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Controllability Distributions and Systems Approximations: a Geometric Approach

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Abstract

Given a nonlinear system we determine a relation between controllability distributions defined for a nonlinear system and a Taylor series approximation of it. Special attention is given to this relation at the equilibrium. It is known from nonlinear control theory that the solvability conditions as well as the solutions to some control synthesis problems can be stated in terms of geometric concepts like controlled invariant (controllability) distributions. Here, by dealing with a k-th Taylor series approximation of the system, we are able to decide when the solvability conditions of this kind of problems are equivalent for the nonlinear system and its approximation. Additionally, we will distinguish some cases when the solution obtained from the approximated system is an approximation of an exact solution for the original problem. Some examples illustrate the results.

1. Introduction

For decades it has been common practice of control engineers to solve nonlinear control synthesis problems by using a linear approximation of the nonlinear system around an operating point and after application of linear control techniques, use the resulting linear solution as a linear approximation of a true solution for the original nonlinear control problem, see e.g. [SM1], [SM2], [THN], and other applications. That this approach is in some cases successful for specific control objectives like input-output decoupling, model matching, etc., is partially understood, see [GN], [HN], [vdW], [RN1], [RN2],[vdS]. However, it is not a general rule that this linearization procedure is always justified. That is, even in the case when a particular nonlinear control problem is solvable for the nonlinear system and for the linearization, still the solutions of the linear problem do not necessarily act as a first order approximation of a solution for the nonlinear problem, as it is stated as a principle in [S], (pag.5).
In a practical situation it is an advantage to realize to what extent the solution for a particular control problem obtained by using a linearization of the nonlinear system, can be used as an approximation of any true solution for the original problem. Certainly, the intuitive idea that a higher order Taylor series approximation for Σ will provide 'better' results than just the standard linearization or the second order approximation and so forth, is not completely false.
In nonlinear control theory, differential geometric concepts as controlled invariant and controllability distributions play a fundamental role in the solution of synthesis problems like disturbance decoupling, input-output decoupling, etc., (see [NvdS],[I],[HG]). Not only the solvability conditions of this kind of problems can be stated in terms of these distributions but also these distributions are fundamental to characterize all solutions for a particular control problem, see e.g. [NvdS],[HG],[I] and [G] in a linear context. We consider nonlinear analytic control systems of the form:

$$\Sigma \begin{cases} \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i = f(x) + g(x)u, \\ y_i = h_i(x), \qquad i \in \underline{m} = \{1,\ldots,m\}, \end{cases}$$

defined on an open neighborhood M of x_0 in $\mathbb{R}^n$, where x_0 is an equilibrium point for Σ, i.e., $f(x_0) = 0$, $h(x_0) = 0$, $h := (h_1,\ldots,h_m)^T$. Without loss of generality we assume troughout $x_0 = 0$.
We approximate the system Σ and thus the vectorfields $f, g_1, \ldots, g_m$, and the output functions $h_1,\ldots,h_m$, by means of a Taylor series expansion of $f, g_1,\ldots,g_m$, respectively, $h_1,\ldots,h_m$, around the equilibrium point 0. Regarding Σ as a system locally defined about 0 we denote the k-th order approximation of Σ as:

$$\Sigma^k \begin{cases} \dot{x} = f^k(x) + \sum_{i=1}^{m} g_i^{k-1}(x)u_i = f^k(x) + g^{k-1}(x)u, \\ y_i^k = h_i^k(x), \qquad i \in \underline{m}, \end{cases}$$

here f^k, respectively h^k, is the k-th order approximation of f, respectively h, about the equilibrium and g_i^{k-1} is the $(k-1)$-th order Taylor series of g_i about 0. Clearly, Σ^k is defined on the same state space as Σ.
Suppose we want to solve the input-output decoupling problem for a given system Σ. It is well known that all solutions to this problem are characterized by means of a set of maximal controllability distributions contained in the kernel of the output map, [HG]. Now, we may handle this problem as a 'control engineer' and attempt to solve the control problem by solving the analog control objective associated with a k-th order aproximation of Σ. With the hope that the solution obtained serves as a k-th order approximation of a true solution for Σ, we would in general, as will be shown, have made an incorrect conclusion, unless, as happens in exceptional cases, the above mentioned maximal controllability distributions agree with those of the system Σ in a neighborhood of the equilibrium.
For this reason we concentrate on the analysis of geometric objects inherent to the system Σ and Σ^k. In particular we will define controllability distributions for both systems and investigate the relation between these objects. Special interest has its relation at the equilibrium. Although we do not treat in detail how a solution based on Σ^kapproximates a true solution for the problem defined for Σ, the analysis of these specific distributions constitute a useful tool to deal with this question.
The problem of approximating a nonlinear system in a suitable manner is quite popular, see e.g. [H], [B], [Cr]. All these references differ substantially from our work in at least two aspects.

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First of all, the approximation we use, though only of interest locally about $x = 0$, is defined on the same state space and not on a possible larger state space. Secondly, in contrast to the forementioned works, we concentrate on tools that are of interest in synthesis problems as input-output decoupling, etc. Of course, a similar study starting with the work of e.g. [II] instead of the Taylor series used here, seems possible. In [K1], a nonlinear change of coordinates and feedback are used to construct linear approximations that are accurate to higher orders, see also [K2],[K3].
The analysis follows basically the same philosophy of [RN2] where controlled invariant distributions defined for Σ^k and Σ^{k+1} were studied.
The main result of the paper is contained in section 4 in which we relate the maximal controllability distributions contained in the kernel of the output map defined for Σ and Σ^k. In section 2 we fix our notation and recall some basic definitions. Section 3 deals with maximal controllability distributions contained in TM (the tangent space of M), defined for Σ and Σ^k. Finally, conclusions are drawn in section 5.

2. Preliminaries

With respect to the nonlinear systems Σ and Σ^k we will impose throughout the following conditions on the input vectorfields $g(g^{k-1})$ and the output maps $h(h^k)$, respectively. Define $G := \text{span}\{g_1,\dots,g_m\}$, $G^k := \text{span}\{g_1^{k-1},\dots,g_m^{k-1}\}$ and $\mathrm{d}h$ respectively $\mathrm{d}k^k$ the codistributions span $\{\mathrm{d}h_1,\dots,\mathrm{d}h_m\}$,respectively, span $\{\mathrm{d}h_1^k,\dots,\mathrm{d}h_m^k\}$.

Assumption 1
Consider the systems Σ and Σ^k, $k \geq 1$. Assume that:

(i) $\dim G = \dim G^k = m$ on M.

