ESTIMATING A TWO-FACTOR MODEL FOR THE FORWARD CURVE OF ELECTRICITY

Maria Kholopova

Graduation committee:

prof.dr. P.J.J.M. van Loon (chairman)	University of Twente
prof.dr. A. Bagchi (promotor)	University of Twente
dr. D.Y. Dupont (assistant promotor)	University of Twente
prof.dr.ir. M.J.F. Wouters	University of Twente
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prof.dr. S. Aihara	Tokyo University of Science
dr.ir. M.H. Vellekoop	University of Twente
dr. D. Huisman	Erasmus University Rotterdam

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by

Maria Kholopova

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in Syktyvkar, Russia

This dissertation has been approved by:

prof.dr. A. Bagchi (promotor)

dr. D.Y. Dupont (assistant promotor)

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Contents

Chapter 1

Introduction

1.1 Motivation

Electricity markets are unique. Electricity has unique characteristics, electricity markets are very young and in most cases still illiquid. The electricity sector has undergone dramatic changes over the past few years. Before deregulation electricity prices were predictable. Electricity industry companies were mostly regulated or state-owned integrated monopolies combining generation, transmission and distribution. Distribution of the electricity may be a natural monopoly, but generation is not.

The deregulation started in the 1990s in the United Kingdom and New Zealand, followed by Sweden, Norway, Australia and few districts and some US states. In the Netherlands the deregulation process started in 2000 and continues at the moment. As a result of deregulation spot prices and prices of derivatives of electricity are now available for trading in many power, electricity and commodity exchanges all over the world.

The need for creating appropriate models for pricing the spot electricity and derivatives presented in the market arose. The natural possibility to price electricity is to use previously developed financial models for stocks, interest rates and commodities. However, these models must be adapted to the particular conditions of power markets, especially the non-storability of power. Moreover, some derivatives products cannot be found in other markets, for example swing options. There are standard products: futures, forwards, swaps, options, but even they have special features, which reflect the physical nature. The commodities underlying these products are also different. Power delivered at any particular hour, block of hours, week, month, and so on, represents very different commodity, because electricity cannot be stored and thus must be studied independently.

What makes electricity so different from other products such as derivatives on stocks, interest rates and commodities?

Electricity markets are very young markets and derivative markets lack liquidity even for such simple products as vanilla options. On the other hand, electricity is widely used by households and industries. There are many complex fundamental price drivers such as generation and transmission restrictions, which makes all electricity products especially difficult to model.

Though the liberalization of electricity market brings a lot of risks for players in the market, it also offers new possibilities for producers, distributors and users of electricity. To use these possibilities one needs to understand all important characteristics of electricity and derivatives products and use the wide array available products to manage risks. As the basis for the management of risks associated with electricity one needs to build new pricing models that can capture all important characteristics of spot and other products used for hedging.

1.2 Goal

The goal of this thesis is to construct an appropriate model for pricing futures and options on futures on spot electricity and to test it on existing data from electricity markets. To do this we consider first the important characteristics of the spot, futures and option markets and give an overview of the models used in financial markets to price stocks, interest rates and commodities. We introduce the two-factor Schwartz and Smith model [29] and show how to modify the model to take the averaging of the spot price over the delivery period into account. We test the model on German and Dutch markets.

1.3 Structure of the thesis

In Chapter 2 we investigate electricity markets in general, highlight the main characteristics of spot electricity prices, futures and options on futures in more details and give an overview of other derivatives traded in the different markets. We concentrate our attention on European Energy Exchange (EEX) located in Leipzig in Germany and Amsterdam Power Exchange (APX).

In Chapter 3 the statistical properties of spot, futures and options prices on the EEX and APX markets are analyzes and statistics for these data are presented. Chapter 4 introduces an overview of classical models for financial derivatives, interest rates and commodities and pros and cons of each model for purpose of spot electricity pricing are pointed out. Chapter 5 presents the two-factor model derived from the Schwartz and Smith model for pricing commodities. In this modified model, which takes into account the averaging of the spot prices over the delivery period, we derive closed-form solutions for futures prices, options prices and risk term premium. Chapter 6 describes how we implement the model from the previous chapter, methods used for model calibration, Kalman filter and optimization techniques. Different possibilities to include seasonality factors into the model, description of data used for calibration and constraints of the implementation are also presented in this chapter. Chapter 7 gives results of implementation. In Chapter 8 we give our conclusions and direction for future research.

Chapter 2

Electricity market

2.1 Deregulation of electricity market

This deregulation of power markets on gas and electricity markets started at the end of 1990s. Before deregulation the electricity industry was highly vertically integrated and had little competition. It was observable all over the world, that generation and transmission industries were integrated within one company. The same was true for distribution and supply companies. National state-owned monopolies of electricity production and distribution dominated the market until recently. But in contrast, during the last decade many governments introduced competition in this sector.

Among the first countries to start deregulating were the UK (1990), Norway (1991), Sweden, Australia and New Zealand (1995), Finland (1997) and Spain (1999). Deregulation in Germany and in the Netherlands started in 1999.

The degree of competition in a given electricity market can be measured by looking at the concentration of suppliers and the size of transportation capacity. The liberalization of electricity market and electricity generation in particular has attracted some new players such as oil companies. However, the main result of the electricity market liberalization has been a wave of mergers and acquisitions in Europe, especially in Germany and the United Kingdom (see [35]). According to the EU's statistical office, Eurostat (see [36]), only ten member states had opened their markets completely by September 2005: Denmark, Germany, Spain, Ireland, the Netherlands, Austria, Portugal, Finland, Sweden and the UK.

Conventional thermal power stations still dominate electricity production, accounting for 58% of installed capacity in the EU, nuclear power accounts for 19% (half of it in France alone), hydropower 18%, and wind turbines 5%. Wind power has made the strongest progress since 2000. It increased its installed capacity by 154%. Wind power is especially well developed in Denmark (23%), Germany (13%) and Spain (12%). Cross-border trade in electricity is still limited by the interconnector capacity (see [36]).

Market deregulation allowed the creation of various financial instruments based on electricity: from short- and long-term futures to options. Using futures contracts, buyer and sellers can fend off the danger of adverse price movements by locking in the prices of a future transaction. Options allow holder to gain from favorable market developments while enjoying some level of protection against unfavorable developments. Different kinds of electricity derivatives could be created but the first step in introducing more complicated financial products to the electricity market is understanding how electricity prices themselves fluctuate; the second step is being able to price the standard derivatives such as futures.

2.2 Structure of the electricity market

The purpose of the electricity industry is to convert primary energy (conventional fuels, wind, water, uranium, etc.) into electricity and transport it to the final consumers (factories, household, etc.). Electricity cannot be stored (except small quantity by means of water reservoirs), moreover, electricity produced at any moment should be immediately consumed, thus supply and demand of electricity have to matched at any moment of time. The whole electricity industry is based on this the most important and special feature of electricity as commodity. The process from producing electricity to usage of electricity by customer can be divided into four main steps: generation, transmission, distribution and retail.

2.2.1 Generation

Electricity can be generated by burning fuel such coal, natural gas, oil, biomass or waste . Electricity could also be generated by the gravitational power of running water from mountain rivers or lakes, by wind power, by solar power or by fission of enriched uranium. Figures 2.1 and 2.2 present the shares of fuels for electricity generation in years 1973 and 2003 and the evolution of electricity generation.

2.2.2 Transmission and distribution

Transmission is the transportation of electricity at high voltage (between 138 and 765 kilovolts). High voltage is used in order to minimize losses of electricity, because losses of electricity are inversely proportional to voltage. Before entering the transformation grid the voltage of electricity stepped up by transformers. After transmission the electricity is stepped down by the transformers and supplied to the customers using lower voltage lines. Transformers, transformation networks and the lower voltage distribution lines are costly investment and are very costly to repair and thus regarded as a natural monopoly

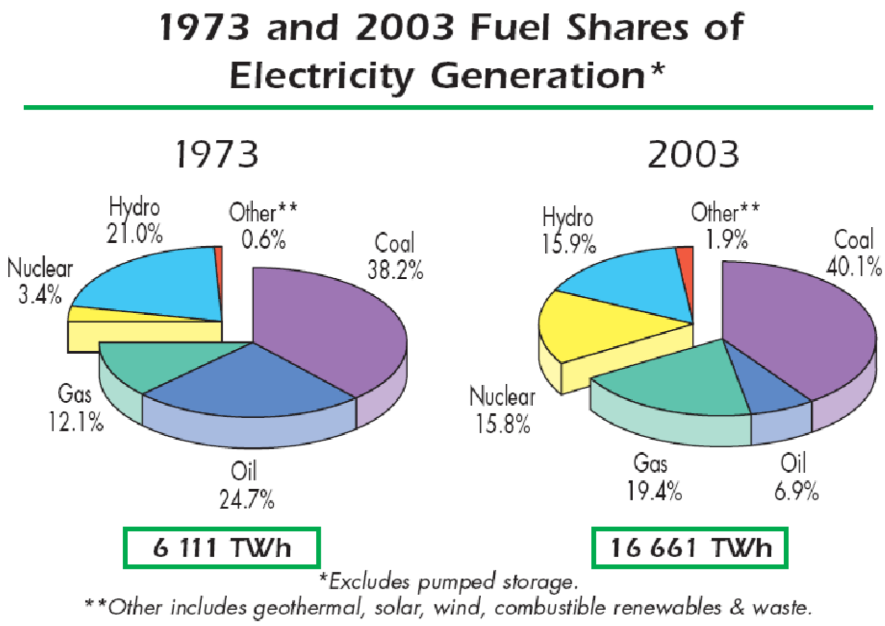

Figure 2.1: Worldwide electricity generation in year 1973 and 2003. (Source: Key World Energy Statistics, 2005 Edition, IEA (International Energy Agency).

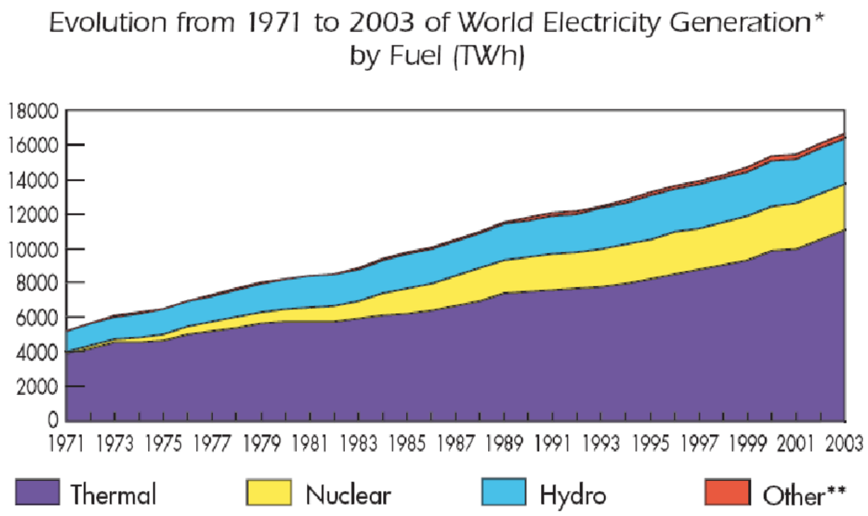

Figure 2.2: Evolution of electricity generation form 1971 to 2003 (Source: Key World Energy Statistics, 2005 Edition, IEA (International Energy Agency).

2.2.3 Retail

Deregulation should transform the retail side of the electricity market. Instead of facing a monopolistic electricity provider, customers should be able to select their preferred provider form a pool of competing bidders.

Deregulation could also allow the emergence of merchant companies, that are corporations that do not own generation assets or distribution networks but purchase the power they sell from third parties and pay fees for using the network to network owners. In practice, however, even though customers are often free to switch provider, few of such merchant companies could establish themselves.

2.2.4 Trading

After deregulation trading mechanism for pricing electricity was established in order to meet supply and demand. Trading is carried out between generating companies and supply (or distribution) companies at Power exchanges such as Amsterdam Power Exchange (APX) and European Energy Exchange (EEX). Contracts traded on these exchanges are standardized. Spot electricity, futures and options are considered standardized products. In addition, OTC (over the counter) contracts are traded. We consider them later in this chapter. In the following few sections we first describe standardized electricity products. The market of electricity is divided into spot market (day-ahead market), adjustment market (also called imbalanced or real-time market) and forward market for trading futures and electricity derivatives (options and OTC derivatives).

Spot price is the intersection point of demand and supply curves. The spot market is usually organized as an auction for delivery of electricity in the near future. Thus before producing electricity most of the electricity is sold via energy exchanges or OTC (over-the-counter) electricity market.

Of course there are certain needs for reserves and imbalanced markets which use these reserves. Reserves are needed because load may vary unexpectedly, for example, because of weather conditions or because the generators may face unexpected outages. The adjustment market is a market which uses reserve capacity to meet sudden demands on electricity or compensate outage of the plant. For the reserves there are usually two possibilities. Firstly, the plants with high flexibility are used such as quickly started gas turbines. They are rewarded with high prices during short periods. Secondly, each generator is obligated to keep constantly 10-20% of their capacity during peak hours for reserves. For example in the UK the capacity charges via so called *plant margins* are applicable to all generators. These margins are vary widely between different generators and electricity markets. See, for example, [7] for more information about the linkage between day-ahead and real-time markets in the Netherlands.

Any transaction on electricity derivatives market may be physical (with actual delivery of electricity) or financial (a cash flow from one party to another and no actual delivery of electricity). Physical transactions still have a crucial importance today, although financial transactions (especially futures) represent a big volume.

In the next sections of this chapter we consider first the unique properties of electricity in more details and then explain more about the spot and futures markets of electricity.

2.3 Electricity as a unique product

As we already pointed out electricity has some specific characteristics which makes this commodity so different from all other commodities, even such related ones as gas and oil. Here are the main characteristics:

1. *Non-storability*

   There are no efficient ways to store electricity. Basically there is only one potential way to store electricity - to use storages (reservoirs) of water on hydro generators. As we saw on the Figure 2.1, only about 15% of all generators are hydro and there are not available in all countries. For example, only 0,17% of electricity production in the Netherlands is hydro[1]. Most of generators are still thermal and we can say that electricity is non-storable. This distinctive characteristic leads to so called spikes in the price dynamics (sudden jumps upwards, shortly followed by steep downward moves to some average level).

   The magnitude of the spikes is huge. For example on the EEX the prices of electricity could jump from an average level of 40-50 euros per megawatt hour (MWh) to 1000 euros or more in just few hours and drop to zero price at night. See Figures 2.3 and 2.4 to compare spike sizes of EEX and APX market. We can see that the price on the APX market is in general more volatile and the spikes higher than 100 euros per MWh happen more often on the Dutch market.

   If we consider averaged daily prices, shown on the Figures 2.5 and 2.6 of Base prices from EEX and APX market respectively, we still can see strong mean-reversion and the spikes, although not of the same amplitude. We can also see increase in average price in the year 2005 which is explained by introducing CO_2 emissions permits in 2005. In general electricity is one of the most volatile products of all commodities.

