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A state-space algorithm for the spectral factorization[1]

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Abstract

This paper presents an algorithm for the spectral factorization of a para-Hermitian polynomial matrix. The algorithm is based on polynomial matrix to state space and vice versa conversions, and avoids elementary polynomial operations in computations; It relies on well-proven methods of numerical linear algebra such as Schur decompositions. *Keywords:* Subspace methods, Numerical methods, Linear systems.

1 Introduction

Polynomial matrices play an important role in linear systems and control theory. The present algorithm for the spectral factorization of a diagonally reduced para-Hermitian polynomial matrix Z is useful in control with quadratic cost functionals. We look for a square polynomial matrix Q and a signature matrix J such that

$$Z(s) = Q^*(s)JQ(s), \quad J = \begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix}. \qquad (1)$$

The roots of Q all lie in the open left-half plane and the column degrees of Q equal the *half-diagonal degrees* of Z. Sufficient but not necessary for the existence of such a factorization is that Z has no roots on the imaginary axis. If the factorization (1) exists such that Q has the correct column degrees then the factorization is said to be *canonical.*

A square polynomial matrix Z is *para-Hermitian* if $Z^* = Z$ where the *adjoint* Z^* is the polynomial matrix defined by $Z^*(s) = Z^{\mathrm{H}}(-s)$. The $m \times m$ para-Hermitian Z is *diagonally reduced* if there exist *half-diagonal degrees* $\delta_1, \delta_2, \cdots, \delta_m$, so that the *leading diagonal coefficient matrix*

$$Z_{\mathrm{L}} = \lim_{|s|\to\infty} D^{-1}(-s)Z(s)D^{-1}(s) \qquad (2)$$

exists and is nonsingular. D is the diagonal matrix $D(s) = \mathrm{diag}(s^{\delta_1}, s^{\delta_2}, \cdots, s^{\delta_m})$.

If Z is not diagonally reduced then it may be made so by a symmetric unimodular transformation [1]. Other algorithms for spectral factorization are described in [2, 3]. The present algorithm is simpler, and does not require elementary polynomial operations.

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2 The algorithm

The algorithm for (1) is viewed as a special case of $Z = SR$, where S has all its roots in the open right-half plane and R has all its roots in the open left-half plane. Additionally, R is column reduced, the column degrees of R equal the half-diagonal degrees of Z, and S has a special form $S = R^*K^{-1}$, with K a constant real nonsingular symmetric matrix. That is, (1) is initially sought in a form

$$Z(s) = R^*(s)K^{-1}R(s). \qquad (3)$$

1. Find $\frac{d}{dt}x(t) = Ax(t)$, $w(t) = Cx(t)$ as an observable state-space realization of the differential equation $Z\left(\frac{d}{dt}\right) w(t) = 0$.

2. Use Schur transformation to transform the coordinates of the realization (A, C) such that

$$\begin{aligned} A &= U^{\mathrm{H}}\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} U, \\ C &= \begin{bmatrix} C_1 & C_2 \end{bmatrix} U, \end{aligned}$$

where A_{11} has all its roots in the open left-half plane and A_{22} has all its roots in the open right-half plane.

3. Convert $C_1(sI - A_{11})^{-1} = R^{-1}(s)E(s)$ such that R is a square polynomial matrix whose column degrees equal those of Z.

4. Then $R(s)Z^{-1}(s)R^*(s) = K$, where K is a constant Hermitian matrix $K = R(0)Z^{-1}(0)R^*(0)$.

5. If $R(s)$ is nonsingular[1], then K is nonsingular and

$$Z(s) = R^*(s)K^{-1}R(s).$$

The desired spectral factorization $Z(s) = Q^*(s)JQ(s)$ with $Q(s) = VR(s)$, follows from the decomposition $K^{-1} = V^{\mathrm{H}}JV$. The constant matrix V is obtained from the Schur decomposition of L^{-1} by permutation.

Separately, if $Z(s) = P^*(s)WP(s)$ as in $\mathcal{H}_\infty$ applications, then we define $WP(s)w = z$. The system $Z(s)w = 0$ is equivalently represented by the two equations $P(s)w = W^{-1}z$ and $P^*(s)z = 0$, or

$$\underbrace{\begin{bmatrix} P(s) & -W^{-1} \\ 0 & P^*(s) \end{bmatrix}}_{T(s)} \begin{bmatrix} w \\ z \end{bmatrix} = 0. \qquad (4)$$

[1] A polynomial matrix is nonsingular if it is square and its determinant is not identically zero.