(ii) $\dim \mathrm{d}h = \dim \mathrm{d}h^k = m$ on M. ∎

The following definitions of involutive and controlled invariant distributions are the starting point in of section 4.
We say that a distribution Δ is involutive if $[X, Y] \subset \Delta$ for all $X, Y \in \Delta$.

Definition 2.1 ([NvdS]): A C^w constant dimensional involutive distribution Δ is said to be *controlled invariant* if there exists a C^w regular static state feedback

$$u = \alpha(x) + \beta(x)v \tag{1}$$

with $\alpha : M \longrightarrow \mathbb{R}^m$, $\beta : M \longrightarrow \mathbb{R}^{m\times m}$, $\beta(x)$ a nonsingular matrix for all $x \in M$ and $v \in \mathbb{R}^m$ such that after applying (1) to Σ the modified vectorfields $\tilde{f} := f + g\alpha$, $\tilde{g}_i := (g\beta)_i$, $i \in \underline{m}$, satisfy $[\tilde{f}, X] \subset \Delta$, $[\tilde{g}_i, X] \subset \Delta$, $i \in \underline{m}$, for all $X \in \Delta$. ∎

In section 4 the notion of maximal controllability distribution in $\ker \mathrm{d}h$ is used. We give now a formal definition of this concept. First, let $C(x)$, the accessibility distribution for Σ, be the the smallest involutive distribution in TM invariant under f which contains the vectorfields $\{g_1,\dots,g_m\}$. A maximal controllability distribution is an involutive distribution that contains a distribution $\hat{G} \subset G$ and which is invariant under $\tilde{f}$ and $\tilde{g}_i$, $i \in \underline{m}$, for some feedback (1).
Denote by $\Delta^*(\Delta_k^*)$, $(\Pi^*(\Pi_k^*))$, the maximal controlled invariant distribution (maximal controllability distribution) contained in $\ker \mathrm{d}h$ ($\ker \mathrm{d}h^k$) of Σ (Σ^k), respectively. An algorithm to compute Π^*, which plays a decisive role in this paper, is taken from [I]:

Algorithm

$$\Pi_0 = \Delta^* \cap G,$$
$$\Pi_{\mu+1} = \Delta^* \cap ([f, \Pi_\mu] + \sum_{i=1}^{m} [g_i, \Pi_\mu] + G), \qquad \mu \geq 0. \quad ∎$$

In connection with this algorithm we make some standard assumptions.

Assumption 2

(i) For each $k \geq 1$, there exists an integer $k^*(k^*(k)) \geq 0$ such that the Algorithm terminates when applied to $\Sigma(\Sigma^k)$.

(ii) Π_i, $i \in \mathbb{N}$, has constant dimension on M.

(iii) The distribution Δ^* has constant dimension on M.

Remark 2.2 The Algorithm provides a sequence of nonincreasing distributions and it terminates whenever $\Pi_{\mu+1} = \Pi_\mu$, for some $\mu \geq 0$, [I]. Moreover, since $\dim \ker \mathrm{d}h = n - m$ and thus $\dim \Delta^* \leq n - m$ then $\dim \Pi^* \leq n - m$.
Actually, $C(x)$ associated to Σ can be computed by applying the Algorithm and replacing Δ^* by TM. $C(x)$ is usually called the strong accessibility distribution of Σ. Here we will investigate in which cases does there exist a relation between $C(x)$ and $C^k(x)$ at the equilibrium. The relevance of knowing such a relation is better appreciated if we recall that the system Σ is said to be locally strong accessible about 0 if and only if $\dim C(0) = n$, [SJ]. Therefore, a relation of $C(0)$ with $C^k(0)$ can be useful to recognize accessibility properties of Σ by a hopefully simpler analysis of Σ^k.
In the next section we distinguish some cases in which we can relate the distributions $C(x)$ of Σ with $C^k(x)$ of Σ^k and their relation at the equilibrium.

3. Accessibility Distributions for Σ and Σ^k

To give an insight of the problem we start by studying the accessibility distribution defined for Σ and its first order approximation Σ^1.
The first order approximation of Σ is given by

$$\Sigma^1 \left\{ \ \dot{z} = Az + B\tilde{u}, \ \ \tilde{y} = Cz, \right.$$

with $A := \frac{\partial f}{\partial x}(0)$, $B := G(0)$, $C := \frac{\partial h}{\partial x}(0)$, $\tilde{u} \in \mathbb{R}^m$ and $\tilde{y} \in \mathbb{R}^m$. The distribution $C^1(z)$ reduces in this case to the well known controllability subspace of Σ^1. That is, if we denote by $R \equiv C^1(z) \equiv C^1(0)$ the controllability subspace of Σ^1, then R is given as:

$$R := < A \mid \mathrm{Im}B >= \text{span}\{B, AB, \dots, A^{n-1}B\}. \tag{2}$$

Our interest now is to compare the controllability subspace R defined for Σ^1 with the maximal controllability distribution $C(x)$ associated with Σ, at the equilibrium. To do so, we may consider the linear subspace R as a flat distribution on TM. In general, we have that $C^1(0) \subseteq C(0)$. If $C^1(0) = C(0)$ always holds, we could deduce local accessibility properties of Σ based on an analysis of the controllability subspace of the linearization. That is, if $\dim C^1(0) = n$ then necessarily $C(0) = C^1(0)$ and thus Σ should be locally strong accessible at 0. This fact can be concluded from [Su]. But since in general, $C^1(0) \neq C(0)$, the next obvious step would be to perform a second order approximation of Σ, denoted as Σ^2, and compare $C^2(0)$ with $C(0)$. Intuitively, we expect that $C^2(0) \subseteq C(0)$. However, it will turn out that this is not always the case. Some examples of this phenomena are in order. For this we introduce some extra notation.
Let C_i^k, $i = 0, 1, \dots$, be a set of distributions associated with Σ^kdefined as:

$$C_0^k(x) = \text{ span}\{g_1^{k-1},\dots,g_m^{k-1}\}(x)$$
$$C_i^k(x) = \text{span}\{[X_i,[\dots,[X_1,g_j^{k-1}]]](x) + \sum_{\ell=0}^{i-1} C_\ell^k(x), j \in \underline{m}\}, \tag{3}$$

for $i = 1, 2, \dots$, and X_j any vectorfield in the set $\{f^k, g_1^{k-1},\dots, g_m^{k-1}\}$. In a similar way we define the distributions $C_i(x)$ $i = 0, 1, \dots$, for Σ. Note that the distributions $C_i^k(0)$, $i = 0, 1, \dots$ defined in (3) correspond to the Algorithm with Δ^* replaced by TM.