2. *Spikes*

[1] Source: Eurostat [36]

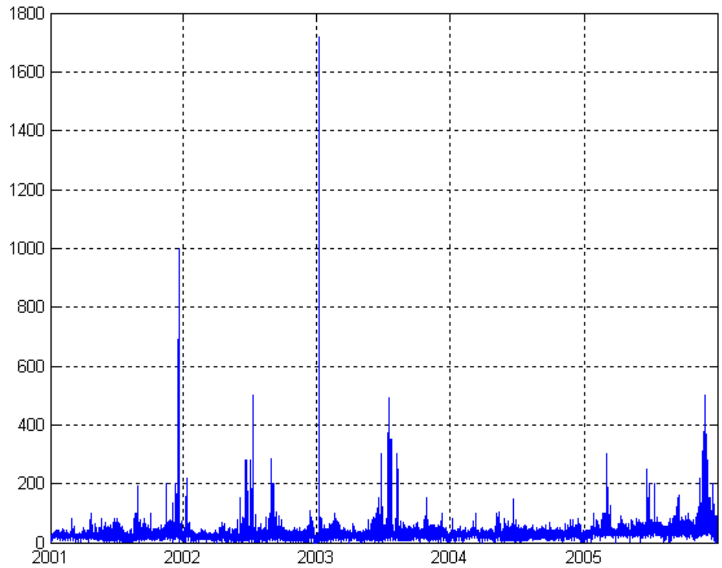

Figure 2.3: EEX hourly prices in euros per MWh, January 2001 - December 2005

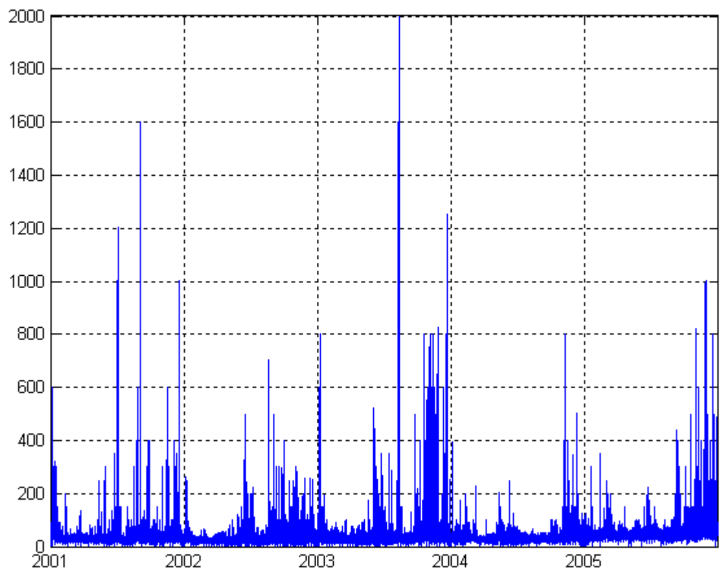

Figure 2.4: APX hourly prices in euros per MWh, January 2001 - December 2005

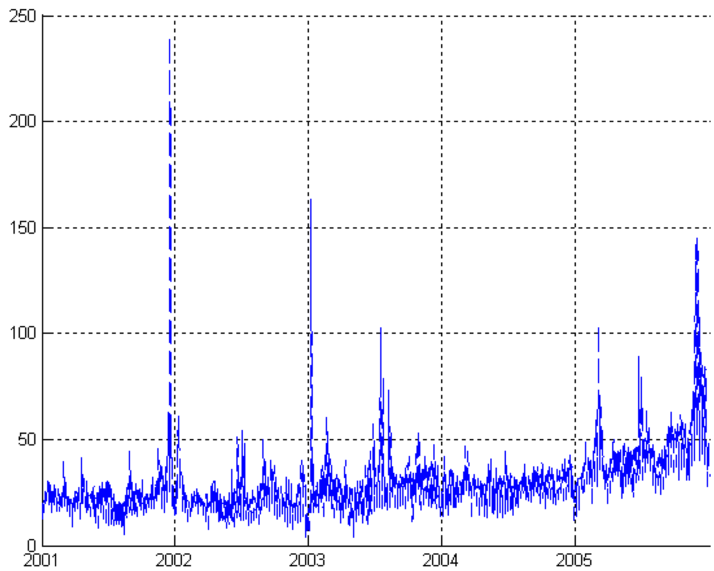

Figure 2.5: EEX Base prices (daily average prices) in euros per MWh, January 2001 - December 2005

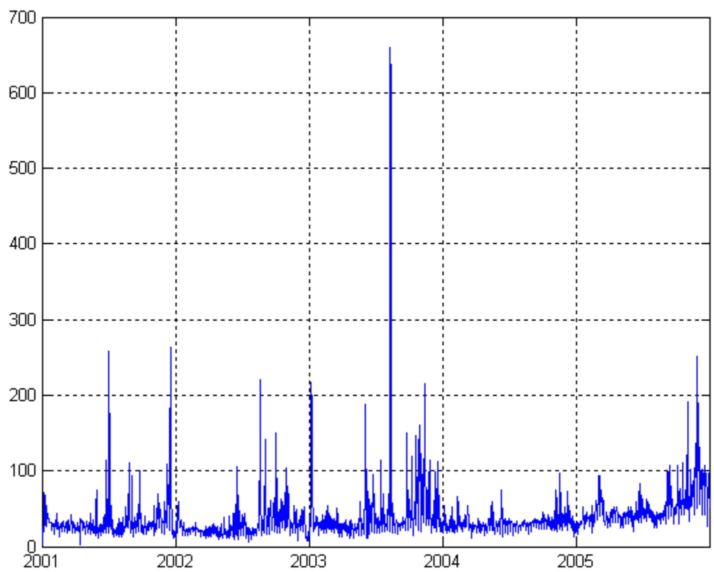

Figure 2.6: APX Base prices (daily average prices) in euros per MWh, January 2001 - December 2005

Theoretically, if there is a total absence of the inventory, the price of electricity could be unbounded, but on the exchanges maximum and minimum prices are imposed. The reasons for such spikes and for prices to be volatile in general are:

(a) Constant need for balancing supply and demand because electricity is to be consumed as soon as generated.

(b) Electricity demand shifts throughout the day.

(c) Demand is fairly price inelastic and is cyclical (weekdays versus weekend, peak hours versus off-peak hours).

(d) The marginal cost of producing electricity is rapidly increasing as production comes close to capacity.

(e) Supply can experience dramatic changes in case of planned or unplanned outage of the plant or any failure in transmission.

3. *Mean reversion*

It can be seen on the Figures 2.3 - 2.6 and could be easily statistically checked that there is mean reversion effect of electricity price. The price moves around some mean level and gets pulled back to this level rapidly after a spike.

4. *Seasonal patterns*

Electricity exhibits the most complicated cycle patterns on different time scales:

(a) Seasonal pattern through the calendar year. The prices are usually higher in winter and in summer because of higher demand due to heating in winter and cooling in summer.

(b) Seasonality within the week: the prices are higher during working days and lower during weekends due to "normal producing cycles ".

(c) Seasonality within the day. Additionally to two mentioned above seasonality patterns, electricity has different prices during different hours of the day. The price is higher during so called peak hours (07.00-23.00 for APX and 08.00-20.00 for EEX) with respect to non-peak hours (23.00-07.00 for APX and 20.00-08.00 for EEX) which is explained by human/industrial activity cycles.

Note, that for example in Norway the spikes of electricity are much lower, but seasonality is much more pronounced. The lower spikes level is due to the fact that in Norway electricity is mostly produced by hydro power, which allows to store some of amount water to produce electricity and use these storages in case of the spot prices are high. High seasonality is due to a natural cycle of water temperature.

If there is also no evident trend in electricity prices. Only in year 2005 due to an introduction of CO_2 emissions regulations the prices of the electricity and derivatives increased substantially.

2.4 Spot market

In this section we briefly describe structure of the spot markets, particular for EEX and APX, and variety of products usually traded on spot markets:

- Price is an intersection of demand and supply

  Electricity price is determined as intersection of demand and supply curves. The development of demand and supply on the APX and EEX spot markets is completely determined by market players themselves. Players are production and distribution companies, large consumers, industrial end-users, brokers and traders.

- Spot market is 24-hour head market

  Spot electricity markets both in Germany (EEX) and the Netherlands (APX) are 24-hour ahead markets. This means that every day an auction takes place based on the bidding from buyers and sellers of electricity and around 12am prices for each hour of the next day are quoted. Thus the electricity spot market is not the same as in classical definition of spot market of some commodity where delivery is carried out immediately. The hourly instruments are subject to physical delivery of electricity of a constant output on the electricity grid.

- Adjustment market

  Because of non-storability of electricity the immediate delivery of electricity is possible only in exceptional cases and carried out on a adjustment market. On EEX and APX mostly hourly contracts are traded, but also the half-hourly contracts on APX are available. (See [7]).

- The spot prices reflect only physical contracts, but they are also bases for underlying for many derivatives on electricity market, which could be either with physical or financial delivery.

- With respect to delivery hours there are three types of contract on the spot market: base load (00.00-24.00 for both markets), peak load (07.00-23.00 during weekdays for APX and 08.00-20.00 during weekdays for EEX) and off-peak load (23.00-07.00 during weekdays and 00.00-24.00 during weekends and holidays for

APX and 20.00-08.00 during weekdays and 00.00-24.00 during weekends and holidays for EEX).

2.5 Derivatives traded on electricity markets

The prices of the future contracts do not converge smoothly to the spot base prices and the spot prices do not converge to adjustment market prices as they do in case of storable commodity. The usual spot-forward relationship, when forwards are expected spot prices, also does not hold. That is why one can consider electricity as dual commodity, where dynamics of the spot and forward prices modelled separately. In this section we give a more detailed overview of futures and options contracts traded on organized exchanges and review the various derivatives products used to hedge risks exposure of the spot prices.

2.5.1 Futures contracts

Before presenting different types of futures we should note that, though there are certain differences between futures and forward contracts, we consider them as the same product. The reason for that lies in our pricing model, which assumes non-stochastic interest rate and which makes forward and future prices for contracts with same fixed maturity and underlying delivery periods to be of the same value (see, for example, [16]).

A *futures contract* is a contract that obligates the seller of the contract to deliver and a buyer of a future contract to receive a given quantity of electricity (1MWh) over a fixed period $[T_0, T]$ at a price K specified in advance. Futures are usually used to assure fixed delivery price of electricity delivered in some future period. The difference between electricity futures and other futures, is that electricity futures use *averaged spot price* $\frac{1}{n}\sum_{t=T_0}^{T} S(t)$ as the underlying of the contract and not one fixed spot price $S(T)$ as in most financial and commodity markets. Here n is the number of days during the delivery period which is taken to calculate the average price.

Futures contracts are traded on the exchanges, they require the payment of margins and they are standardized products in terms of their characteristics (maturity, delivery period, quantity of underlying electricity).

Consider first the futures at time t on a spot electricity price $S(T)$ with a delivery date T fixed in advance. The price of such a future at time t we denote by $F(t, T)$. At time T, the futures price $F(T, T)$ is equal to the spot price $S(T)$. But what is the price of future before delivery date T?

There is risk and a corresponding *risk premium* attached to spot market. Usually if we consider future on spot price, the short-term futures are upward-biased estimators of the spot prices. This is used in case when t is close to time T or $T - t$ is small.

That is why we express future price at time t as conditional expectation of the future spot price $S(T)$ plus some risk-term premium over the period $[t, T]$:

$$F(t, T) = \mathbb{E}_t[S(T)] + \pi(t, T).$$

We suggest that the risk-premium $\pi(t, T)$ is positive when $T - t$ is small and negative if $T - t$ is big. Risk premium may be negative because long-term future/forward contracts usually sold by generator as protection against variable demand, especially by plants which do not have flexibility in the load, such as nuclear plants.

Futures are traded at the electricity exchanges or at OTC (over-the-counter) markets. Currently the most active electricity futures exchanges in Europe are

1. Nordic Power Exchange (NordPool) in Norway which covers Norway, Denmark, Finland and Sweden.

2. European Energy Exchange (EEX) in Germany where futures with delivery in the Netherlands, Germany and France and Phelix financial futures are traded.

3. Endex (European Energy Derivatives Exchange) in Amsterdam where Dutch Power and Belgian Power futures are traded.

4. Paris Power Exchange (Powernext) which trades futures with delivery in France.

The NordPool exchange was one of the first exchanges in Europe to trade forwards and at the moment it is the most liquid market, not only because of longer trading history (operated from 1990s), but also because a big part of electricity traded on NordPool is produced by hydro power, which could be stored and for this reason the market is less volatile, closer in its characteristics to the usual financial market.

Different kinds of futures/forwards exist. We consider futures traded on Dutch and German exchanges because we will use data from these markets to estimate the model parameters.

- Futures are standardized products on both markets. Futures contract is a contract to deliver electricity with the fixed rate of one MWh (Megawatt per hour) during fixed delivery period. Thus, for example, a month April 2005 base future will deliver in total 1MWh$\times$24hours$\times$30days= 720MWh of electricity in April 2005.

- There are futures with physical or cash settlement, called physical and financial futures accordingly. Examples of financial futures on EEX market are Phelix

Base and Phelix Peak futures. All Dutch-Power, German-Power and French-Power futures on EEX are futures with physical delivery. On Endex only futures with physical delivery in the Netherlands and Belgium are traded.

- Physical futures differ by geographical factor. For example on the EEX there are futures with delivery in the Germany, Netherlands and France (German, Dutch and French future contracts respectively). Endex trades futures for physical delivery in the Netherlands and Belgium. Some markets use another method to hedge basic risk which is the difference in the delivery prices between different locations or price areas. For example NordPool has so called Contracts for Difference (CfD), which are forward contracts on the difference between the delivery price for specific area and the system price (NordPool market price).

- There are base, peak and off-peak loads futures. They have different underlying. For example Phelix Base month futures have delivery for 24 hours per day during delivery month and use Phelix Base load averages over month price as underlying, Phelix future Peak contracts have delivery during peak hours between 8.00 and 20.00 during working days and use the averages of the corresponding Phelix Peak load prices as underlying. On the Dutch market futures on Base, Peak and Off-peak hours are traded.

- Futures differ in delivery periods. For example EEX has futures with monthly, quarterly and yearly deliveries. In some markets futures with longer delivery periods such as quarter and year delivery futures are fulfilled by cascading. Futures with shorter delivery (such as month) are fulfilled by cash settlement. On the NordPool weekly contracts are also available.

  Cascading means the automatic splitting of the long-term contracts into contracts with shorter maturity on the last trading day. For example, a year contract on EEX is split two days before the start of its delivery period (calendar year, from January to December) into three monthly contracts with delivery in January, February and March and three quarterly contracts for second, third and fourth quarter. Then two trading days before entering into second quarter delivery (before 1st of April), second quarter contract again splits into three month contracts with delivery in April, May and June, and so on.

  Available market data provides relatively good information about the short end of term structure, but a much less detailed picture of the long end and the important seasonal component.

- After signing the Kyoto agreement which commits countries to curb CO_2 emissions, new EU-allowance futures began to trade on European markets (both established power markets like the EEX or Powernext and new markets like ECX).

For example, on the EEX market there are two European carbon futures: for the first period of three years 2005-2007 and for the second period of 5 years 2008-2012. One contract allows emission of one ton of carbon dioxide or equivalent. Regulations of CO_2 emissions have had a huge impact on electricity markets.

- The number of futures tradable in the market varies from market to market. For example on the EEX market at any day there are 19 types of futures traded: futures based on the current and the six closest months, the seven closest quarters and the six coming years. At Endex (future market with APX index as underlying) 6 monthly, 6 quarterly and 3 yearly delivery futures are traded.

There are a number of margins which could vary from market to market. On EEX for example there are additional margins, margin calls, intra-day margins. Smallest price change is fixed (0,01 euro per MWh for EEX and APX markets).

