there are two possibilities for the elementary factor T.

   (a) *Case 21.* The first possibility is that T has a single nonunit kth column of degree two, so that T is of the form

$$T(\mathsf{s}) = \begin{bmatrix} I & -a_1 - b_1\mathsf{s} & 0 \\ 0 & (\mathsf{s}-\zeta)(\mathsf{s}-\bar{\zeta}) & 0 \\ 0 & -a_2 - b_2\mathsf{s} & I \end{bmatrix}, \tag{18}$$

   with a_1, a_2, b_1 and b_2 real constant coefficient vectors to be determined. Extraction of this factor takes care of both the zero ζ and its complex conjugate $\bar{\zeta}$. Extraction on the right in the form

$$A = A'T \tag{19}$$

   results in a matrix A' whose elements equal those of A except those in the kth column. Writing the complex zero ζ and the corresponding (complex) null vector e in Cartesian form as

$$\zeta = \sigma + j\omega, \qquad e = p + jq, \tag{20}$$

   denoting the kth column of A' as p'_k, and defining the two constant vectors a and b by

$$a = \begin{bmatrix} a_1 \\ 1 \\ a_2 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ 0 \\ b_2 \end{bmatrix}, \tag{21}$$

   it may be found that

$$[a \quad b]\begin{bmatrix} p_k & q_k \\ \sigma p_k - \omega q_k & \sigma q_k + \omega p_k \end{bmatrix} = [p \quad q], \tag{22}$$

   which may be solved for a and b. The kth column p'_k of A' may be obtained from the equality

$$A(\mathsf{s})(a + b\mathsf{s}) = p'_k(\mathsf{s})(\mathsf{s}-\zeta)(\mathsf{s}-\bar{\zeta}). \tag{23}$$

   The nondiagonal elements of the kth column of the remaining factor A'' in the extraction $A = T^*A''T$ equal the corresponding elements of p'_k, while the nondiagonal elements of the kth row of A'' follow by adjugation. The kth diagonal element p''_{kk} of A'' may be found from the equality

$$p'^*_k(\mathsf{s})(a + b\mathsf{s}) = p''_{kk}(\mathsf{s})(\mathsf{s}-\zeta)(\mathsf{s}-\bar{\zeta}). \tag{24}$$

   (b) *Case 22.* The second situation that may arise when the zero ζ is complex is that the elementary factor T has two nonunit columns (columns k and l) of degree one, so that it has the form

$$T(\mathsf{s}) = \begin{bmatrix} I & -a_1 & 0 & -b_1 & 0 \\ 0 & \mathsf{s}-\alpha & 0 & -\beta & 0 \\ 0 & -a_2 & I & -b_2 & 0 \\ 0 & -\gamma & 0 & \mathsf{s}-\delta & 0 \\ 0 & -a_3 & 0 & -b_3 & I \end{bmatrix}, \tag{25}$$

   with a_1, a_2, a_3, b_1, b_2, and b_3 real constant vectors to be determined, and α, β, γ, and δ real constants to be found. Extraction on the right in the form $A = A'T$ now results in a polynomial matrix A' whose elements equal the corresponding elements of A except those in the kth and lth columns. Writing the complex zero ζ and the corresponding complex null vector e in Cartesian form as

$$\zeta = \sigma + j\omega, \qquad e = p + jq, \tag{26}$$

and denoting the two constant vectors a and b by

$$a = \begin{bmatrix} a_1 \\ 1 \\ a_2 \\ 0 \\ a_3 \end{bmatrix}, \qquad b = \begin{bmatrix} b_1 \\ 0 \\ b_2 \\ 1 \\ b_3 \end{bmatrix}, \tag{27}$$

it may be found that

$$[a \quad b] \begin{bmatrix} p_k & q_k \\ p_l & q_l \end{bmatrix} = [p \quad q], \tag{28}$$

with p_k and p_l the kth and lth elements of p, and a similar notation for q. This expression allows solving for a and b. The constants α, β, γ, and δ follow from the identity

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \sigma I - \omega \begin{bmatrix} p_k & q_k \\ p_l & q_l \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} p_k & q_k \\ p_l & q_l \end{bmatrix}^{-1}, \tag{29}$$

with I the 2×2 unit matrix. In the right extraction $A = A'T$, the elements of A' equal the corresponding elements of A except those of the kth and lth columns. The kth and lth columns p'_k and p'_l of A' may be solved from the identity

$$A(s)[a \quad b] = [p'_k(\mathsf{s}) \quad q'_k(\mathsf{s})] \begin{bmatrix} \mathsf{s} - \alpha & -\beta \\ -\gamma & \mathsf{s} - \delta \end{bmatrix}. \tag{30}$$