Example 3.1 Consider the nonlinear systems

$$\Sigma:\ \dot{x}=g_1(x)u_1+g_2(x)u_2,\quad \Sigma^2:\ \dot{x}=g_1^1(x)u_1+g_2^1(x)u_2;$$

$$g_1(x)=(1+x_2,1,1,0,0)^T,\ g_2(x)=(x_1^2,1+x_3+x_5^2,0,1,x_4^2)^T,$$

$$g_1^1(x)=(1+x_2,1,1,0,0)^T,\ g_2^1(x)=(0,1+x_3,0,1,0)^T.$$

In this example we have that $C(0)\subset C^2(0)$ but not $C(0)=C^2(0)$, i.e., at the equilibrium point $(x_0,u_0)=(0,0)$, the maximal controllability distribution associated with Σ is 'strictly included' in the analogue controllability distribution defined for Σ^2. To see this note that $C_0^2(0)=C_0(0)$, $C_1^2(0)=C_1(0)$, $C_2^2(0)=C^2(0)$ and $C_2(0)=C(0)$ with

$$C^2(0)=\text{span}\{(\,1\ \ 1\ \ 1\ \ 0\ \ 0\,)^T,(\,0\ \ 1\ \ 0\ \ 1\ \ 0\,)^T,$$
$$(\,1\ \ -1\ \ 0\ \ 0\ \ 0\,)^T,(\,2\ \ 0\ \ 0\ \ 0\ \ 0\,)^T,\},$$

$$C(0)=\text{span}\{(\,1\ \ 1\ \ 1\ \ 0\ \ 0\,)^T,\ (\,0\ \ 1\ \ 0\ \ 1\ \ 0\,)^T,$$
$$(\,1\ \ -1\ \ 0\ \ 0\ \ 0\,)^T\}$$

then $C(0)\subset C^2(0)$ but not $C(0)=C^2(0)$. ∎

The following example shows that in some cases we are not able to establish any relation of maximal controllability distributions defined for a nonlinear system and an approximation of it.

Example 3.2 Consider the systems Σ and Σ^2 with

$$g_1(x)=(1+x_2,1,1,0,1)^T,\quad g_2(x)=(0,1+x_3,0,1,x_5^2)^T,$$
$$g_1^1(x)=(1+x_2,1,1,0,1)^T,\quad g_2^1(x)=(0,1+x_3,0,1,0)^T.$$

For Σ^2 and Σ as above we have that $C^2(0)\neq C(0)$, with which we mean that neither $C^2(0)\subset C(0)$ or $C(0)\subset C^2(0)$ are true.
A few calculations show that $C_0^2(0)=C_0(0)$, $C_1^2(0)=C_1(0)$, $C_2^2(0)$= $C^2(0)$, $C_2(0)=C(0)$, with

$$C^2(0)=\text{span}\{(\,1\ \ 1\ \ 1\ \ 0\ \ 1\,)^T,(\,0\ \ 1\ \ 0\ \ 1\ \ 0\,)^T,$$
$$(\,1\ \ -1\ \ 0\ \ 0\ \ 0\,)^T,(\,2\ \ 0\ \ 0\ \ 0\ \ 0\,)^T\},$$

$$C(0)=\text{span}\{(\,1\ \ 1\ \ 1\ \ 0\ \ 1\,)^T,(\,0\ \ 1\ \ 0\ \ 1\ \ 0\,)^T,$$
$$(\,1\ \ -1\ \ 0\ \ 0\ \ 0\,)^T,(\,2\ \ 0\ \ 0\ \ 0\ \ -2\,)^T\}$$

Therefore we have $C(0)\neq C^2(0)$ with no inclusion relation among them. ∎

An explanation of the underlying idea in the Examples 3.1 and 3.2 is in order. Consider the systems $\Sigma(\Sigma^k)$ and the set of vectorfields associated to them $\{f,g_1,\ldots,g_m\}(\{f^k,g_1^{k-1},\ldots,g_m^{k-1}\})$, respectively. It will be shown that for every Lie bracket up to order $(k-1)$ of the vectorfields $f,g_1,\ldots,g_m$ there exist a Lie bracket of the same order with the vectorfields $f^k,g_1^{k-1},\ldots,g_m^{k-1}$ such that they agree when evaluated at the equilibrium. This is a rough statement of what is claimed in Proposition 3.3, below. From this proposition a fundamental conclusion will be made, namely, suppose the system Σ is locally strong accessible at 0, i.e., dim $C(0)=n$, then there exist an integer $s^*\geq 0$, such that the Lie Brackets up to order s^* of the vectorfields $f,g_1,\ldots g_m$ span $C(0)$. Then it suffices to take the (s^*+1)-th Taylor series approximation of Σ to ensure that Σ^{s^*+1} is also locally strong accessible at 0 and that $C^{s^*+1}(0)=C(0)$. Evenmore, if the distributions $C_i(x)$, $i\in \underline{m}$, are assumed to be of constant dimension, then $s^*\leq n$. Now, if the strong accessibility assumption on Σ is dropped, it is reasonable to expect that the equality of $C(0)$ and $C^k(0)$ for any $k\geq 0$, will in general not be obtained.
In the Examples 3.1 and 3.2, observe that Σ, in both cases, is not locally strong accessible at 0. Only the third order approximation of Σ will be such that $C(0)=C_3(0)$ because the system Σ equals its third order approximation.
The next result relates the distributions $C_i^k(x)$ with $C_i(x)$ for $i=0,1,\ldots,k-1$. Making an abuse of notation, for any distribution $D(x)$, $(D(x))^k$, denotes the k-th order Taylor series approximation of every vectorfield $X(x)\in D(x)$. Similarly, for $X(x)\in D(x)$, we let $(X(x))^k$ be the k-th order approximation of $X(x)$ around the equilibrium.

Proposition 3.3 Consider the analytic systems Σ, Σ^k, for a fixed $k\geq 1$ and the distributions C_i, C_i^k, $i=0,1,\ldots$, as defined in (3). Then

$$(C_i^k(x))^{k-1-i}=(C_i(x))^{k-1-i},\quad i=0,1,\ldots,k-1.$$

i.e., for each $X(x)\in C_i(x)$ there exist a $\tilde{X}(x)\in C_i^k(x)$ such that $(X)^{k-1-i}=(\tilde{X})^{k-1-i}$ and vice versa. ∎

Proof. See the appendix where a similar result for controllability distributions is proven. The proof follows by taking $\Delta^*=TM$. ∎

From Proposition 3.3 follows that $C_i^k(0)=C_i(0)$, $i=0,\ldots,k-1$.