Forward curve, i.e. the curve which represents the set of all available forward prices as a function of their maturity, is a subject of careful analysis for the participants of futures market, because the futures prices provide a measure of expected price of power in the future. The forecasting ability of the futures prices to predict future spot prices is one of the main factors of great value for futures markets. Futures result from competitive trading and represent expected values of the underlying supply and demand at various point in the future plus term premium. The arrival of news has a large influence not only on the spot but also on future prices. Naturally all the participants of the electricity market are particularly interested in the relationship between current spot price, future prices and available inventory, but this is not the subject of the present study.

2.5.2 Options on electricity

Valuing option is a big issue for market participants. Although a lot of literature is available on pricing European options on forwards, there is hardly any literature on valuing option contracts written on forwards for some delivery period. There are few formulae available for approximations of the Asian options which have an average of a stock price as underlying. On the electricity market however, options have futures as an underlying of the contract and these futures from their part are based on the averages of the spot price.

Probably it is one of the reasons why most European markets are very illiquid on option trading. Of course as market develops options are expected to be more liquid and new derivatives are expected to appear for trading at the exchanges. That is why we have to know how we should price the options even if one can say that they are not that important at the moment because there is no liquidity. At the moment on Endex in Amsterdam there are no available quotes for options, so we use in our analysis only option prices from EEX market.

All the options on electricity markets are usually cash settled, thus have a financial future and not a physical futures as an underlying. Basically there are two types of options available on electricity market: European options on futures and Asian options on spot.

- European-type options on financial futures

  There are two kinds of European options exist: European Call option and European Put option.

  The buyer of a European option has the right (but not the obligation) to buy the underlying asset at the fixed time in the future at a fixed price called strike price. For example, let us consider at time t the price of the electricity future contracts $F(t, T_0, T)$ with delivery between T_0 and T. This futures contract is based on the average of the spot prices during the period $[T_0, T]$. This futures contract in its part is an underlying for the European options. A payoff function of European Call option at time T_e, called expiry of the option, with strike K on this future contract is equal to $max(F(T_e, T_0, T) - K, 0)$.

  The buyer of a European Put option has the right (but not the obligation) to sell the underlying asset at the fixed time in the future at fixed price K. A payoff function of European Put option at expiry time T_e with strike K on the same future contract is equal to $max(K - F(T_e, T_0, T), 0)$. The person who buys an option is said to be in a *long position*, the seller of the option is said to be in a *short position*.

  European-type options on electricity market usually have the base, peak or off-peak futures as underlying. Accordingly they are also distinguished by delivery period of the underlying future. On the EEX there are Phelix Base options with month, quarter and year Phelix Base futures as underlying. Options have same delivery periods as underlying futures have. There are Call and Put options with different strikes. Number of strikes increases with the time coming closer to maturity of the option, which is the only exercise day of these options, which is the same as last trading day of the corresponding option. For all month options on EEX with delivery period other than January and quarter options with delivery in second, third and forth quarter the last trading day is 4 days before the delivery starts. For month and quarter Phelix options with delivery in January the last trading day is third Thursday of preceding December and for year contracts it is the second Thursday of preceding December. Of course exercise day has an influence on the price of the option. For example the month option with delivery in January and December exercise different number of days before delivery, which makes pricing formulas for these months to be different. There are also European-type options based on Phelix Peakload futures available.

- Options on the spot price

  Some other markets have also Asian options based on spot price averaged over delivery period. The payoff of such Call option is equal to $max(\frac{1}{n}\sum_{t=T_0}^{T} S(t) - K, 0)$, where n is the number of days during the delivery period and number of points taken for averaging.

2.5.3 Derivatives traded on OTC

The OTC contracts are the contracts which are not traded on the exchange and could be very complicated and client oriented. For example they don't have fixed number of the strikes and delivery periods to choose from and differ in contract specifications in order to satisfy the wishes of both parties entering the contract.

Bulk contracts

One of the simple contracts is called the bulk forward contract which is sold by generating companies to big electricity consumers in order to fix price, amount of electricity delivered and location of delivery in advance. These forward contracts have physical delivery, payoffs of these contracts are identical to futures traded on the market, but payment of these contract differs from client to client.

Floating contracts

Floating contracts are usually long-term contracts which have fixed amount of electricity to be delivered, but have floating price. At every period owner pays short-term floating price. Floating contract is a substitute for constantly buying electricity of fixed quantity on the spot market.

Caps and Floors

Cap provides the electricity price protection for the buyer of electricity above a predetermined level - the cap price - for a specific period of time. A floor guarantees a minimum price - floor price. Caps can be considered as floating contracts, but with a maximum level of this floating price, thus always more expensive than floating contract. Floorlet can be considered as floating contracts with minimum level which actually equivalent to the selling a series of put options to the seller of floor.

Swaps and Swaptions

Electricity Payer Swap is a financial contract between to parties which obligates the buyer to pay a fixed price for the underlying electricity and receive the floating spot electricity price over the contracted time period. The electricity Receiver Swap is a contract which obligates the buyer to receive fixed price for electricity in exchange to floating spot market price. This contract typically has fixed quantity of electricity and uses spot price at location of either producer or consumer of electricity. These contracts are widely used to hedge against short- or medium-term uncertainty in the market.

The payoff P of vanilla Payer Swap at any future swap date can be expressed as

$$P(t) = S(t) - K,$$

where $S(t)$ is a spot price at time t, K is fixed price. The swap is a contract which consists of number of such payoffs in the future.

There exist variable volume swaps, differential swaps, crack swaps, participation swaps, double-up swaps, extendable swaps and so on. For example *locational swaps* can use the spot price of electricity exchange located in the other location, not the delivery location, as floating price of a contract. For an overview about these swaps one can check for example [9].

Swing options

The swing option gives its holder the flexibility in the quantity of electricity to be delivered to him. The volume of electricity delivered can *swing* from some minimum to some maximum limits. These options give the holder of the option some security against the uncertainty in the volume of electricity to be consumed. The amount of electricity consumed by industrial parties can vary because of uncertainty in production volumes.

Thus the basic swing contract allows the holder of this contract to swing the amount of the base load of energy delivered with maximum (up-swing) and minimum (down-swing) daily and global (over whole delivery period) amount defined. So, the amount of electricity can vary daily between some daily limits, global limits for whole contract also exist. There are also restrictions on how many times (days) during the time in the contract the holder of this option is allowed to swing the amount of energy delivered.

Contracts for Difference (CfD)

Contracts for Difference (CfD) are forward contracts on the difference between specific area delivery price and some system price. A Contract for Difference is a purely financial transaction that involves no physical delivery. The contract must precisely specify the term, the underlying quantity and the prices of electricity. Compensation is paid for price differences over periods agreed in advance - monthly, quarterly or half-yearly. Such contracts are traded for example on NordPool for difference between price in particular country, say Sweden, and system price of NordPool.

Cross commodity derivatives

There are two main classes of cross commodity derivatives: spark-spread options and locational options

- *spark spread options* (also called heat rate linking derivative).

  It is the most important cross commodity option on electricity market. The spark spreads are derivatives which are linking electricity with a particular fuel used to generate electricity. The *spread* between electricity price and price of fuel is of interest because it is the main product to determine the economic value of generation assets. The amount of fuel needed for generating given amount of electricity is given as *heat rate*, which is number of units of fuel requested for generation of one megawatt hour (MWh) of electricity. The lower the heat rate the more efficient the generation and the higher is the price of the spark spread, which is defined as the difference between electricity price and the product of the heat rate and fuel price. The option written on this difference is called spark spread option.

  The payoff of the European Call spark spread option at time T is:

  $$\max(S_E(T) - H_F S_F(T), 0),$$

  where $S_E(T)$ is the price of electricity at time T, $S_F(T)$ is the price of fuel F and H_F is a heat rate for the fuel F. Heat rate is sometimes called "strike" of the spark option.

- *locational spread options*

  Locational spread options are the options written on the differences in the electricity prices at different locations (different electricity markets). These differences exist due to transmission constraints and transmission costs associated with the price of electricity.

  The payoff of the locational spread option at time T is

  $$\max(S_1(T) - S_2(T), 0),$$

  where $S_1(T)$ is the price of electricity at time T at location 1 and $S_2(T)$ is the price of electricity at time T at location 2.

Interruptible contract

Interruptible electricity contracts are the contracts issued by distributors or suppliers of electricity, that allow for interruptions to electric service. In exchange for a possibility of interruptions usually a reduction in the price of electricity delivered is offered. Sometimes financial compensation at the time of interruption is a substitute for a price reduction, this compensation depending on how far in advance notification about future interruption was announced. These contracts allow distributors of electricity to shift supply of electricity at peak hours or electricity failure from parties with interruptible contracts to the parties with non-interruptible contracts in order to meet demand and minimize the costs.

Weather derivatives

Most of derivatives considered before are used to hedge either price or volume risk. Although weather derivatives do not specifically include electricity prices in their payoff, they can be use on electricity market to hedge risks of changing demand and supply. Weather, more precisely outside temperature is one of the main factors of demand changes. Water temperature and precipitations influence supply side. Contracts based on heating degree days (HDD) or cooling degrees days (CDD) are traded, although liquidity is still lacking in Europe.

We did not include many other derivatives used for hedging risks arising in the electricity market.

Chapter 3

Data analysis

In this chapter we present a brief analysis of the data we use to estimate our model (see Chapter 5). We compare statistics, present seasonality and volatility analysis for the spot and futures on two different markets: EEX and APX/Endex, and present results of implied volatility estimations for EEX options market.

3.1 Spot analysis

3.1.1 Descriptive statistics

First we present descriptive statistics for the hourly, peak, off-peak and base prices for the EEX and APX spot markets. We calculate mean, median, standard deviation, maximum and minimum value, skewness and excess kurtosis for electricity spot prices (S) and for the log returns of electricity spot prices ($\Delta \ln(S)$). Empirical mean, standard deviation, skewness and kurtosis are four moments of empirical distribution.

The (sample) mean is

$$\text{Mean}(S) = \frac{1}{n} \sum_{k=1}^{n} S_k,$$

where S_k is the spot price at time $k = 1, \ldots, n$.

The median is such a value, that half of S_k are greater than the median, and half of S_k are less than median. The median is less sensitive to outliers than the mean.

The (sample) standard deviation of S is:

$$\text{Std.Dev.}(S) = \sqrt{\frac{1}{n-1} \sum_{k=1}^{n} (S_k - \text{Mean})^2} =: \sigma.$$

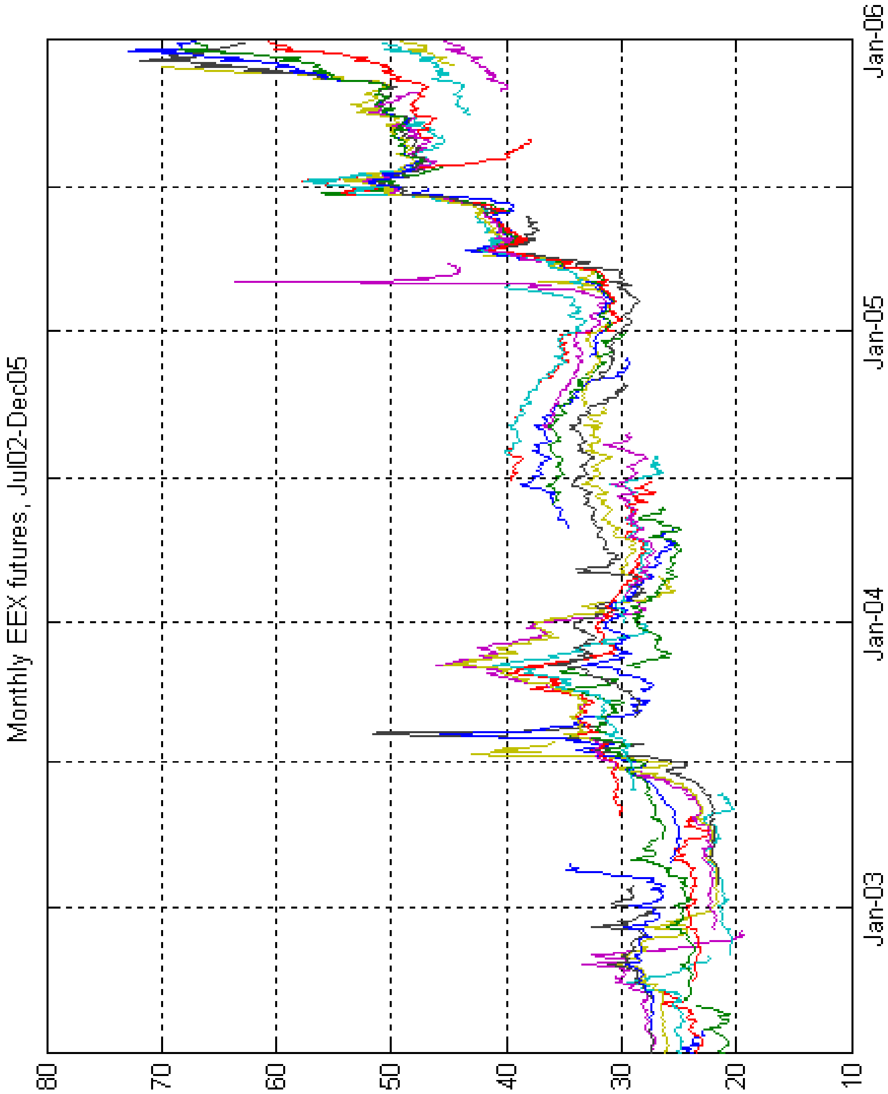

Figure 3.1: Monthly EEX futures prices, July 2002 - December 2005

	EEX			APX		
	S	$\Delta \ln(S)$	ΔD^2	S	$\Delta \ln(S)$	ΔD
Mean	30.12	0.00	0.00	38.75	0.00	0.00
Median	26.55	-0.01	0.00	28.50	-0.02	0.01
Std.Dev.	23.54	0.28	0.20	60.27	0.49	0.47
Maximum	1719.72	11.51	4.30	2000.00	7.84	7.81
Minimum	0.01	-10.60	-3.92	0.01	-8.16	-3.92
Skewness	15.72	0.72	-0.01	14.17	0.52	0.13
Kurtosis	762.23	249.38	49.50	320.80	94.18	110.38

Table 3.1: Descriptive statistics of Hourly EEX and APX prices, log returns and deseasonalized log returns, January 2001 - December 2005

Skewness is calculated by

$$\text{Skewness}(S) = \frac{1}{(n-1)\ \sigma^3} \sum_{k=1}^{n} (S_k - \text{Mean})^3 ,$$

and excess kurtosis is calculated as

$$\text{Kurtosis}(S)^1 = \frac{1}{(n-1)\ \sigma^4} \sum_{k=1}^{n} (S_k - \text{Mean})^4 - 3.$$

The descriptive statistics of EEX and APX electricity hourly prices from January 2001 till December 2005 and log returns are shown in Table 3.1. (Statistics for the price levels are given for indicative purposes. The price levels may not be stationary and unconditional moments may not exist.)

Skewness is a measure of the degree of asymmetry of a distribution. The skewness of a symmetric distribution is zero, and positive skewness indicates that the random variable is skewed to the right, which mean that right (higher value) tail is longer. As we can see from the Tables 3.1 and 3.2 that in both markets (EEX and APX) skewness of log returns is positive, thus the prices for both markets are right skewed.