The elements of the factor A'' in the extraction $A = T^* A'' T$ equal the elements of A, except all elements of the kth and lth columns and rows. The elements of the kth column of A'', except those in the kth and lth positions, equal the corresponding elements of p'_k. Similarly, the elements of the lth column of A'', except those in the kth and lth positions, equal the corresponding elements of p'_l. The elements of the kth and lth rows of A'', except those in the kth and lth positions. follow by adjugation. The elements p''_{kk}, p''_{kl}, p''_{lk}, and p''_{ll} of A'', finally, may be obtained from the identity

$$\begin{bmatrix} p'^{*}_k(\mathsf{s}) \\ p'^{*}_l(\mathsf{s}) \end{bmatrix} [a \quad b] = \begin{bmatrix} p''_{kk}(\mathsf{s}) & p''_{kl}(\mathsf{s}) \\ p''_{lk}(\mathsf{s}) & p''_{ll}(\mathsf{s}) \end{bmatrix} \begin{bmatrix} \mathsf{s} - \alpha & -\beta \\ -\gamma & \mathsf{s} - \delta \end{bmatrix}. \tag{31}$$

The following rules determine which case applies and which column k or columns k and l are to be selected. Let δ_j, $j = 1, 2, \cdots$, be the degree of the jth diagonal entry of A. Next, given the null vector e corresponding to the root ζ, define the *active index set* $\mathcal{A}$ as

$$\mathcal{A} = \{i : \ e_i \neq 0\}, \tag{32}$$

and the *highest active index set* $\mathcal{M} \subseteq \mathcal{A}$ as

$$\mathcal{M} = \{i \in \mathcal{A} : \ \delta_i \geq \delta_j \text{ for } j \in \mathcal{A}\}, \tag{33}$$

with e_i the ith element of e.

1. If the zero ζ is real, case 1 applies. The column number k may be any element of $\mathcal{M}$. For numerical reasons it is best to select k such that $|e_k|$ maximizes e_j, $j \in \mathcal{M}$.

2. If ζ is complex, case 21 or case 22 applies. Write ζ and e in Cartesian form as

$$\zeta = \sigma + j\omega, \qquad e = p + jq. \tag{34}$$

Define the matrix E as

$$E = [p_{\mathcal{M}} \quad q_{\mathcal{M}}], \tag{35}$$

where $p_{\mathcal{M}}$ is the column vector whose entries consist of those elements p_i of p such that $i \in \mathcal{M}$, and $q_{\mathcal{M}}$ is similarly defined. Then if rank $E = 1$, case 21 applies, while if rank $E = 2$, case 22 is applicable.

   (a) In case 21, k is any element of $\mathcal{M}$. Again, for numerical reasons it is best to choose k such that $|e_k|$ maximizes $|e_j|$, $j \in \mathcal{M}$.

   (b) In case 22, k and l are any two different elements of $\mathcal{M}$. For numerical reasons it is best to choose k and l such that the condition number of

$$\begin{bmatrix} p_k & q_k \\ p_l & q_l \end{bmatrix} \tag{36}$$

   is maximal.

For spectral factorization, the zero ζ of each factor that is extracted on the right is chosen to have negative real part. Then the corresponding factor that is extracted on the left has $-\zeta$ as its zero. The extraction procedure is repeated until the supply of zeros is exhausted. The order in which factors corresponding to the successive zeros are extracted is not important. Eventually, A is reduced to the form

$$A = (T^{(m)}T^{(m-1)} \cdots T^{(1)})^* A^{(0)} (T^{(m)}T^{(m-1)} \cdots T^{(1)}), \tag{37}$$

with $T^{(1)}, T^{(2)}, \cdots, T^{(m)}$ the elementary factors that successively have been extracted.

For standard spectral factorization (where A is positive definite on the imaginary axis and $J = I$), after each extraction the remaining factor is diagonally reduced, and the final factor $A^{(0)}$ is a symmetric real constant matrix. $A^{(0)}$, in its turn, may be factored as

$$A^{(0)} = T^{(0)\mathrm{T}} K T^{(0)} \tag{38}$$

by successively using each diagonal element of $A^{(0)}$ to clear the corresponding row and column, and normalizing this diagonal element to ± 1 depending on its sign. For ordinary spectral factorization this amounts to Cholesky factorization (with $K = I$).