Example 3.4 Consider the systems Σ and Σ^2 of Examples 3.1 and 3.2. Observe that for these systems we have $C_i^2(0)=C_i(0)$, for $i=0,1$, in both cases. Only these relations are valid but not for $i=2$ because it yields the results illustrated in previous examples. ∎

Although in general, $C^k(0)\neq C(0)$, for an arbitrary $k\geq 1$, it is desirable to know if there exists a minimal nonnegative integer s^* for which $C^{s^*}(x)$, the accessibility distribution associated with Σ^{s^*}, yields $C^{s^*}(0)=C(0)$. An answer to this question is given in

Corollary 3.5 Consider the nonlinear system Σ. Assume Σ is locally strong accessible at 0 and let τ be the smallest nonnegative integer for which dim $C_\tau(0)=$ dim $C(0)=n$, on M. Then

$$C^{s^*}(0)=C(0),\quad s^*=\tau+1,$$

with $C^{s^*}(x)$ the accessibility distribution associated with Σ^{s^*}. ∎

The concept of accessibility distribution is closely related to that of maximal controllability distribution contained in the kernel of the output map. In the next section we define similar objects for Σ and $\Sigma^k$and find a relation among them.

4. Controllability Distributions contained in the Kernel of the Output Map defined for Σ and Σ^k

In this section we will investigate the relation, if any, of the maximal controllability distribution contained in the kernel of the output map defined for Σ at the equilibrium, with the analogous object defined for Σ^k. To some extent, a controllability distribution Π can be seen as the nonlinear analogue of a controllability subspace R contained in the kernel of the output of a linear system of the form Σ^1. One difference lies in the fact that the dynamics of a linear system restricted to R are controllable and thus stabilizable ([Wo]), while in the nonlinear setting, the dynamics restricted to Π are not necessarily stabilizable, see e.g. [vdW],[Br]. Observe also that by definition, a controllability distribution is locally controlled invariant.
By means of the next example we will emphasize the importance of knowing the relation at the equilibrium of these kind of distributions between a nonlinear system and an approximation of it. In particular, we study the case when for a nonlinear system and the linear approximation of it, the Triangular Decoupling Problem (TDP) is solvable and compare the solutions obtained from the

linearization with the solutions of the original nonlinear problem. This comparison is based on specific controllability distributions defined for both systems. Therefore we briefly review the main ideas for solving the TDP.

Given the system Σ, the problem consists in finding a regular static state feedback as defined in (1) such that the closed loop system Σ together with (1) is triangular decoupled. That is, v_1 affects y_1 and possibly $y_2, \ldots, y_m$; v_2 influences y_2 and possibly $y_3, \ldots, y_m$ but not y_1, etc., (see [N], [RN1], for details).

Necessary and sufficient conditions to solve the TDP for Σ are given in terms of $(\Delta^p)^*$, the maximal controlled invariant distribution contained in

$$\bigcap_{i=1}^{p} \ker dh_i, \qquad p = 0, 1, \ldots, m, \tag{4}$$

with $(\Delta^0)^* := C(x)$, the accessibility distribution of Σ.

Theorem 4.1([RN1])
Assume the system Σ satisfies $\dim dh(C(x)) = m$, for all $x \in M$, and suppose that $(\Delta^p)^*$, $(\Delta^p)^* \cap G$, with $(\Delta^p)^*$ the largest controlled invariant distribution in (4), have constant dimension for $p = 0, 1, \ldots, m$. Then the TDP is locally solvable on M if and only if

$$\dim\left((\Delta^p)^* \bigcap G\right) = m - p, \qquad p = 0, 1, \ldots, m. \qquad \blacksquare$$

These facts are used in Example 4.2 below.

Example 4.2 ([RN1]) Consider the nonlinear control system

$$\begin{array}{ll} \dot{x}_1 = u_1 + x_1^2, & y_1 = x_1, \\ \dot{x}_2 = u_1 + e^{-x_2}, u_2 & y_2 = x_2. \\ \dot{x}_3 = u_1 + e^{x_2} - x_2 - 1 + x_1, & \end{array} \tag{5}$$

The control objective is to solve the TDP for (5). It can be checked that the conditions stated in Theorem 4.1 are satisfied and thus the TDP is locally solvable for (5).

The linearization of (5) around $(x_0, u_0) = (0, 0)$ is given by

$$\begin{array}{ll} \dot{z}_1 = \tilde{u}_1, & \tilde{y}_1 = z_1, \\ \dot{z}_2 = \tilde{u}_1 + \tilde{u}_2, & \tilde{y}_2 = z_2. \\ \dot{z}_3 = \tilde{u}_1 + z_1, & \end{array} \tag{6}$$

Now, the solutions for the linear TDP are characterized in terms of the maximal controllability distribution contained in the kernel of $d\tilde{y}_1$, denoted as $\tilde{R}_2^*$ and given by $\tilde{R}_2^* = \text{span}\{(010)^T\}$. Any feedback law $\tilde{u} = Fz + G\tilde{v}$ which leaves the system (6) triangular decoupled is of the form

$$\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix} = \begin{pmatrix} f_{11} & 0 & f_{13} \\ f_{21} & f_{22} & f_{23} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} + \begin{pmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \tag{7}$$

with $g_{11} \neq 0, g_{22} \neq 0$. For the nonlinear system (5) it can be shown that $\Pi_2^* = \text{span}\left\{\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right\}$, where Π_2^* is the maximal controllability distribution contained in the kernel of dy_1. But any feedback which leaves the nonlinear system (5) triangular decoupled must have the form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \alpha_1(x_1) \\ \alpha_2(x) \end{pmatrix} + \begin{pmatrix} \beta_{11}(x_1) & 0 \\ \beta_{21}(x) & \beta_{22}(x) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{8}$$

with $\beta_{11}(x) \neq 0$, $\beta_{22}(x) \neq 0$. Then, when $f_{13} \neq 0$ in (7), (7) does not correspond to a linearization of (8), i.e., the linear solution is not a first order approximation of a true solution for the nonlinear TDP. Basically, the reason for which the linear feedback law (7) is not a linearization, in general, of (8) is because of the following: for a system with $m = 2$ which is triangular decouplable, any feedback of the form (1) leaving the system triangular decoupled must satisfy, [N2],[RN1]:

$$X(\alpha_1)(x) = 0, \quad X(\beta_{11})(x) = 0 \quad \beta_{12}(x) = 0, \tag{9}$$