Kurtosis is a measure of the ”peakedness” and also "fatness" of tails of the probability distribution, since the probabilities integrate to one. It shows whether the data are peaked or flat relative to a normal distribution. The kurtosis for a standard normal distribution is three. That is, data sets with high kurtosis tend to have a distinct peak near the mean and have heavy tails. As we can see from Table 3.1 kurtosis for both

[1] From here on we use a term Kurtosis for an excess kurtosis of the Sample, which is the Sample kurtosis minus 3, which means when 'excess kurtosis' is positive, there is greater kurtosis than in the normal distribution.

[2] Here deseasonalized logarithms D_t of the hourly prices S_t are calculated as follows: at the hour i, the deseasonalized prices $D_t^i = \ln(S_t^i) - A^i$, where A^i is the average of the logarithms of all ith hour prices. Corresponding returns of deseasonalized log prices ΔD are defined as $\Delta D_t = D_t - D_{t-1}$.

	EEX		APX	
	S	$\Delta \ln(S)$	S	$\Delta \ln(S)$
Mean	30.12	0.00	38.77	0.00
Median	27.84	-0.04	31.35	-0.03
Std.Dev.	15.59	0.33	33.48	0.44
Maximum	240.26	2.37	660.34	3.54
Minimum	3.12	-1.96	2.05	-2.53
Skewness	3.92	0.88	8.70	0.80
Kurtosis	32.53	4.48	130.78	6.45

Table 3.2: Descriptive statistics of Base EEX and APX prices and log returns, January 2001 - December 2005

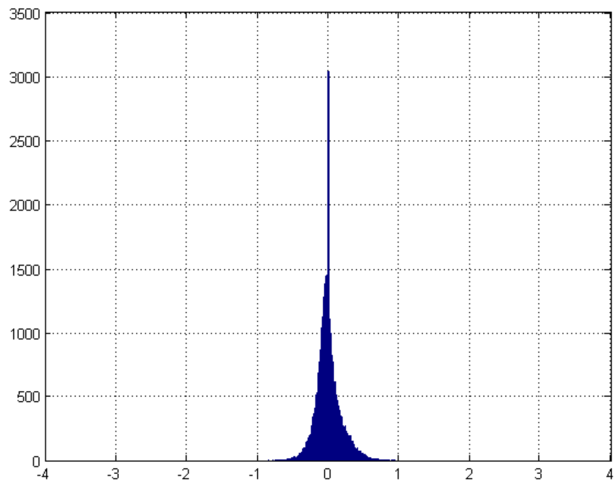

Figure 3.2: Histogram of the log returns of EEX hourly prices, January 2001 - December 2005

markets is very high, which means the tails of distribution are fat. Higher kurtosis means more of the variance is due to infrequent extreme deviations, or spikes, which are distinctive characteristic of electricity prices.

The histograms of the log hour prices for EEX and APX are presented in the Figure 3.2 and Figure 3.5. Big spike in the middle of histograms for log returns is an effect of rounding the prices to ± 0.01. From these graphs we can see that log returns of hourly prices have very peaked distribution, which is also confirmed by high kurtosis values in the Table 3.1.

The descriptive statistics of EEX electricity Base Phelix prices from January 2002 till December 2005 and for APX daily prices from July 2001 till December 2005 are shown in the Table 3.2.

Corresponding histograms of the log returns of Base EEX and APX prices are presented in the Figure 3.6 and Figure 3.7. These graphs also show peaked and visibly

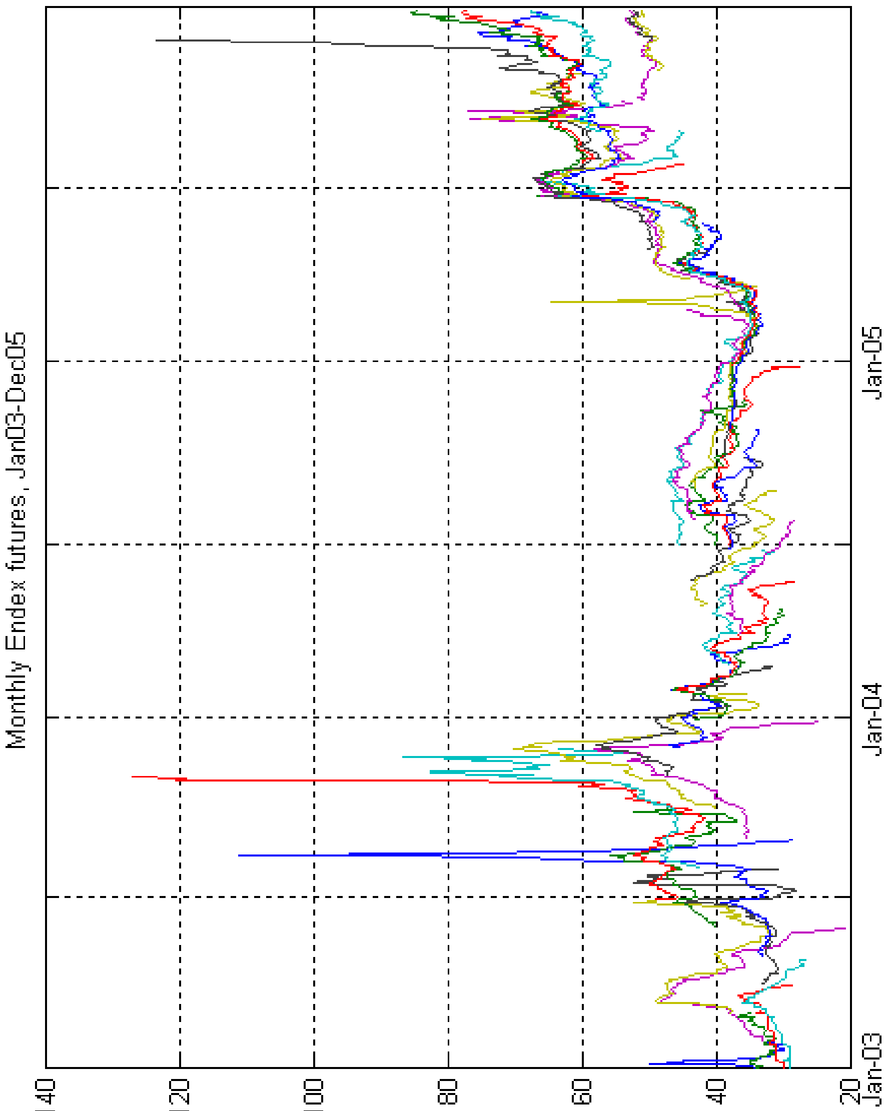

Figure 3.3: Monthly Endex futures prices, January 2003 - December 2005

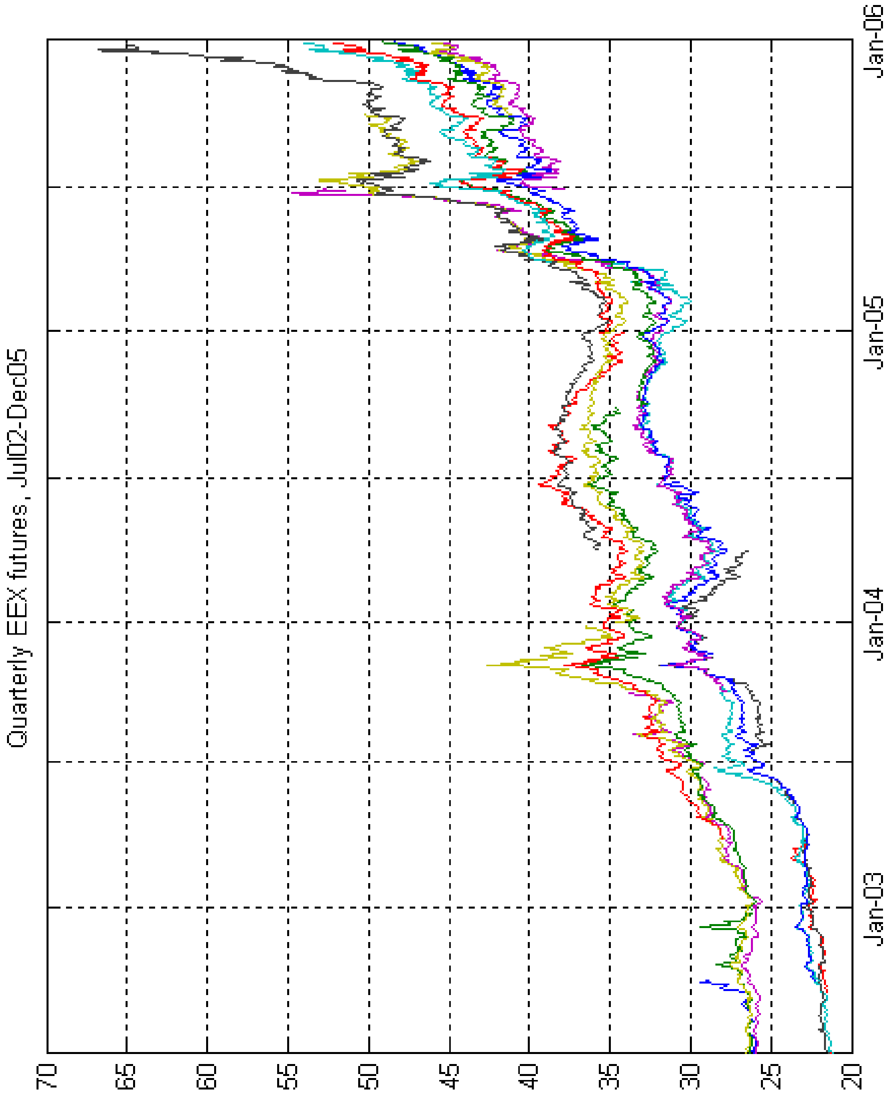

Figure 3.4: Quarterly EEX futures prices, July 2002 - December 2005

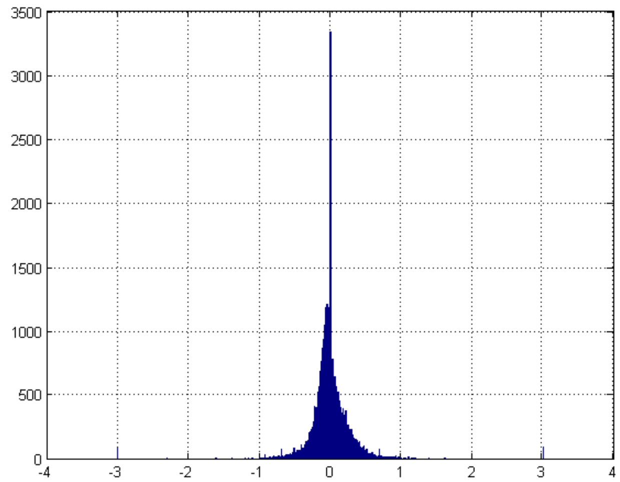

Figure 3.5: Histogram of the log returns of APX hourly prices, January 2001 - December 2005

right skewed distributions, especially for APX market. As expected, log returns of hourly prices for both markets have higher volatility, skewness and much higher kurtosis than corresponding log returns of Base prices.

If we consider deseasonalized hourly prices with respect to hour effect in the following way:

$$D_t = \ln(S(t)) - A_t,$$

where D_i are deseasonalized logarithms of the price at ith hour, $\ln(S(t))$ is the logarithm of the ith hour price and A_t is the mean of the logarithm of ith hour price for both markets (EEX and APX), then we can calculate descriptive statistics for these deseasonalized prices and we present then in the Table 3.1.

Because the price levels exhibit more positive spikes than negative and the log returns can exhibit both positive and negative spikes, the distributions of the log returns and the deseasonalized log returns are more symmetric (their skewness closer to zero).

Histograms of the changes of the deseasonalized hourly data for EEX and APX market are presented on the Figure 3.10 and Figure 3.11. As we can see from the histograms and the statistics for skewness and kurtosis, after deseasonalizing the data are less skewed (more symmetric) and have lower kurtosis.

3.1.2 QQ plots

The quantile-quantile (QQ) plot is a graphical technique for determining if two data sets come from populations with a common distribution. The normal QQ plot graphically compares the distribution of a given variable to the normal distribution (represented by a straight line). A QQ plot is a plot of the quantiles of the first data set against

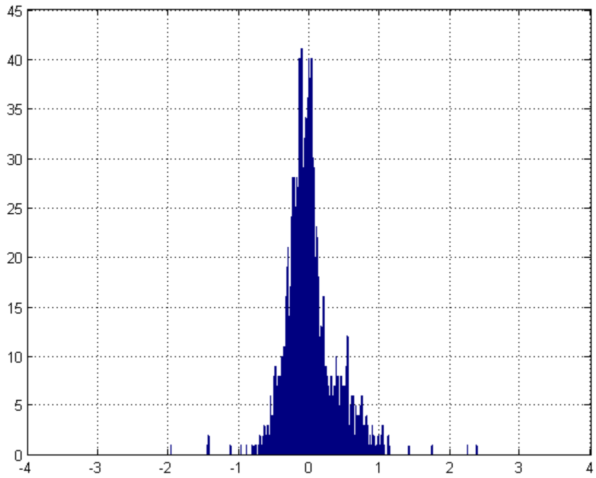

Figure 3.6: Histogram of the log returns of EEX Base prices, January 2001 - December 2005

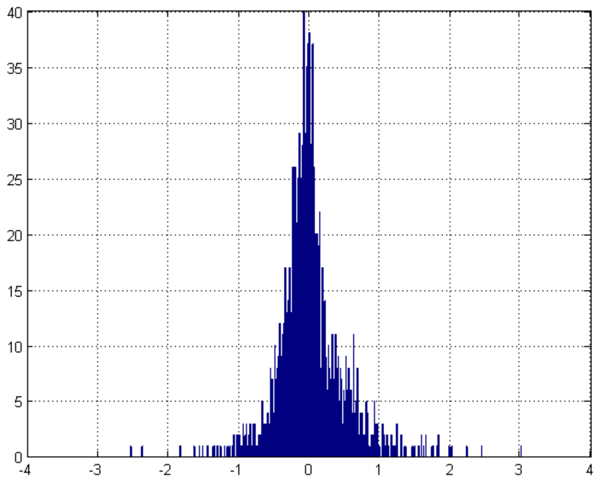

Figure 3.7: Histogram of the log returns of APX Base prices, January 2001 - December 2005