If A is not positive-definite on the imaginary axis, *generically* $A^{(0)}$ is a symmetric constant matrix that may be factored as $A^{(0)} = T^{(0)\mathrm{T}} K T^{(0)}$. After suitable permutation of the columns and rows of K one obtains $K = J$. *Nongenerically* $A^{(0)}$ is a nonconstant *unimodular* para-Hermitian polynomial matrix, which may be J-factored as explained for the second algorithm discussed in this paper. If this situation arises, we call the J-spectral factorization *singular*. As the example below illustrates, J-spectral factorizations that are "near-singular" lead to spectral factors with very large coefficients.

Example 4 By way of example we consider the J-spectral factorization of the polynomial matrix[7]

$$A\mathsf{s}) = \begin{bmatrix} (1 - \frac{2}{\lambda^2}) - 2(1 - \frac{1}{\lambda^2})\mathsf{s}^2 & -1 + (-1 + \frac{2}{\lambda^2})\mathsf{s}\sqrt{2} \\ -1 - (-1 + \frac{2}{\lambda^2})\mathsf{s}\sqrt{2} & (1 - 2\frac{c^2}{\lambda^2}) + \frac{2}{\lambda^2}\mathsf{s}^2 \end{bmatrix}, \tag{39}$$

[7]Taken from Section 4.12 of [7].

with λ a real constant. Since

$$\det A(s) = \frac{4}{\lambda^2}(1 - \frac{1}{\lambda^2})(1 + s^4), \tag{40}$$

the roots of A are $\frac{1}{2}\sqrt{2}(\pm 1 \pm j)$. We assume that $]\lambda] > 1$ so that A is nonsingular. Selecting the root $\zeta = \frac{1}{2}\sqrt{2}(-1 + j)$, we find the corresponding null vector

$$e = \begin{bmatrix} 2 + j(-2 + \lambda^2) \\ (-2 + \lambda^2) + j(-2 + 2\lambda^2) \end{bmatrix}. \tag{41}$$

The active and highest active index set for $]\lambda] > 1$ are both given by $\mathcal{A} = \mathcal{M} = \{1, 2\}$. Writing e in Cartesian form as $e = p + jq$ it is easily found that the determinant of the matrix

$$E = [p_{\mathcal{M}} \; q_{\mathcal{M}}] = [p \; q] = \begin{bmatrix} 2 & -2 + \lambda^2 \\ -2 + \lambda^2 & -2 + 2\lambda^2 \end{bmatrix} \tag{42}$$

is $\Delta_\lambda = -8 + 8\lambda^2 - \lambda^4$. The roots of the determinant are $\pm\sqrt{4 \pm 2\sqrt{2}}$. For λ not equal to one of these four values the determinant is nonzero, so that the extraction corresponding to the root ζ is of type 22.

We select the column numbers k and l as $k = 1$, $l = 2$. Going through the necessary computational steps (where the assistance of Mathematica is very helpful) the elementary factor T_λ of type 22 and the "remaining factor" A''_λ defined by $A = T^*_\lambda A''_\lambda T_\lambda$ may be found to be given by

$$T_\lambda(s) = \begin{bmatrix} s - \frac{8-4\lambda^2-\lambda^4}{\Delta_\lambda\sqrt{2}} & -\frac{8-4\lambda^2+\lambda^4}{\Delta_\lambda\sqrt{2}} \\ -\frac{-8+12\lambda^2-5\lambda^4}{\Delta_\lambda\sqrt{2}} & s - \frac{8-12\lambda^2+3\lambda^4}{\Delta_\lambda\sqrt{2}} \end{bmatrix}. \tag{43}$$

$$A''_\lambda(s) = \begin{bmatrix} 2 - \frac{2}{\lambda^2} & 0 \\ 0 & -\frac{2}{\lambda^2} \end{bmatrix}. \tag{44}$$

From this it easily follows that the desired J-spectral factorization is $A_\lambda = C^*_\lambda J C_\lambda$, with

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{45}$$

$$C_\lambda(s) = \begin{bmatrix} (s - \frac{8-4\lambda^2-\lambda^4}{\Delta_\lambda\sqrt{2}})\sqrt{2 - \frac{2}{\lambda^2}} & -\frac{8-4\lambda^2+\lambda^4}{\Delta_\lambda\sqrt{2}}\sqrt{2 - \frac{2}{\lambda^2}} \\ -\frac{-8+12\lambda^2-5\lambda^4}{\Delta_\lambda\sqrt{2}}\sqrt{\frac{2}{\lambda^2}} & -(s - \frac{8-12\lambda^2+3\lambda^4}{\Delta_\lambda\sqrt{2}})\sqrt{\frac{2}{\lambda^2}} \end{bmatrix}. \tag{46}$$

Inspection shows that as λ approaches one of the roots of $\Delta_\lambda = 0$ most of the coefficients of the spectral factor C_λ approach infinity.