for all vectorfields $X(x) \in \Pi_2^*$. Note that the feedback (8) satisfies (9). The feedback (7) also must satisfies conditions of the same type, as (9), namely: $\tilde{X}(\tilde{\alpha}_1) = 0$, $\tilde{X}(\tilde{\beta}_{11}) = 0$, $\tilde{\beta}_{12} = 0$, for all $\tilde{X} \in \tilde{R}_2^*$, with $\tilde{\alpha}_1(z) := f_1 z_1 + f_2 z_2 + f_3 z_3$, $\tilde{\beta}_{11}(z) := g_{11}, \tilde{\beta}_{12}(z) = g_{12}$. But since the distributions $\tilde{R}_2^*$ and Π_2^* do not agree at the equilibrium then (7) does not correspond to the linearization of (8). On the other hand, whenever these distributions agree at 0 we are able to approximately solve locally the original nonlinear TDP by using the linear feedback (7). $\blacksquare$

To state the main result of this section an assumption concerning Δ^* and Δ_k^* is made.

Assumption 3
Consider the systems Σ and Σ^k. Suppose that $\Delta_k^*(0) = \Delta^*(0)$. $\blacksquare$

Remark 4.3 If this assumption is dropped it seems not to be possible to conclude any relation between $\Pi^*(0)$ and $\Pi_k^*(0)$; at least not by analysis of the Algorithm. This can be readily seen from the first step of the Algorithm applied to Σ and Σ^k. Sufficient conditions for the systems Σ and Σ^k to satisfy Assumption 3 are analogous to those given in [RN2], where the systems Σ^k and Σ^{k+1} were treated. Observe also that for the Example 4.2, Assumption 3 is fulfilled.

Theorem 4.4
Consider the systems Σ, Σ^k, $k \geq 1$ and the constant dimensional distributions Δ^*, Δ_k^*, together with the Algorithm. Provided Assumptions (1-3) are satisfied then

(i) $(\Pi_j^k(x))^{k-1-j} = (\Pi_j(x))^{k-1-j}$, $j = 0, 1, \ldots, k-1$.

(ii) $\Pi_j^k(0) = \Pi_j(0)$, $\quad j = 0, 1, \ldots, k-1$.

(iii) If $k^*(k) = k^* = j$ for a fixed j in the set $\{0, 1, \ldots, k-1\}$ then $\Pi_k^*(0) = \Pi^*(0)$.

(vi) If $k^*(k) < k^*$ and $k^*(k) \leq j$ for some fixed j in the set $\{0, 1, \ldots, k-1\}$ then $\Pi_k^*(0) \subseteq \Pi^*(0)$,

where Π_j^k (Π_j) corresponds to the j-th step of the Algorithm applied to Σ, (Σ^k). $\blacksquare$

Proof. The proof is given in the Appendix.

Remark 4.5 Shortly said, Theorem 4.4 (i) assures that for every element in $\Pi_j(x)$ there exist an element in $\Pi_j^k(x)$ which agree up to terms of order $k-1$ with respect to x. Part (ii) states that given the fact that the maximal controlled invariant distributions of Σ and Σ^kagree at $x = 0$, (see also [RN2]), then the first $k-1$ steps of the Algorithm applied to Σ and Σ^k are equivalent at the equilibrium.

Part (iii) assures that Π_k^* and Π^* agree at the equilibrium provided a restrictive condition holds: the Algorithm terminates at the same step $j \leq k-1$ when applied to Σ and Σ^k. Part (vi) contemplates the case when the Algorithm terminates for Σ^k before it stops for Σ and at the same time $k^*(k)$ satisfies $k^*(k) \leq k-1$. Hence, a relation of $\Pi^*(0)$ and $\Pi_k^*(0)$ has only be established in the cases specified in $(i)-(iii)$. Given the examples in Section 3, it is not possible to show such a relation in general.

5. Conclusions

A relation, at an equilibrium point, between maximal controllability distributions defined for a nonlinear system and the k-th order Taylor series approximation of it is given, provided particular maximal controlled invariant distributions defined for both systems agree at the equilibrium. We found a minimal nonnegative integer s^* for which locally strong accessibility of the s^*-th order Taylor series approximation of a nonlinear system implies this property on the original nonlinear system provided Assumption 1 is satisfied. Whenever the solutions for a nonlinear synthesis control problem are characterized in terms of maximal controllability distributions we have identified some cases in which it is possible to locally approximately solve the nonlinear problem by using a solution obtained from the associated problem for the k-th order Taylor series approximated system.

References

[B] Bressan A.; 'Local asymptotic approximations of nonlinear control systems', *Int. J. of Contr.*, 41, 1985, pp. 1331-1336.

[Br] Brockett R.W.; 'Asymptotic stability and feedback stabilization', **Differential Geometric Control Theory**, R.W. Brockett, R.S. Millman and H.J. Sussmann, editors, Birkhäuser, Boston, 27, 1983, pp. 181-191.

[Cr] Crouch P.E.; 'Solvable approximations to control systems', *SIAM J. Contr. Optim.*, 22, 1984, pp. 40-54.

[G] Gilbert E.G.; 'The decoupling of multivariable systems by static state feedback', *SIAM J. Control*, Vol. 17, No.1, 1969, pp. 50-63.

[GN] Gras L.C.J.M., Nijmeijer H.; 'Decoupling in nonlinear systems: from linearity to nonlinearity', *IEE Proc.*, 1989, Vol.136, Pt.D., No.2, pp. 53-62.

[H] Hermes H.; 'Nilpotent and higher-order approximations of vector field systems', *SIAM Review*, Vol. 33, No. 2, June 1991, pp. 238-264.

[HG] Ha I.J., Gilbert E.G.; 'A complete characterization of decoupling control laws for a general class of nonlinear systems', *IEEE Trans. Aut. Contr.*, AC-31, 1986, pp. 823-830.

[HN] Huijberts H.J.C., Nijmeijer H.; 'Local nonlinear model matching: from linearity to nonlinearity', *Automatica*, 26, 1990, pp. 175-183.

[I] Isidori A.; **Nonlinear control systems**, (Second Edition), Springer, Berlin, 1989.

[K1] Krener A.J.;'Approximate linearization by state feedback and coordinate change', *Systems and Contr. Letters*, 5, 1984, pp. 181-185.

[K2] Krener A.J., Karahan S., Hubbard M., Frezza R.; 'Higher order linear approximations to nonlinear control systems', Proc. 26-th CDC, Los Angeles, 1987, pp. 519-523.