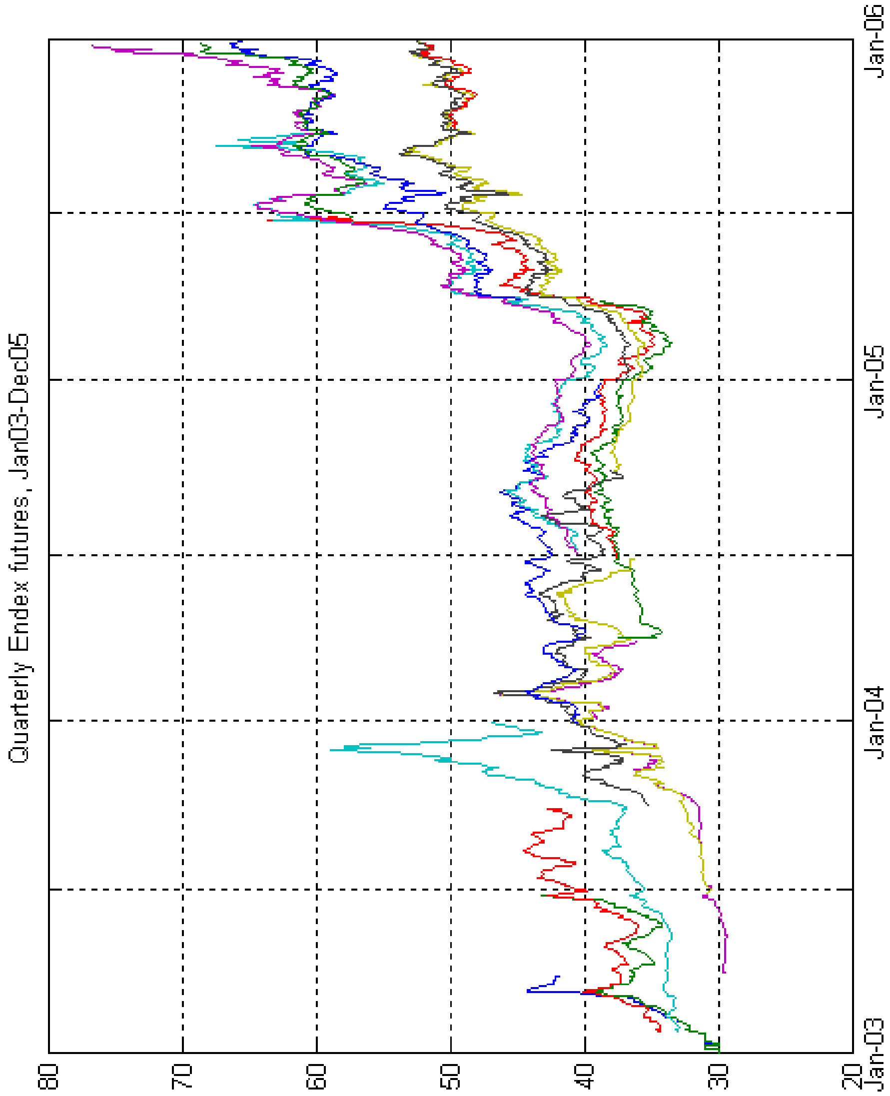

Figure 3.8: Quarterly Endex futures prices, January 2003 - December 2005

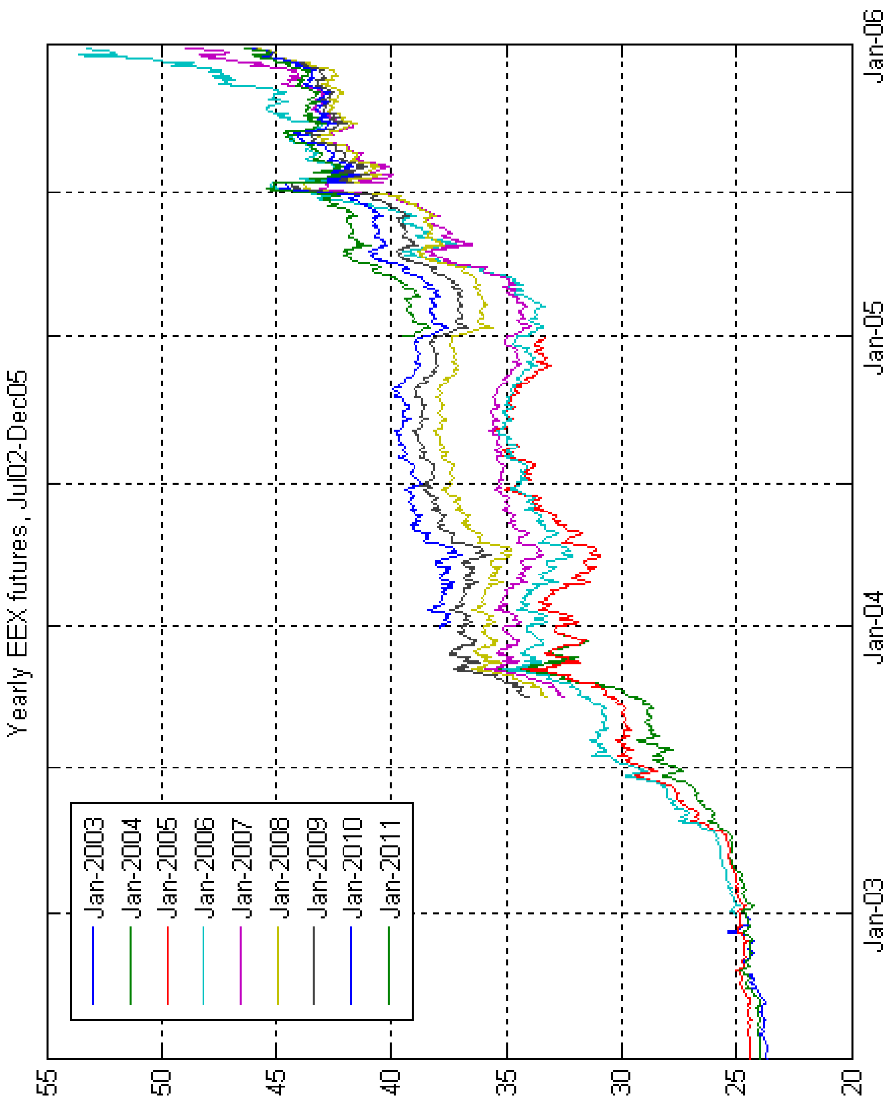

Figure 3.9: Yearly EEX futures prices, July 2002 - December 2005

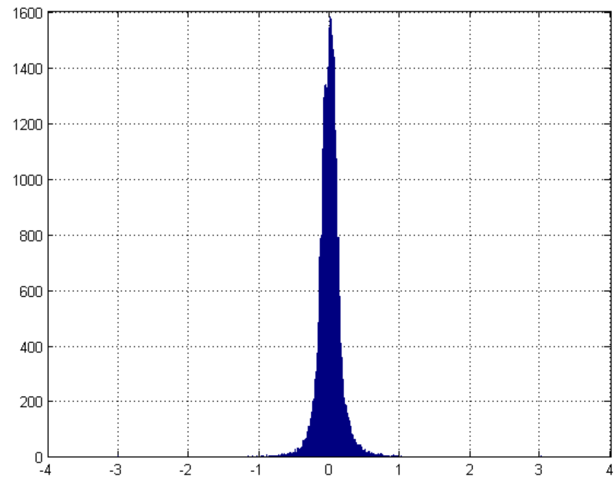

Figure 3.10: Histogram of deseasonalized log returns of EEX hour prices, January 2001 - December 2005

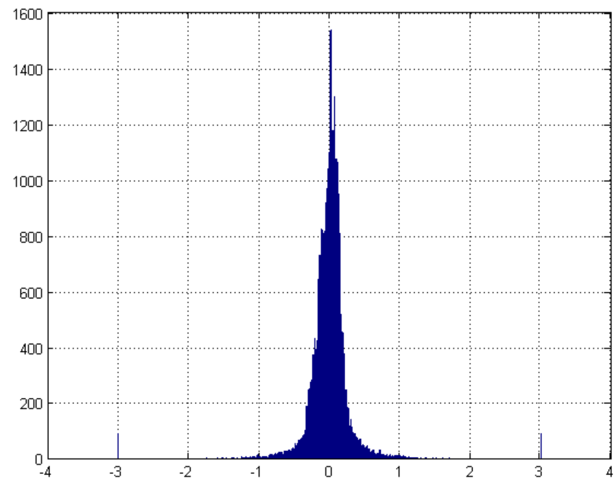

Figure 3.11: Histogram of deseasonalized log returns of APX hour prices, January 2001 - December 2005

the quantiles of the fitted normally distributed data set. By a quantile, we mean the fraction (or percent) of points below the given value. That is, the 0.3 (or 30%) quantile is the point at which 30% percent of the data fall below and 70% fall above that value.

QQ plots of log returns of EEX and APX hour prices are presented on the Figures 3.15 and 3.16. QQ plots of EEX and APX base prices are presented on the Figures 3.17 and 3.18.

These graphs reveal that the hourly log returns have fatter tails than Normal distribution. The tails of daily log returns are less fat than the tails of hourly log returns. These graphs also show that daily and hourly log returns on the APX have fatter tails than those on the EEX.

3.1.3 Volatility estimations

Volatility is found by calculating the annualized standard deviation of changes in price over a given period. Volatility of the spot process can be estimated in two ways. A standard way of estimating volatility for a given underlying is to use the price of an option on that underlying. This volatility is called *implied volatility*. Unfortunately we do not have options based on the spot price. We only can observe quotes of the options based on futures. We calculate implied volatilities based on these options in the next section 3.2 where we present data analysis of futures prices.

Another approach to estimating volatilities is to apply techniques of time series analysis to historical data. Volatilities calculated in this manner are called *historical volatilities*.

Historic volatilities

There are different ways of computing historic volatility. Hourly log volatility is the standard deviation of the log of the ratio of prices during the same hour on consecutive days. We analyze the prices of the same hour of a day to exclude the seasonality across the day from the hourly prices. Daily log volatility is the logarithm of the ratio of the average prices on consecutive days. And the rolling volatility is calculated over 30 days using the standard deviation of the log-ratio of daily average prices.

Because in our future analysis we use only daily prices, we do not calculate hourly volatilities here, but only daily and rolling historic volatilities.

We already calculated daily volatilities in the section 3.1.1, where we presented the descriptive statistics for the spot prices. We used EEX Base and APX Base spot price for the calculations of standard deviation of log return and the results are presented in the section above where we presented descriptive statistics. Standard deviation of log returns for EEX Base spot prices was 0.33 and for APX Base spot prices was 0.44. These volatilities are *daily volatilities*. Because volatilities are usually quoted on an

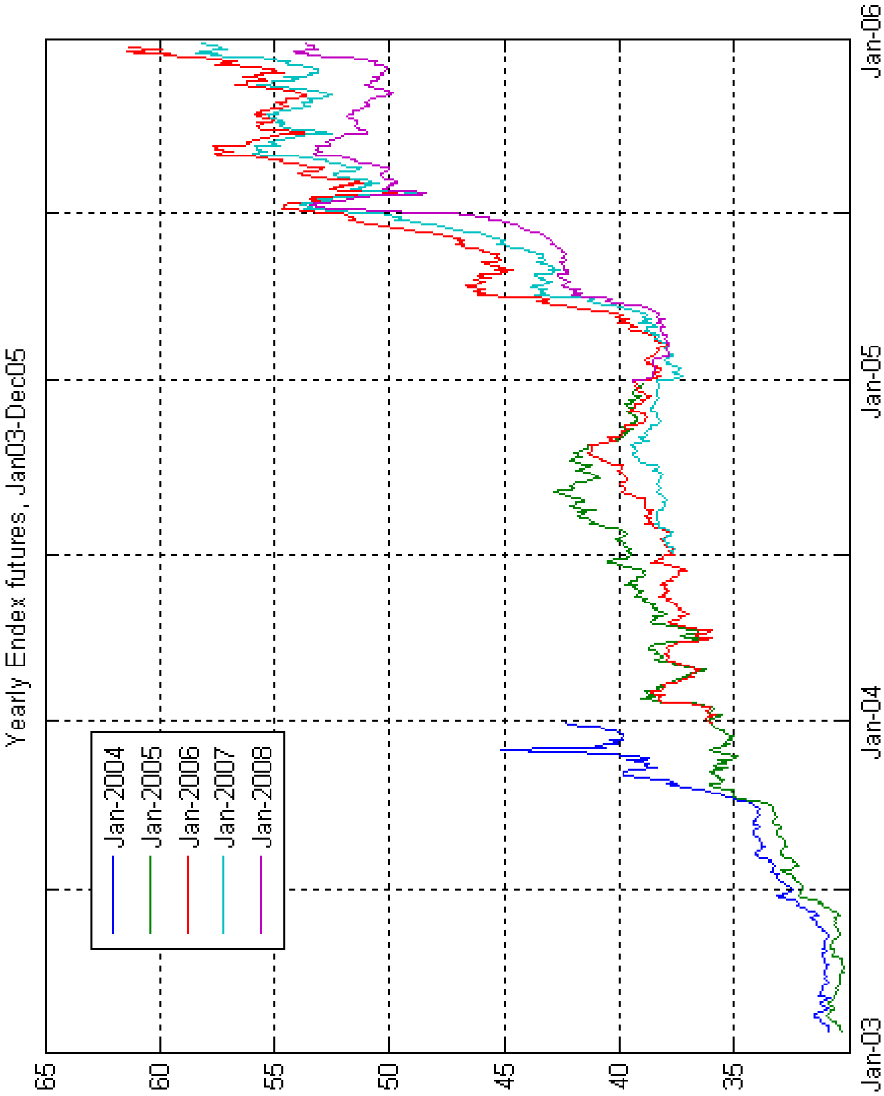

Figure 3.12: Yearly Endex futures prices, January 2003 - December 2005

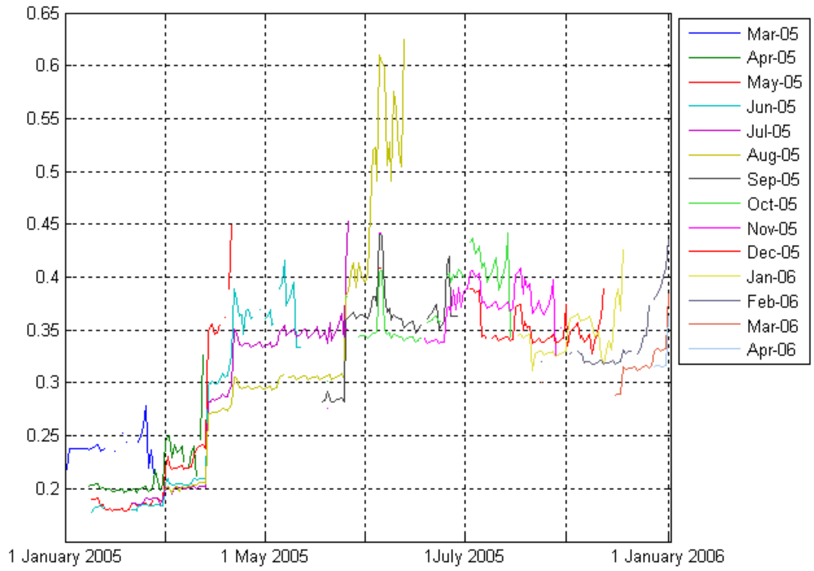

Figure 3.13: ATM implied volatility derived from EEX monthly options prices (with interest rate $r = 0.03$), January 2005 - December 2005

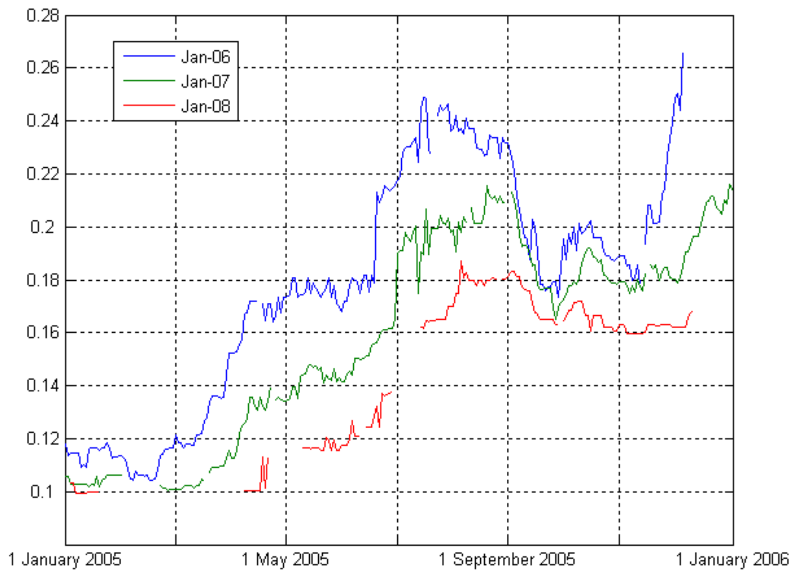

Figure 3.14: ATM implied volatility derived from EEX yearly options prices (with interest rate $r = 0.03$), January 2005 - December 2005