Next assume that λ equals one of the roots of $\Delta_\lambda = 0$, for instance, $\lambda = \sqrt{4 + 2\sqrt{2}}$. For this value of λ the factorization is singular. Again taking $\zeta = \frac{1}{2}\sqrt{2}(-1 + j)$, the rank of the matrix $E = [p \; q]$ now is one, so that the extraction is of the type 21. We choose $k = 1$. With or without Mathematica it is straightforward to find that

$$[a \; b] = \begin{bmatrix} 1 & 0 \\ 1 + \sqrt{2} & 0 \end{bmatrix}, \tag{47}$$

so that the elementary factor of type 21 we extract is

$$T_o(s) = \begin{bmatrix} 1 + s\sqrt{2} + s^2 & 0 \\ -1 + \sqrt{2} & 1 \end{bmatrix}. \tag{48}$$

The "remaining factor" $A''_o = (T^*_o)^{-1} A T_o^{-1}$ may be found to be given by

$$A''_o(s) = \begin{bmatrix} 0 & \frac{1}{2}\sqrt{2} \\ \frac{1}{2}\sqrt{2} & \frac{1+\sqrt{2}+s^2}{2+\sqrt{2}} \end{bmatrix}. \tag{49}$$

This factor is nonconstant unimodular, which confirms that the factorization is singular. Using the unimodular factorization

$$\begin{bmatrix} 0 & \alpha \\ \alpha & p \end{bmatrix} = U^*_o \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} U_o, \qquad U_o = \sqrt{\frac{\alpha}{2}} \begin{bmatrix} 1 & \frac{p}{2\alpha} + 1 \\ 1 & \frac{p}{2\alpha} - 1 \end{bmatrix} \tag{50}$$

with α any positive number and p any symmetric scalar real polynomial, it may be found that the singular factorization we are looking for is $A = C^*_o J C_o$, where numerically

$$C_o = \begin{bmatrix} -1.55865 + 0.840896s + 0.297302s^2 & 0.891905 + 0.123146s^2 \\ 1.31235 + 0.840896s + 0.297302s^2 & -0.297302 + 0.123146s^2 \end{bmatrix}. \tag{51}$$

Note that the column degrees of this factor are both two, instead of one as for the nonsingular case.

4 Diagonalization

The second algorithm to be presented here is based on the diagonalization[8] of given matrix. It avoids the need to compute the zeros but it consists of more involved polynomial operations. It works under the assumption:

Assumption (6) *Let for every nonzero invariant polynomial* $i_l(A)(s)$ *of* $A(s)$ $i = 1, \ldots, n$, *holds that if it possesses zeros on* $\text{Re}\, s = 0$, *then they must be of even multiplicity.*

which clearly equals the condition of Theorem 1. In fact, the algorithm provides a constructive proof of Theorem 1 as well.

Algorithm 1 *For a given* $A(s)$ *satisfying (4.A) we calculate the desired* $C(s)$ *as well as the corresponding* K *in the following steps:*

STEP 1. *Find a diagonal form* $D(s) = \text{diag}\{d_l(s)\}$ *of* $A(s)$ *and the corresponding unimodular matrices* $V(s)$ *and* $W(s)$ *such that*

$$A(s) = V(s)D(s)W(s) \tag{52}$$

and

(a) $D(s)$ *is para-Hermitian* ($D^*(s) = D(s)$)

(b) $D(s)$ *is nonnegative definite on* $\text{Re}\, s = 0$

In addition, if some $d_l(s)$ *appears to be a zero polynomial (due to the singularity of given* $A(s)$ *), replace it by 1 while zeroing the corresponding column of* $V(s)$ *at the same time. Such a way,* $D(s)$ *is always nonsingular while* $V(s)$ *is a generalized unimodular matrix*[9] *(singular if* $A(s)$ *is so).*

STEP 2. *Perform scalar spectral factorizations of diagonal entries of* $D(s)$

$$d_l(s) = f_l^*(s) f_l(s) \tag{53}$$

and form a (nonsingular) Hurwitz polynomial matrix

$$F(s) = \text{diag}\{f_l(s)\} \tag{54}$$

Clearly

$$D(s) = F^*(s)F(s) \tag{55}$$

STEP 3. *Compute matrices*

$$X(s) = W^{-*}(s)V(s) \tag{56}$$

[8] In fact, this method appears to be a polynomial counterpart of the rational algorithm from [11].