[K3] Krener A.J.; 'Extended quadratic controller normal form and dynamic state feedback linearization of nonlinear systems', to appear in *SIAM J. Contr. Optim.*

[N] Nijmeijer H.; 'The triangular decoupling problem for nonlinear control systems', *Nonl. Analysis Th. Meth. Appl.*, 1984, Vol.8, No.3, pp. 273-279.

[NvdS] Nijmeijer H., van der Schaft A.J.; **Nonlinear dynamical control systems**, Springer, New York, 1990.

[RN1] Ruiz A., Nijmeijer H.; 'Further results in the nonlinear triangular decoupling problem', Proc. European Contr. Conf. (ECC), Grenoble, France, 1991, pp. 1490-1495.

[RN2] Ruiz A., Nijmeijer H.; 'Nonlinear control and systems approximations: a geometric approach', Proc. of the Nonlinear Control System Design Symposium (NOLCOS'92), June 1992, Bordeaux, France, pp. 106-111.

[S] Sontag, E.; **Mathematical control theory**, Springer New York, 1990.

[SM1] Shimizu K., Matsubara M.; 'Decoupling and disturbance rejection control of a binary distillation column with pressure variation', *J. Chem. Eng. Japan*, Vol. 17, No. 3, 1984, pp. 262-269.

[SM2] Shimizu K., Matsubara M.; 'Triangular decoupling for a binary distillation column under pressure control', *IEEE Trans. Aut. Contr.*, Vol. AC-30, No.5, 1985, pp. 474-477.

[SJ] Sussmann H.J., Jurdjevic V.; 'Controllability of nonlinear systems', *J. Differential Equations*, 12, 1972, pp. 95-116.

[Su] Sussmann H.J., 'A general theorem on local controllability ', *SIAM J. Contr. Optim.*, 25(1), 1987, pp.158-194.

[THN] Takamatsu T., Hashimoto I., Nakai Y.; 'A geometric approach to multivariable control systems design of a distillation column', *Automatica*, 15, 1979, pp. 387-402.

[vdS] van der Schaft A.J.,; 'Relation between H_∞ optimal control of a nonlinear system and its linearization', Proc. 30-th CDC, Brighton, 1991, pp.1807-1809.

[vdW] Van der Wegen L.L.M.; **Local disturbance decoupling with stability for nonlinear systems**, *Lectures Notes in Control and Information Sciences*, Vol. 166, Springer, 1991.

[Wo] Wonham W.M.; **Linear multivariable control: a geometric approach**, (Third Edition), Springer, New York, 1985.

Appendix

The proof of Theorem 4.4 is done basically by analysis of the Algorithm applied to Σ and Σ^k. The proof is done by induction and the first two steps will be stated in the next lemmas. Throughout, we suppose that Assumptions (1-3) hold for Σ and Σ^k.

In the subsequent discussion the fact that $\Delta^*(0) = \Delta_k^*(0)$, (Assumption 3) plays a decisive role. From Theorem 3.4 of [RN2] it can be seen that whenever $\Delta^*(0) = \Delta_k^*(0)$ then we can always express $\Delta^*(x)$ and $\Delta_k^*(x)$ as

$$\begin{aligned}\Delta^*(x) &= \text{span}\,\{d_1^{k-1}(x) + d_{1r}(x), \ldots, d_p^{k-1}(x) + d_{pr}(x)\}\\ \Delta_k^*(x) &= \text{span}\,\{d_1^{k-1}(x) + d_{1s}^{k-1}(x), \ldots, d_p^{k-1}(x) + d_{ps}^{k-1}(x)\},\end{aligned} \tag{A.1}$$

for some integer p such that $n \geq p > 0$ and where $d_{ir}(x)$, $(d_{is}(x))$, $i \in \underline{p}$, denotes a possible infinite (finite) series of terms of order k and higher with respect to (w.r.t.) x, respectively. Note that $d_{ir}(x)$ and $d_{is}(x)$ have in general different terms of order k and higher w.r.t. x. In what follows, notice that the terms $d_{ir}(x)$, $d_{is}(x)$, $i \in \underline{p}$, have no relevant importance in the manipulations. The reason for this is, loosely speaking, that we are interested in the first $k-1$ steps of the Algorithm and this involves somehow, only the first $k-1$ terms of any element in Δ^*, Δ_k^*. Concerning the first step in the Algorithm we have

Lemma A.1 Consider the constant dimensional distributions Π_0, Π_0^k and Δ^*, Δ_k^* as in (A.1). Then $(\Pi_0(x))^{k-1} = (\Pi_0^k(x))^{k-1}$. ■

Proof. The proof is done if we show that for all $X \in \Pi_0$ there exists a vectorfield $\tilde{X} \in \Pi_0^k$ such that $(X(x))^{k-1} = (\tilde{X}(x))^{k-1}$ and conversely. Let $X \in \Pi_0 = \Delta^* \cap G$. X can be expressed as

$$\begin{aligned}X(x) &= \alpha_1(x)d_1^{k-1}(x) + \cdots + \alpha_p(x)d_p^{k-1}(x)\\ &\quad+\alpha_1(x)d_{1r}(x) + \cdots + \alpha_p(x)d_{pr}(x)\\ &= \beta_1(x)g_1^{k-1}(x) + \cdots + \beta_m(x)g_m^{k-1}(x)\\ &\quad+\beta_1(x)g_{1r}(x) + \cdots + \beta_m(x)g_{mr}(x)\end{aligned} \tag{A.2}$$

where we have written $g_i(x) = g_i^{k-1}(x) + g_{ir}(x)$, $i \in \underline{p}$, with $g_{ir}(x)$ a possible infinite series of terms of order k and higher w.r.t. x and α_i, β_j, $i \in \underline{p}$, $j \in \underline{m}$ some set of analytic functions. The $(k-1)$-th order approximation of X in (A.2) is:

$$\begin{aligned}(X(x))^{k-1} &= (\alpha_1(x)d_1^{k-1}(x) + \cdots + \alpha_p(x)d_p^{k-1}(x))^{k-1}\\ &= (\beta_1(x)g_1^{k-1}(x) + \cdots + \beta_m(x)g_m^{k-1}(x))^{k-1}.\end{aligned} \tag{A.3}$$

Note that in (A.3) we have: $\alpha_1(x)d_1^{k-1}(x) + \cdots + \alpha_p(x)d_p^{k-1}(x) \in \Delta_k^*(x)$, $\beta_1(x)g_1^{k-1}(x) + \cdots + \beta_m(x)g_m^{k-1}(x) \in G^k(x)$. Hence it follows from (A.3) that $(X(x))^{k-1} \in (\Delta_k^* \cap G^k)^{k-1}(x)$, i.e., there exists a vectorfield $\tilde{X} \in \Delta_k^* \cap G^k$ such that $(X(x))^{k-1} = (\tilde{X}(x))^{k-1}$, stated differently $(\Pi_0(x))^{k-1} \subset (\Pi_0^k(x))^{k-1}$. The converse inclusion can be shown as follows.