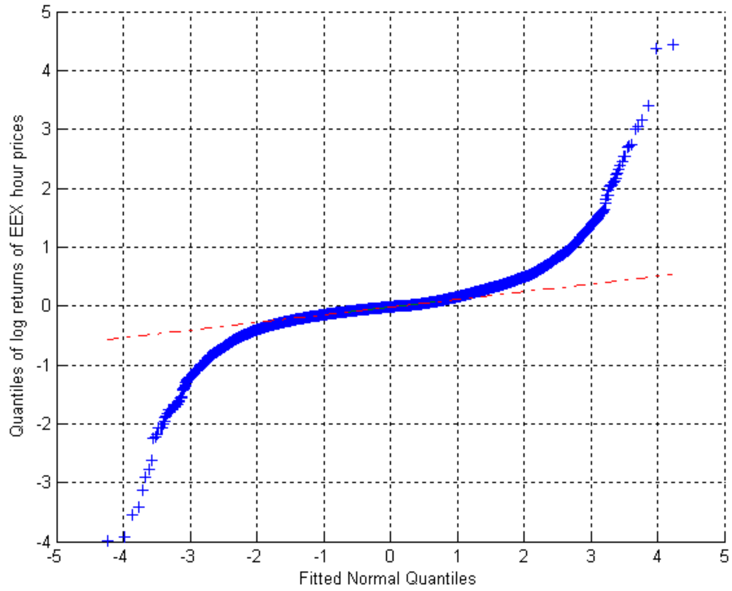

Figure 3.15: QQ Plot of log returns of EEX hourly prices versus Normal, January 2001 - December 2005

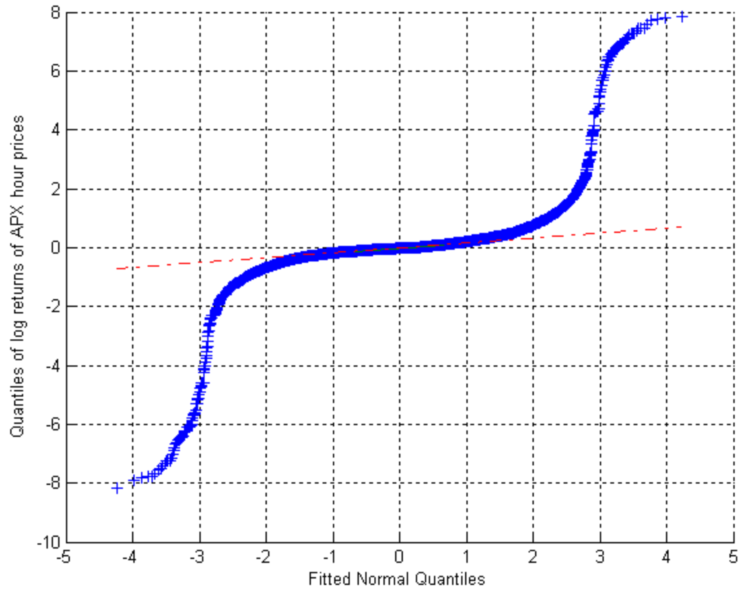

Figure 3.16: QQ Plot of log returns of APX hourly prices versus Normal, January 2001 - December 2005

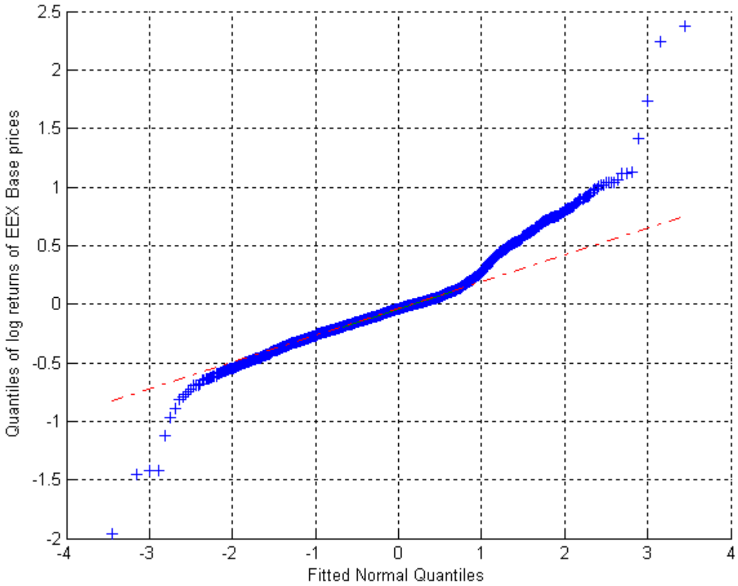

Figure 3.17: QQ Plot of log returns of Base EEX prices versus Normal, January 2001 - December 2005

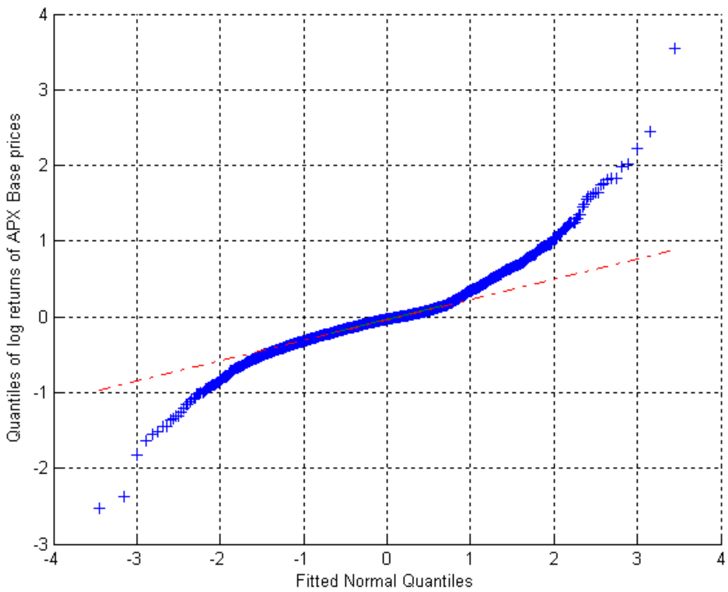

Figure 3.18: QQ Plot of log returns of Base APX prices versus Normal, January 2001 - December 2005

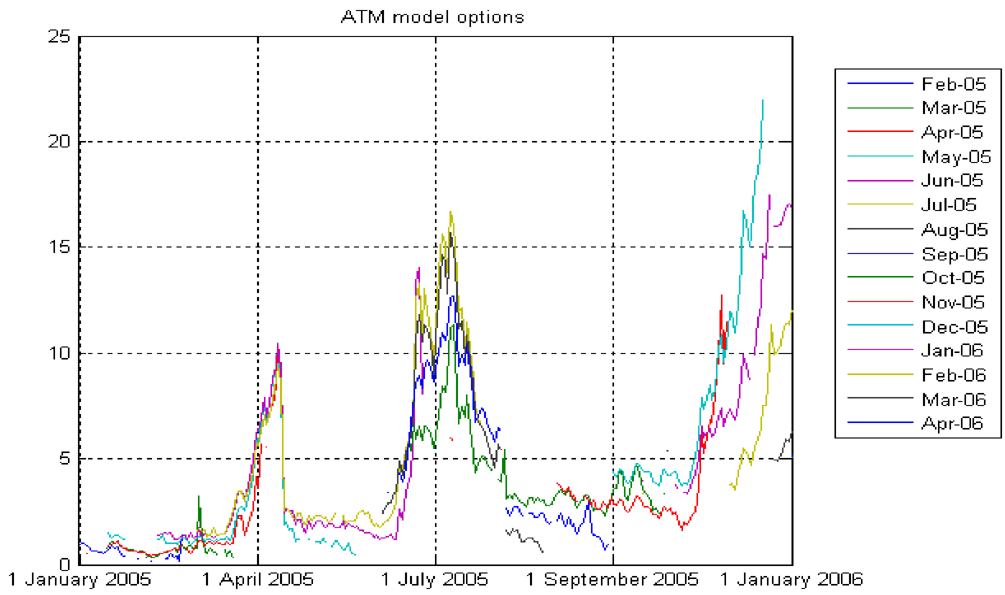

Figure 3.19: ATM Call options prices, EEX data

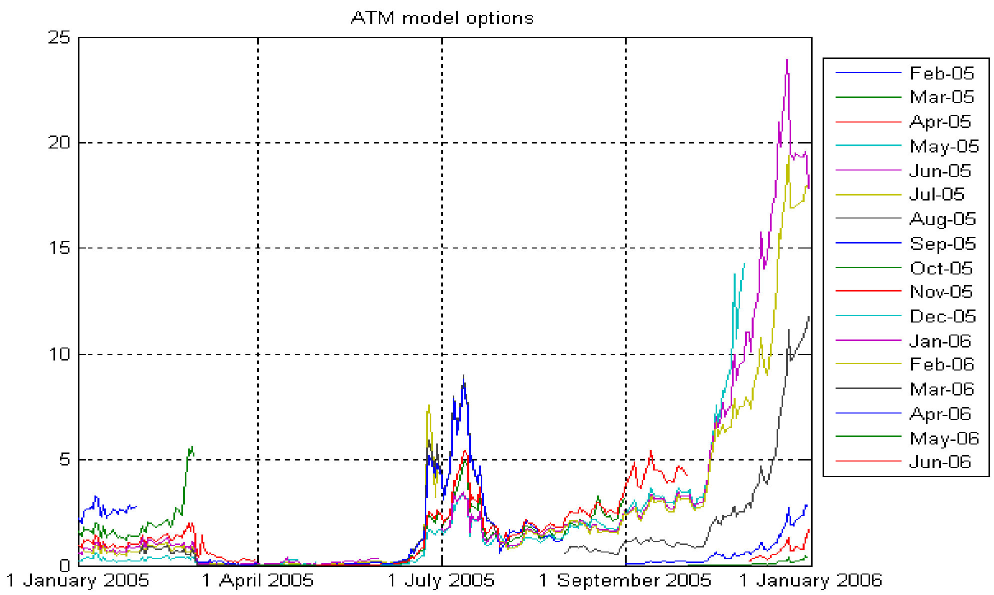

Figure 3.20: Modeled ATM Call options prices, calculated using estimated parameters

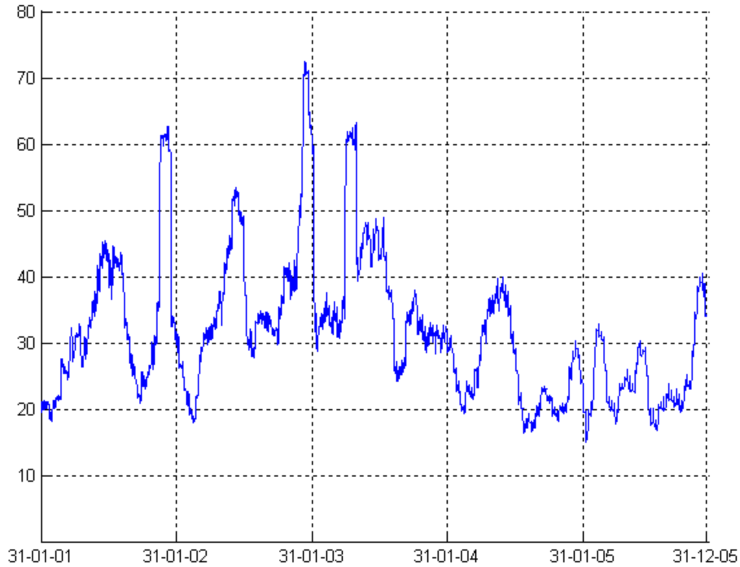

Figure 3.21: Rolling 30-day volatility of EEX base prices, January 2001 - December 2005

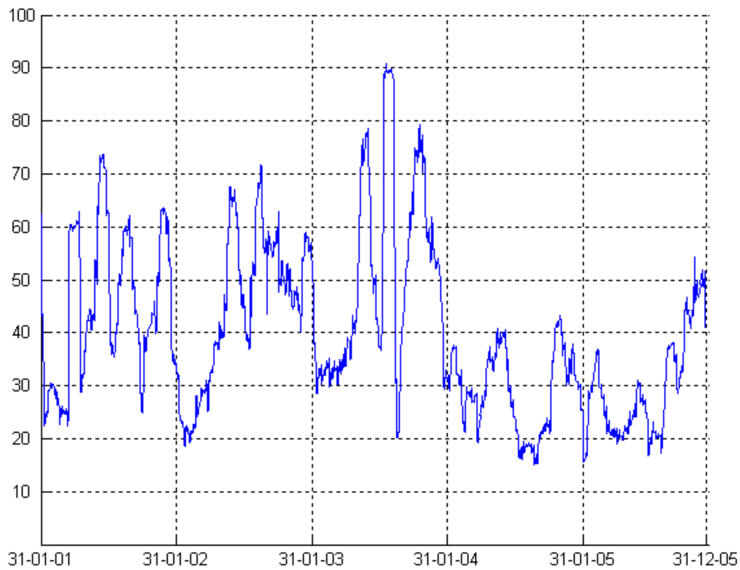

Figure 3.22: Rolling 30-day volatility of APX base prices, January 2001 - December 2005

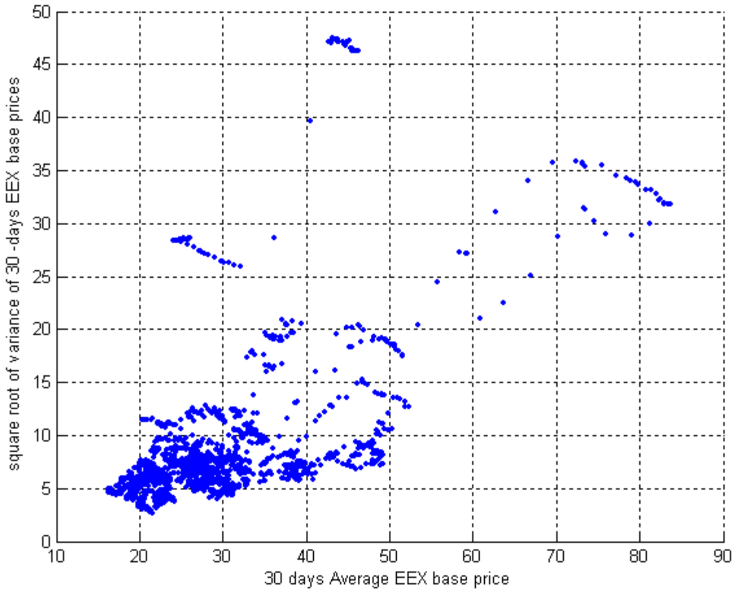

Figure 3.23: 30-day moving average of EEX Base prices versus their variance, January 2001 - December 2005

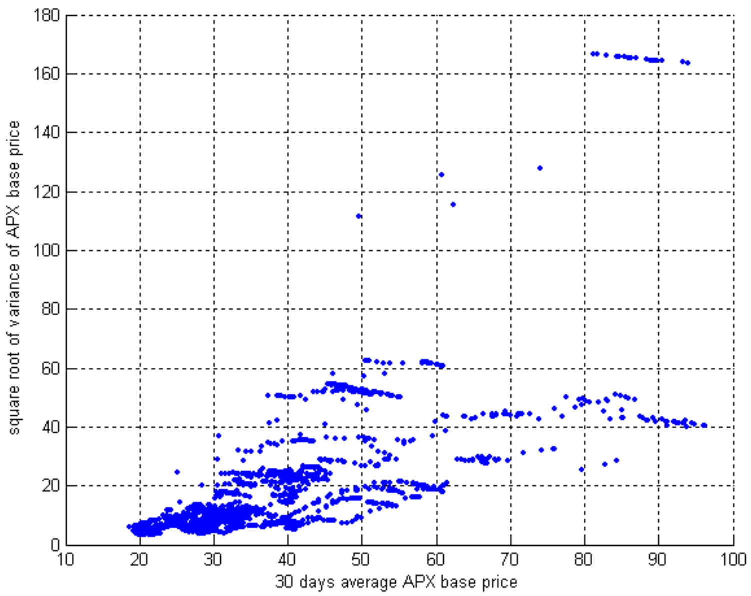

Figure 3.24: 30-day moving average of APX Base prices versus their variance, January 2001 - December 2005

annual basis such daily historical volatilities are routinely converted to an annual basis by applying the square root of time rule. The resulting volatilities are referred to as *annualized volatilities.* Annualized volatilities for EEX and APX Base prices are thus $0.33 * \sqrt{365} \simeq 6.30$ and $0.44 * \sqrt{365} \simeq 8.41$ accordingly.