[9] A (possibly singular) polynomial matrix is called *generalized unimodular* if all its invariant polynomials are just real constants (possibly zeros).

$$Y(s) = F^{-*}(s)X(s)F^{*}(s) \quad (57)$$

both of which appear to be polynomial and generalized unimodular. Moreover, $Y(s)$

is para-Hermitian ($Y^*(s) = Y(s)$).

STEP 4. *Find a nonsingular (unimodular) polynomial matrix* $Z(s)$ *along with a (possibly singular) real matrix* K *so that*

$$Y(s) = Z^*(s)KZ(s) \quad (58)$$

RESULT *A desired Hurwitz spectral factor is now*

$$C(s) = Z(s)F(s)W(s) \quad (59)$$

PROOF:
See [9] □

Particular steps of the algorithm are standard as discussed in [9].

Example 5 Consider now the same $A(s)$ as in Example 4, i.e.

$$A(s) = \begin{bmatrix} -0.25 - 0.75s^2 & -s \\ s & 0.5 + 0.5s^2 \end{bmatrix}$$

Its diagonal form (52) is found to be

$$D(s) = \begin{bmatrix} 0.5 & 0 \\ 0 & -0.25 + s^2 - 0.75s^4 \end{bmatrix}$$

After two scalar spectral factorizations we get the diagonal factor (54)

$$F(s) = \begin{bmatrix} 1 & 0 \\ 0 & 0.5774 + 1.5774s + s^2 \end{bmatrix}$$

Then the matrix $X(s)$ from (56) is

$$X(s) = \begin{bmatrix} 0.5 + 0.5s^2 & 0.375s \\ -0.125s - 0.375s^5 & -0.75 + 0.6563s^2 - 0.2813s^4 \end{bmatrix}$$

while $Y(s)$ from (57)

$$Y(s) = \begin{bmatrix} 0.5 + 0.5s^2 & 0.2165s + 0.375s^3 \\ -0.2165s - 0.375s^3 & -0.75 + 0.6563s^2 - 0.2813s^4 \end{bmatrix}$$

Performing finally the unimodular factorization (58) we result in

$$J = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$Z(s) = \begin{bmatrix} -2.7321 & 3.2321 - 2.049s \\ -2.639 - 0.7071s & 3.3461 - 1.1427s - 0.5303s^2 \end{bmatrix}$$

so that the desired Hurwitz spectral factor (59) is

$$C(s) = \begin{bmatrix} 1.866 - 0.866s & -2.7321 \\ 1.9319 & -2.639 - 0.7071s \end{bmatrix}$$

which possesses its zeros in -1 and -0.5774.

5 Conclusions

Unlike the standard case, the J-spectral factorization copes with polynomial matrices which are indefinite (rather than positive definite) on the imaginary axis. This change is reflected in a variety of different properties tackled in the paper concerning existence and uniqueness *(Sec 2)* and numerical aspects *(Sec. 3 and 4)* of the solution.

The two numerical algorithms which are proposed are the only methods of J-spectral factorization available so far. The first algorithm *(Sec. 3)*, which is based on the ***generalized symmetric factor extraction***, features simplicity and it is easy to implement via MATLAB . Its stumbling block appears to be the need to calculate zeros (roots) of the given polynomial matrix. In presence of an internal singularity *(Sec. 3)*, it behaves as follows: the precision drops when approaching such a situation, but once it is reached the algorithm provides a precise yet partial factor. Fortunately, this is exactly all we need to compute a reduced $\mathcal{H}_\infty$-optimum regulator (see [7]).

The second algorithm *(Sec. 4)*, which is based on the ***diagonalization***, applies for a more general class of matrices (including those with zeros on the imaginary axis). It also obviates the need to compute zeros which is a quality one can enjoy namely in the presence of multiple zeros close to or on the imaginary axis. Finally, it handles matrices of any signature and it is believed to be numerically attractive even in the standard (positive definite) case. On the other hand, it consists of more complex polynomial operations. In its present form, the method does not look after the internal singularity. When the given matrix

is close to it, the precision cuts down, of course, as during the first procedure but once the

phenomenon is reached, the algorithm provides precise full factor (of higher degrees).

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