Let Π_0 be given by span $\{X_1, \ldots, X_\ell\} = \Pi_0$, $\ell \leq p$. By the foregoing, there exist vectorfields $\tilde{X}_1, \ldots, \tilde{X}_\ell \in \Pi_0^k$, so that $(X_i(x))^{k-1} = (\tilde{X}_i(x))^{k-1}$, $i \in \underline{\ell}$. In particular $X_i(0) = \tilde{X}_i(0)$. Now, since the dimension of Π_0 is constant on M, the vectorfields $X_i(0)$, $i \in \underline{\ell}$ are independent. Hence we have

$$\text{span}\,\{\tilde{X}_1(0), \ldots, \tilde{X}_\ell(0)\} = \Pi_0^k(0). \tag{A.4}$$

But since Π_0^k has also constant dimension on M, then dim $(\Pi_0^k) =$ dim (Π_0). Thus the vectorfields $\tilde{X}_i$, $i \in \underline{\ell}$, are so that $\Pi_0^k(x) =$ span $\{\tilde{X}_1(x), \ldots, \tilde{X}_\ell(x)\}$.

Next, suppose there exists a vectorfield $\tilde{Y} \in \Pi_0^k$ for which

$$(\tilde{Y}(x))^{k-1} \notin (\Pi_0(x))^{k-1}. \tag{A.5}$$

Since $\tilde{Y}(x) \in \Pi_0^k(x)$ we rewrite $\tilde{Y}(x)$ as $\tilde{Y}(x) = \sum_{i=1}^{\ell} \gamma_i(x)\tilde{X}_i(x)$ with $\tilde{X}_i(x)$, $i \in \underline{\ell}$, satisfying (A.4), for some analytic functions $\gamma_i(x)$, $i \in \ell$. But for each $\tilde{X}_i(x) \in \Pi_0^k(x)$ there exists a vectorfield $X_i(x) \in \Pi_0(x)$ such that $(\tilde{X}_i(x))^{k-1} = (X_i(x))^{k-1}$. Hence $(\tilde{Y}(x))^{k-1}$ can be written as a linear combination of vectorfields $(X_i(x))^{k-1}$, $i \in \underline{\ell}$. Thus equation (A.5) does not hold and the proof is completed. ■

The following Lemma is instrumental to relate the second step of the Algorithm for Σ and Σ^k. This step is:

$$\begin{aligned}\Pi_1 &= \Delta^* \cap ([f, \Pi_0] + \sum_{i=1}^{m}[g_i, \Pi_0] + G),\\ \Pi_1^k &= \Delta_k^* \cap ([f^k, \Pi_0^k] + \sum_{i=1}^{m}[g_i^{k-1}, \Pi_0^k] + G^k).\end{aligned} \tag{A.6}$$

With respect to (A.6), define the distributions:

$$D_1 := [f,\Pi_0] + \sum_{i=1}^{m}[g_i,\Pi_0] + G,$$
$$D_1^k := [f^k,\Pi_0^k] + \sum_{i=1}^{m}[g_i^{k-1},\Pi_0] + G^k. \quad (A.7)$$

Lemma A.2 Assume the distributions distributions D_1, D_1^k are constant dimensional on M. Then $(D_1)^{k-2} = (D_1^k)^{k-2}$. ∎

A useful result which is often used in the proofs is

Proposition A.3 Consider the vectorfields X, $\tilde{X}$, Y, $\tilde{Y}$, in M, such that $(X(x))^j = (\tilde{X}(x))^j$, $(Y(x))^k = (\tilde{Y}(x))^k$, with $j \geq k \geq 0$. Then $([X,Y](x))^{k-1} = ([\tilde{X},\tilde{Y}](x))^{k-1}$. ∎

Proof. The proof is immediate.

Proof of Lemma A.2 First we show that given any $Y \in D_1$ there exists a $\tilde{Y} \in D_1^k$ such that $(Y)^{k-2} = (\tilde{Y})^{k-2}$, which implies that

$$(D_1)^{k-2} \subset (D_1^k)^{k-2}. \quad (A.8)$$

Take any $X \in \Pi_0$, $\tilde{X} \in \Pi_0^k$, such that $(X)^{k-1} = (\tilde{X})^{k-1}$, see Lemma A.1. Then using Proposition A.3

$$([g_i,X](x))^{k-2} = ([g_i^{k-1},\tilde{X}](x))^{k-2}, \quad i \in \underline{m}. \quad (A.9)$$

To see this we rewrite $g_i(x)$ as $g_i(x) = g_i^{k-1}(x) + g_{ir}(x)$, $i \in \underline{m}$, where $g_{ir}(x)$ is a possible infinite series of terms of order k and higher w.r.t. x. Then we have

$$[g_i^k,X](x) = [g_i^{k-1},X](x) + [g_{ir},X](x), \quad (A.10)$$

and the last term of the right hand side of (A.10) has terms of order $(k-1)$ w.r.t. x and higher. By a similar procedure and using Proposition A.3 we obtain that

$$([f,X](x))^{k-2} = ([f^k,\tilde{X}](x))^{k-2}. \quad (A.11)$$

where we have rewritten $f(x)$ as $f(x) = f^k(x) + f_r(x)$, and $f_r(x)$ is a possible infinite series of terms of order $k+1$ w.r.t. x and higher. Observe finally, that given any $g \in G$ there exists a $\tilde{g}^{k-1} \in G^k$ such that $(g(x))^{k-2} = (\tilde{g}^{k-1}(x))^{k-2}$. Just recall that $G = \text{span}\{g_1,\ldots,g_m\}$ and $G^k = \text{span}\{g_1^{k-1},\ldots,g_m^{k-1}\}$.
The converse of (A.8) is shown next. Let span $\{X_1,\ldots,X_\ell\} = D_1$, (dim $D_1 = \ell$). From the discussion above notice that there exist vectorfields $\tilde{X}_1,\ldots,\tilde{X}_\ell \in D_1^k$ such that $(X_i(x))^{k-2} = (\tilde{X}_i(x))^{k-2}$, $i \in \underline{\ell}$. Moreover, it follows from the above that $X_i(0) = \tilde{X}_i(0)$, $i \in \underline{\ell}$. Assume D_1 and D_1^k have constant dimension on M and proceed further with similar arguments as done in the converse proof of Lemma A.1. ∎