The realization of (4) is in the form

$$\frac{d}{dt}x(t) = Ax(t), \quad \begin{bmatrix} w(t) \\ z(t) \end{bmatrix} = Cx(t). \tag{5}$$

Omitting the equation for z results in the desired observable realization of $P^*(s)WP(s)w = 0$.

3 The state-space realization

The state-space realization does not require computation of elementary polynomial operations. The algorithm is a special case of [4]. The polynomial matrix operator $Z(s)$ is considered as a matrix polynomial $Z(s) = Z_z s^z + \cdots + Z_1 s + Z_0$ in the differential operator $s = \frac{d}{dt}$. The 'external' variables $w : \mathbb{R} \to \mathbb{R}^m$ are from the set of all infinitely often differentiable functions.

1. Introduce 'internal' variables ξ that convert $Z(s)w = 0$ to an externally equivalent[2] form

$$\begin{aligned} P(s)\xi &= 0, \\ Q\xi &= w. \end{aligned} \tag{6}$$

$$\begin{bmatrix} P(s) \\ Q \end{bmatrix} = \left[\begin{array}{cccccc} I & -sI & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I & \ddots & 0 \\ Z_z & \cdots & \cdots & Z_1 & Z_0 & -sI \\ 0 & \cdots & \cdots & \cdots & 0 & I \\ \hline 0 & \cdots & \cdots & 0 & I & 0 \end{array}\right]$$

Permutation of the 'internal' variables ξ transforms (6) into

$$\begin{bmatrix} P(s) \\ Q \end{bmatrix} = \left[\begin{array}{cc} sI - A & -B \\ C & 0 \\ \hline H & 0 \end{array}\right]. \tag{7}$$

2. Compute a state-to-input feedback matrix F such that the maximal $(A, \operatorname{span} B)$-controlled invariant subspace contained in $\ker C$ becomes $(A + BF)$-invariant[3]. Use the relevant basis transformation to convert the system description into

$$\begin{bmatrix} P(s) \\ Q \end{bmatrix} = \left[\begin{array}{ccc} sI - A_{11} & -A_{12} & -B_1 \\ 0 & sI - A_{22} & -B_2 \\ 0 & C_2 & 0 \\ \hline H_1 & H_2 & 0 \end{array}\right] \tag{8}$$

where A_{11} determines the maximal controlled invariant subspace. Correspondingly, (A_{22}, B_2, C_2) is a strongly observable state-space realization.

3. According to [7], Theorem 1.8, a state-space realization is strongly observable if and only if it has no zeros. Since

$$U(s) := \begin{bmatrix} sI - A_{22} & -B_2 \\ C_2 & 0 \end{bmatrix} \tag{9}$$

is a square polynomial matrix with no finite zeros, $U(s)$ is unimodular. $U(s)$ guarantees existence of a transformation matrix such that an equivalent system description is in the form

$$\begin{aligned} \frac{d}{dt}x(t) &= A_{11}x(t) \\ w(t) &= H_1 x(t). \end{aligned} \tag{10}$$

where A_{11} and H_1 are directly inherited from (8). By construction, (10) is externally equivalent to $Z\left(\frac{d}{dt}\right) w(t) = 0$.

4 Conclusions

Given a diagonally reduced para-Hermitian polynomial matrix Z, the result of the algorithm is a square column-reduced polynomial matrix Q with the column degrees equal to the half-diagonal degrees of Z and such that the roots of Q all lie in the open left-half plane; $Z = Q^* J Q$ with J the signature.

In case of a nearly noncanonical factorization [2], the algorith stops at $Z(s) = R^*(s)K^{-1}R(s)$ with R square column-reduced polynomial matrix with correct column degrees and roots. K is a constant real nearly singular symmetric matrix. The nearly noncanonical form is applicable in $\mathcal{H}_\infty$-optimization [3].

The contribution is that the algorithm avoids computation of elementary polynomial operations, and relies on standard numerical linear algebra for constant matrix computations.[4] The principal application of the algorithm is that of a building block for higher-level algorithms including $\mathcal{H}_2$- and $\mathcal{H}_\infty$-optimization [3].

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[2]Operational form for external equivalence is explained in [5].

[3]A standard reference for invariant and controlled invariant subspaces in control theory is [6].

[4]This is important for safeguarding the numerical properties of the algorithm. Current implementation is based on MATLAB kernel and the software appendix to [6].