The 30-day rolling volatility for EEX and APX base prices are shown in Figure 3.21 and Figure 3.22.

As we can see that the volatility changes with time. It changes from around 20 percent to almost 90 percent. One also can see so called "clustering effect" of the volatility on the Figures 3.21 and 3.22, which suggests using GARCH model for estimations of volatility. It is not clear if volatility has seasonality, but it is usually suggested that volatility is price dependent. To check for this property of volatility we plot 30-day moving average of a spot price versus square root of the variance of the same 30 days. Thus for each time point t we consider the spot prices process $S(t)$ and then we plot $\left(\sum_{i=0}^{29} S(t-i)\right)/30$ versus $\mathrm{Var}\{S(t-i)\}_{i=0}^{i=29})$.

This log-log plots for EEX and APX Base prices are presented n the Figures 3.23 and 3.24.

On these pictures we can see that variance of the price is correlated with the price levels, but there is no obvious linear dependency between this two quantities.

3.1.4 Seasonality

As we mentioned before, electricity exhibits the complicated seasonality patterns. We distinguish within-day hourly and weekly seasonal patterns, and a seasonality over the years. We use term "seasonality" for regularities in the prices.

Electricity price has different behavior during working hours, so called Peak hours (07:00-23:00 for APX and 08:00-20:00 for EEX), and during night hours (23:00-07:00 for APX and 20:00-08:00 for EEX). To show the within-day seasonality we present the graphs with the hourly EEX and APX prices for seven consecutive days in December 2005 (Figures 3.25 and 3.26).

As we can see peak prices are much higher than off-peak hours for both markets. December 10th and 16th are Sunday and Saturday correspondingly and the prices are lower during these weekends and higher during working days. EEX and APX prices increase at about 07.00 when day activity is starting. EEX prices show two peaks around 11:00-12:00 and 17:00-18:00. APX price shows more pronounced peak at about 18:00. Evening peaks happen usually during the time when most people leave their offices and start more intensive consumption of electricity at home. After 20:00 prices are low again.

There are also weekly cycles in the base spot prices. Base prices are most of the time lower during Saturdays and Sundays and higher during weekdays.

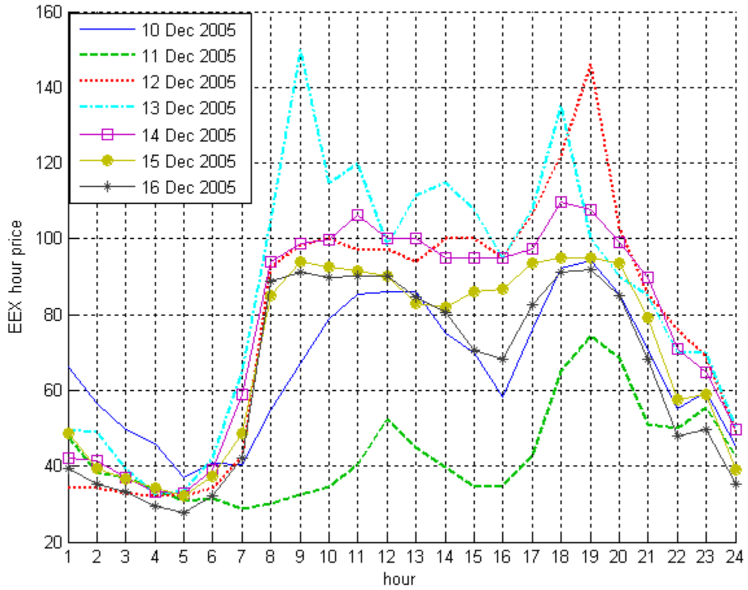

Figure 3.25: EEX hourly prices for 10th-16th of December 2005

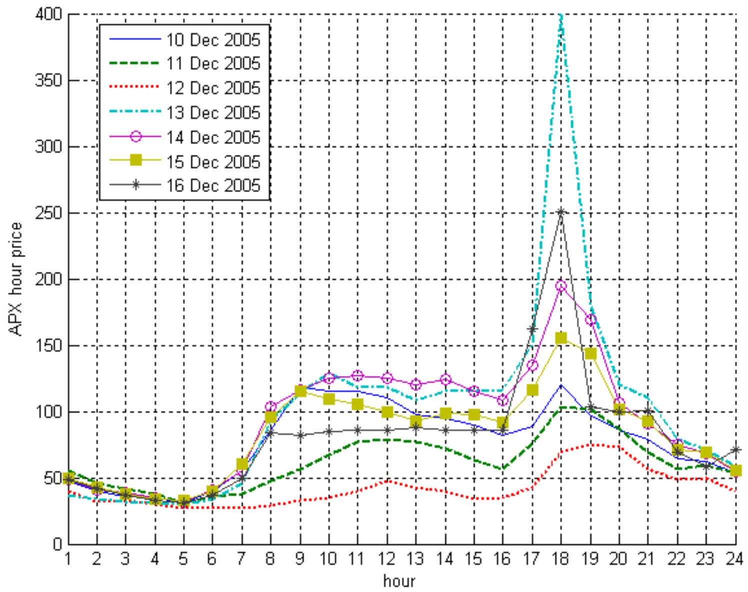

Figure 3.26: APX hourly prices for 10th-16th of December 2005

It is difficult to say if electricity has some regularity in the yearly Base price for either EEX or APX market. There are different levels of the price through the year, but it is not clear if these differences have some regularity.

3.2 Futures analysis

In this section we present statistics for the futures prices, which will help us to choose an appropriate model for futures price modeling. We consider EEX futures based on the averages of EEX Base prices and Endex futures which use averaged APX base spot electricity prices and underlying.

First we present a graphical representation of all monthly, quarterly and yearly futures for EEX and Endex market. Monthly futures are presented on the Figures 3.1 and 3.3. Quarterly futures are presented on the Figures 3.4 and 3.8. Yearly futures are presented on the Figures 3.9 and 3.12.

We can clearly see that monthly futures are more volatile especially during the delivery month. The prices of the yearly futures are monthly smoothly together. We can also see from these Figures that EEX futures in general are less volatile than the Endex futures, which is explained by the less volatile EEX spot Base prices with respect to APX spot Base prices as we saw in the section 3.1.

Because futures in general, and futures with one month delivery in particular, have too short a history to analyze their behavior properly, we should generate some uniform futures series from the data available in the market.

We use market data to generate futures with fixed time to delivery. We will create three different data series for each market with one month to delivery, one quarter to delivery and one year to delivery and will analyze these time series. For example, we generate one-month-to-delivery future as follows: in July 2002 we use the price of a one-month future delivering in August 2002, in August 2002 we use the price of the month future with delivery in September, and so forth. Of course natural problems arise during switching from one future series to another. Monthly futures are based on the averages over the delivery month and these averages differ from month to month. We will get jumps in the prices, which are not observable in the prices and which do not reflect reality of the market. We will exclude these data points and will not use them for our analysis. More precisely, for calculation of descriptive statistics of the rolling one-month-to-delivery future we exclude the prices for the first and the last days of the month. For graphical representation we linearly interpolate these two points (the prices on the first and last day of the month) in order to have equally dispersed quotes.

These rolling EEX and Endex futures are presented on the figures 3.27 and 3.28.

As for the rolling one-month-to delivery futures we exclude two points on the border from one quarter to another (and two points which separate two consecutive years) and

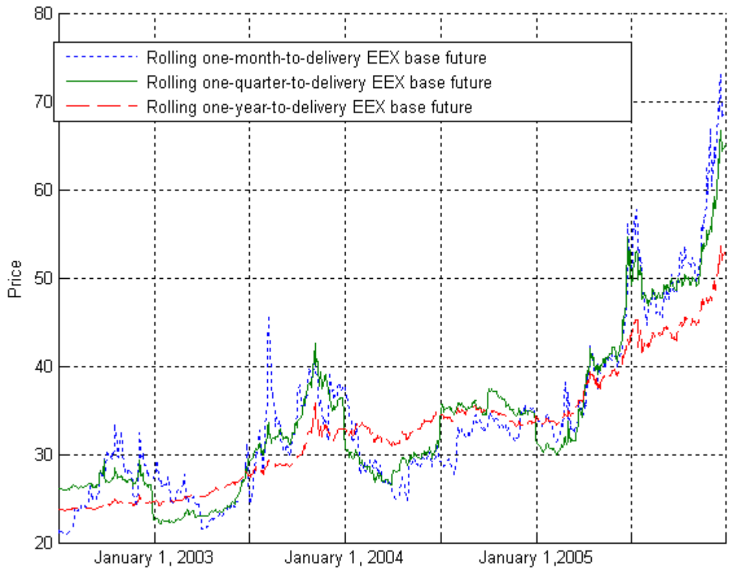

Figure 3.27: Rolling EEX futures, July 2002 - December 2005

linearly interpolate these missing points for graphical representation.

We present descriptive statistics for rolling EEX and Endex one-month-to-delivery futures in the Table 3.3.

QQ plots for rolling one-month-to-delivery futures are presented on the Figures 3.29 and 3.30

Now we consider rolling one-quarter-to-delivery future and present descriptive statistics for rolling EEX and Endex futures in the Table 3.4.

QQ plots for rolling one-month-to-delivery futures are presented on the Figures 3.31 and 3.32

Descriptive statistic for rolling one-year-to-delivery EEX and Endex futures are

	EEX		Endex	
	S	$\Delta \ln(S)$	S	$\Delta \ln(S)$
Mean	34.12	0.0007	44.27	0.0006
Median	32.04	0.0003	41.09	0.0002
Std.Dev.	9.56	0.012	10.33	0.014
Maximum	68.47	0.09	77.82	0.11
Minimum	21.55	-0.07	30.00	-0.08
Skewness	1.15	0.49	1.06	0.93
Kurtosis	0.93	9.90	0.40	13.20

Table 3.3: Descriptive statistics for rolling one-month-to-delivery EEX futures (July 2002 - December 2005) and Endex (January 2003 - December 2005) and log returns

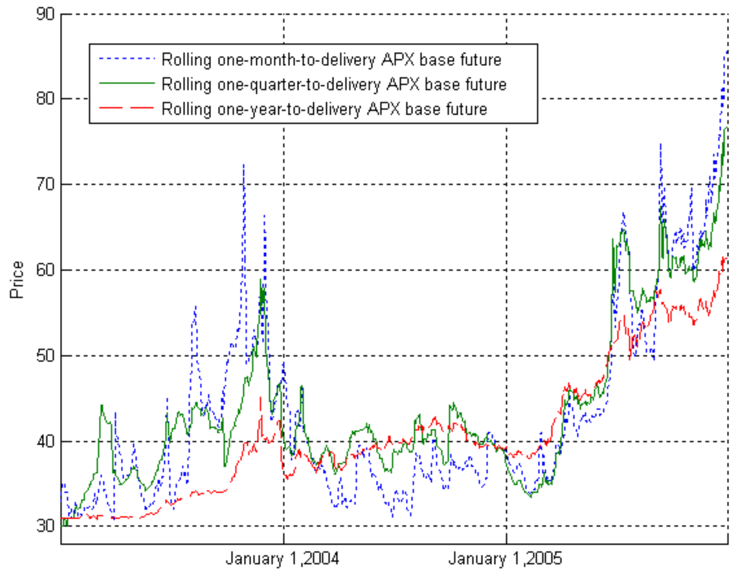

Figure 3.28: Rolling Endex futures, January 2003 - December 2005

	EEX		Endex	
	S	$\Delta \ln(S)$	S	$\Delta \ln(S)$
Mean	33.92	0.0009	43.81	0.001
Median	31.37	0.0004	41.13	0.0003
Std.Dev.	9.11	0.0092	9.42	0.012
Maximum	66.74	0.09	76.77	0.08
Minimum	22.15	-0.06	30.00	-0.09
Skewness	1.11	0.95	1.25	0.77
Kurtosis	0.81	13.43	0.79	11.59

Table 3.4: Descriptive statistics for rolling one-quarter-to-delivery EEX futures (July 2002 - December 2005) and Endex (January 2003 - December 2005) and log returns

	EEX		Endex	
	S	$\Delta \ln(S)$	S	$\Delta \ln(S)$
Mean	32.77	0.0006	41.03	0.0008
Median	32.90	0.0004	39.39	0.0005
Std.Dev.	6.87	0.006	7.75	0.007
Maximum	53.55	0.03	61.47	0.05
Minimum	23.65	-0.07	30.83	-0.06
Skewness	0.63	-1.79	0.79	-0.83
Kurtosis	-0.17	22.25	-0.26	16.91

Table 3.5: Descriptive statistics for rolling one-year-to-delivery EEX and Endex futures and their log returns

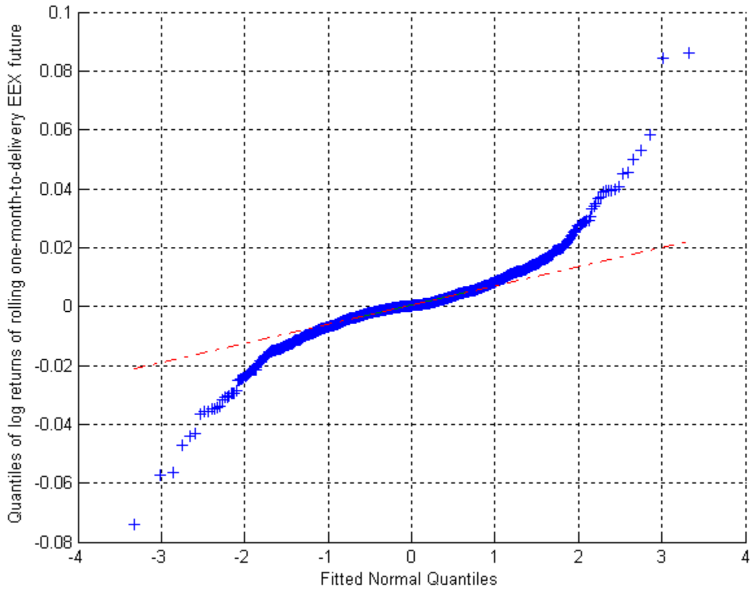

Figure 3.29: QQ plot of log returns of rolling one-month-to-delivery EEX future versus Normal, July 2002 - December 2005