The relation between the second step of Algorithm for Σ and Σ^k is stated in the

Lemma A.4 Consider the systems Σ, Σ^k, the distributions Δ^*, Δ_k^* as in (A.1) and the second step of the Algorithm. Then $(\Pi_1)^{k-2} = (\Pi_1^k)^{k-2}$. ∎

Proof. Recall from Lemma A.1 that $(\Pi_0(x))^{k-1} = (\Pi_0^k(x))^{k-1}$, i.e., for any $X \in \Pi_0$ there exists a $\tilde{X} \in \Pi_0^k$ such that $(X(x))^{k-1} = (\tilde{X}(x))^{k-1}$ and conversely. Let now $D_1 = \text{span}\{X_1,\ldots,X_\ell\}$, where each X_i, $i \in \underline{\ell}$ has a possible infinite Taylor series about 0. Assume that dim $(\Pi_1) =$ dim $(\Delta^* \cap D_1) \neq 0$. Let $Y \in \Delta^* \cap D_1$, then Y can be expressed as

$$\begin{aligned} Y(x) &= \alpha_1(x)d_1^k(x) + \cdots + \alpha_p(x)d_p^k(x) \\ &\quad + \alpha_1(x)d_{1r}(x) + \cdots + \alpha_p(x)d_{pr}(x) \\ &= \beta_1(x)X_1^{k-2}(x) + \cdots + \beta_\ell(x)X_\ell^{k-2}(x) \\ &\quad + \beta_1(x)X_{1s}(x) + \cdots + \beta_\ell(x)X_{\ell s}(x), \end{aligned} \quad (A.12)$$

where d_i, d_{ir}, are as in (A.1), X_i, $i \in \underline{\ell}$, has been rewritten as $X_i(x) = X_i^{k-2}(x) + X_{is}(x)$ with $X_{is}(x)$ is a possible infinite series of terms of order $k-1$ and higher w.r.t. x and α_i, β_j, $i \in \underline{p}$, $j \in \underline{\ell}$ are some set of analytic functions. Take the $(k-2)$-th order approximation of Y and proceed in a similar way as in the proof of Lemma A.1. ∎

Remark A.5 Note that from the definition of D_1 and D_1^k is clear to see that $D_1(x) \subset C_1(x)$, $D_1^k(x) \subset C_1^k(x)$. Recall from Proposition 3.3 that $C_i^k(0) = C_i(0)$ for $i = 0,1,\ldots,k-1$. This fact plays a decisive role in the analysis of the sucessive steps of the Algorithm. Also notice that from Lemma A.4 it turns out that $\Pi_1^k(0) = \Pi_1(0)$.

Proof of Theorem 4.4 The proof of (i) is done by induction. The first two steps have been proven in Lemma A.1 and A.4. We shall proof in the $(k-1)$-th step that:

$$(\Pi_{k-1})^0 = (\Pi_{k-1}^k)^0, \quad (A.13)$$

i.e., for each $X \in \Pi_{k-1}$ there exists a $\tilde{X} \in \Pi_{k-1}^k$ so that $(X)^0 = (\tilde{X})^0$, or shortly, $X(0) = \tilde{X}(0)$. Assume that in the $(k-2)$-th step we have $(\Pi_{k-1})^{k-1-(k-2)} = (\Pi_{k-1})^1 = (\Pi_{k-1}^k)^{k-1-(k-2)} = (\Pi_{k-1}^k)^1$, i.e., for every $Y \in \Pi_{k-1}$ there exists a $\tilde{Y} \in \Pi_{k-1}^k$ such that $(Y)^1 = (\tilde{Y})^1$, in other words, X and $\tilde{X}$ agree up to the first order term w.r.t. x. In the $(k-1)$-th step of the Algorithm applied to Σ and Σ^k define the distributions

$$D_{k-1} := [f,\Pi_{k-2}] + \sum_{i=1}^{m}[g_i,\Pi_{k-2}] + G,$$
$$D_{k-1}^k := [f^k,\Pi_{k-2}^k] + \sum_{i=1}^{m}[g_i^{k-1},\Pi_{k-2}^k] + G^k, \quad (A.14)$$

and assume D_{k-1}, D_{k-1}^k to be of constant dimension on M. To prove that

$$(D_{k-1})^0 = (D_{k-1}^k)^0 \quad (A.15)$$

proceed in an analogous way as done in the proof of Lemma A.1 . To show (A.13) suppose dim $\Pi_{k-1} =$ dim $(\Delta^* \cap D_{k-1}) \neq 0$, then each vectorfield $Y \in (\Delta^* \cap D_{k-1})$ can be expressed as

$$\begin{aligned} Y(x) &= \alpha_1(x)d_1^k(x) + \cdots + \alpha_p(x)d_p^k(x) + \\ &\quad \alpha_1(x)d_{1r}(x) + \cdots + \alpha_p(x)d_{pr}(x) \\ &= \beta_1(x)X_1^0(x) + \cdots + \beta_{\ell_1}(x)X_{\ell_1}^0(x) + \\ &\quad \beta_1(x)X_{1s}(x) + \cdots + \beta_{\ell_1}(x)X_{\ell_{1s}}(x), \end{aligned} \quad (A.16)$$

where $d_{ir}(x)$ are as in (A.1) and $X_{is}(x)$, $i \in \underline{\ell_1}$, is a possible infinite series of terms of order one and higher w.r.t. x and α_j, β_i, $j \in \underline{m}$, $i \in \underline{\ell_1}$, some set of analytic functions. Take the 0-th order approximation of Y and continue in a similar way as in the proof of Lemma A.1.
Parts (ii)-(vi) are direct consequences of (i). ∎

Remark A.6 If we attempt to compare the k-th step of the Algorithm applied to Σ and Σ^k we will notice that, if we define the distributions D_k, D_k^k in a similar way as in (A.14) the equality $D_k(0) = D_k^k(0)$ is in general not true.

Remark A.7 A very particular situation arises when we deal with Σ^1 (the standard linearization of Σ) and Σ. Since in this case we have that $C_j^1(0) \subset C_j(0)$ for $j = 0,1,\ldots$ If we pursue an analysis of the Algorithm for these systems as done above, we will conclude that $\Pi_1^*(0) \subset \Pi^*(0)$.