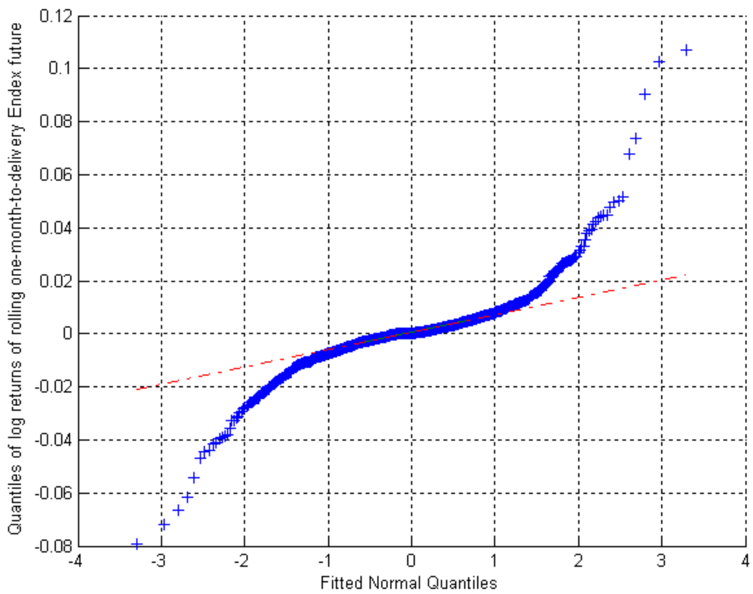

Figure 3.30: QQ plot of log returns of rolling one-month-to-delivery Endex future versus Normal, January 2002 - Decemebr 2005

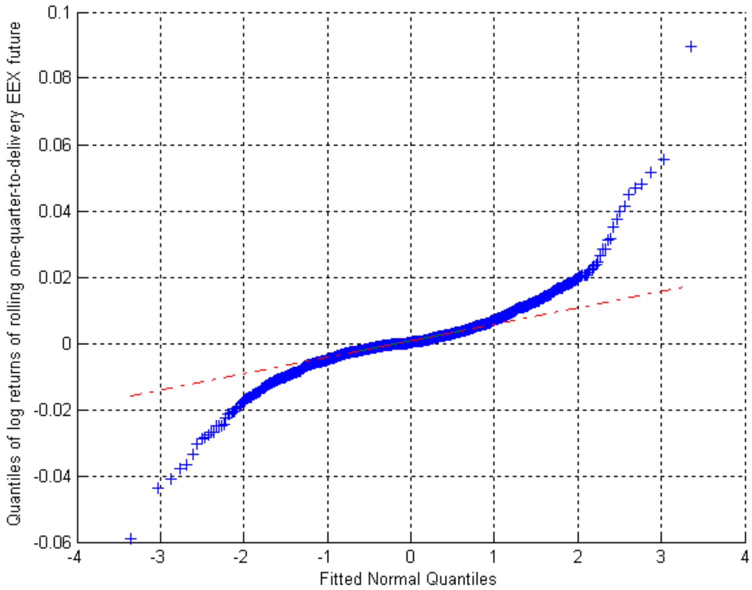

Figure 3.31: QQ plot of log returns of rolling one-quarter-to-delivery EEX future versus Normal, July 2002 - December 2005

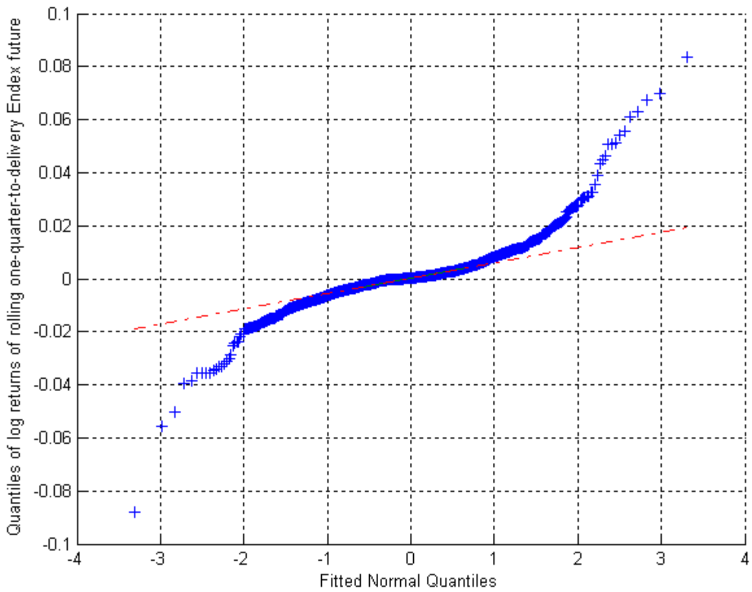

Figure 3.32: QQ plot of log returns of rolling one-quarter-to-delivery Endex future versus Normal, January 2003 - December 2005

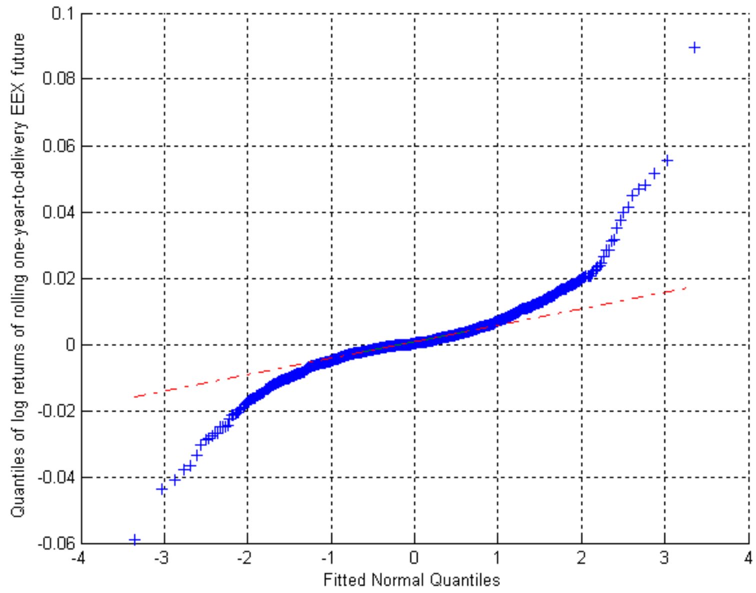

Figure 3.33: QQ plot of the rolling one-year-to-delivery EEX future versus Normal, July 2002 - December 2005

presented in the Table 3.5.

Finally, QQ plots of the log returns of the rolling one-year-to-delivery EEX and Endex futures are presented on Figures 3.33 and 3.34.

As we can see from the standard deviation values, monthly futures are more volatile than quarterly futures and quarterly futures are more volatile than yearly futures. From the QQ plots we see that the tails of monthly futures are fatter than the tails of quarterly and yearly futures. Amazingly, though yearly and quarterly contracts have lower values of kurtosis and skewness for the price levels, they do not have lower kurtosis and skewness values for the log returns, which could be explain by artificial nature of data used for estimation. More precisely, removing two points at the beginning and the end of each delivery period does not completely smooth out the high differences in the prices between different contracts. Compare, for example, prices of Endex futures for the calendar year 2004 at the end of December 2003 and the Endex future for the calendar year 2005 at the beginning of January 2005 (see Figure 3.12). Thus the price of rolling one-year-to-delivery futures drops from 42.38 euro to 36.03 euro per Megawatt hour in just three days. This change is more visible in the log returns of the prices. Of course such changes in the price make our estimated descriptive statistics for the log returns to be not accurate and we should account for these ”rolling errors”.

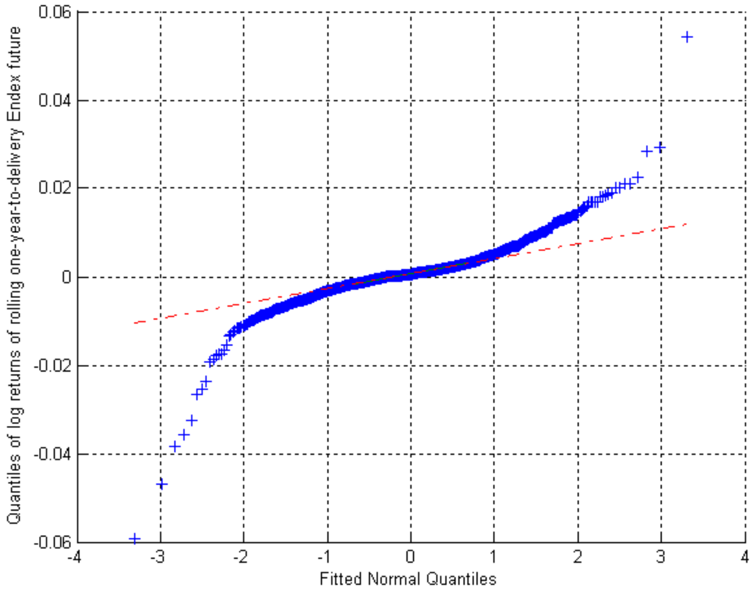

Figure 3.34: QQ plot of the rolling one-year-to-delivery Endex future versus Normal, January 2003 - December 2005

3.2.1 Futures seasonality

As we saw in the section 3.1.4 analyzing spot prices electricity has complicated cyclic patterns. It is not clear from the graph wish show different spot prices dynamics if the spot price is seasonal over the year. Futures on the other hand exhibit seasonality. First of all, the futures on electricity are quoted once a day and based on the averages of the spot price. Thus naturally we do not see effects of daily and weekly seasonality patterns. But the effect of seasonality over calendar year is visibly more pronounced on the graphs of future prices then on the graphs of spot prices.

Because of the fact that futures prices have several time dimensions (current "running " time and time of delivery), it is difficult to uniformly define what is seasonality of the futures prices. One can ignore the dependence of the futures prices on delivery period and consider uniform seasonality function depending on current time. As in [6] and [4] we consider the monthly futures price $F(t,T)$, where T is the expiry month, to depend on two seasonal factors: $a(t)$ - an average price of all month futures traded at the market on day t - and $b(T)$ - average of all the prices of one specific future expiring at time T. Thus our seasonal function $F(t,T) = a(t) + b(T) + \varepsilon(t,T)$ (where $\varepsilon(t,T)$ is the error), depends linearly on two components in different time dimensions.

We consider the above method and different possibilities to model seasonality function in more details in Chapter 6. Figures 3.35 and 3.36 show estimated $a(t)$ and $b(T)$ for EEX and Endex monthly futures.

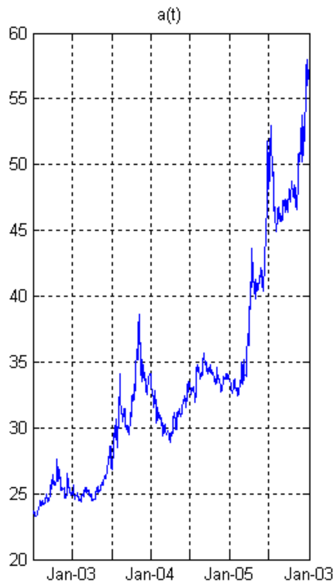

Figure 3.35: Common time factor $a(t)$ and expiry effect factor $b(T)$ for EEX monthly futures, July 2002 - Decemeber 2005

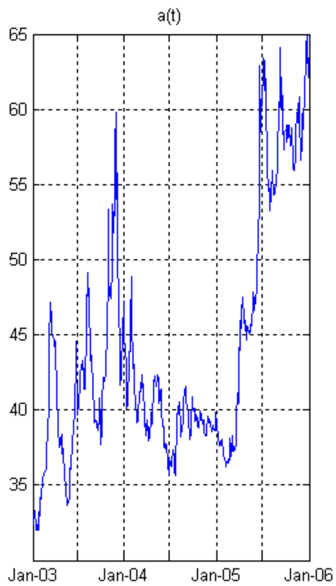

Figure 3.36: Common time factor $a(t)$ and expiry effect factor $b(T)$ for Endex monthly futures, January 2003 -December 2005

3.3 Options analysis

Option prices are available only for EEX market. There are Put and Call options on monthly, quarterly and yearly futures. These options are traded only from the end of 2004, we will consider only quotes from 1st of January 2005 till 30th of December 2005. To analyze options behavior we calculate the implied volatility from the options quotes. Implied volatility for an option price C by definition is such a value of the volatility parameter σ in Black-Scholes formula for calculating options values that gives calculated option value to be equal C. We used formulae 5.34 and 5.35 to calculate implied volatilities from given Call and Put quotes from the market. As we mentioned before, options market is very illiquid, we often can see option quotes of 0.01 euro, which means that option prices reach their minimum allowed price in the market and there are arbitrage opportunities. Almost a fifth part of all quotes have either Call or Put price equal to 0.01 euros, or Call-Put parity does not hold. We remove these quotes before calculating the implied volatilities. Also most of the options quotes do not actually have any traded volume or open interest. This means that a lot of quotes actually are set be the exchange and are not results of fair trading.

Implied at-the-money (ATM) volatilities derived from monthly and yearly EEX Call options are presented in the Figures 3.13 and 3.14.

Here we removed all unrealistic values of the implied volatility (less than 10% for monthly options and less than 5% for yearly quotes) and did not calculated the implied volatilities from the option values too close to expiry of the option (few days before expiry), because volatility parameter grows dramatically in just few days (up to 10 times) and makes it difficult to calculate volatility precisely.

As we can see from these figures, implied volatility is higher for the monthly options than for yearly options, which is explained by the higher volatility of the underlying monthly futures with respect to yearly futures as was shown in the section 3.2. It is also similar time dependence in volatility values for monthly and yearly options. Thus the stochastic volatility models we present in the next chapter 4 or at least deterministic but time-dependent volatility function seems to be appropriate for better volatility modeling.

Chapter 4

Pricing models for electricity

4.1 Modeling approaches for electricity prices

There are three different ways to model electricity prices. We use the work of Anderson [1] to present these different approaches for electricity modeling. According to these approaches all models are divided into three groups of models:

1. *reduced-form models* or, as we call them, financial models, where the price of electricity is modeled directly
2. *fundamental models*, which use production fundamentals to determine the marginal cost of electricity
3. *hybrid models*, which are mixtures of financial and fundamental models.

The fundamental models were developed and used under the regulated electricity market system. These models use optimization procedure to minimize total costs of production of electricity under constrains from demand side and environment. Those models use three main sets of input parameters:

1. Parameters for detailed specification of the loads for specific regions,
2. Generation characteristics, such as fuel costs, heat rates, failure characteristics, type of generation and capacities,
3. Parameters of environmental, operational and transmission constraints.

These models are usually very detailed and have non-linear relationships between all the inputs and drivers in the price process. These models also have to take into account uncertainties about each of these parameters. The fundamental approach has two main disadvantages:

First, for each different scenario these models should be re-specified and optimized, which makes simulations impractical because of high computational expenses.

Second, there is no mechanism to impose existing electricity prices into the computations. This is because all the prices are based on the cost of production under specified constraints and do not depend on the market data. This makes is hard to fit forward curves for example.

In contrast, financial models specify spot prices of electricity directly. As can be observed in the market, the spot and futures/forward electricity prices are random and thus can be modeled by stochastic models with specific properties which match characteristics of the price process we are looking for and fit historical data. These models try to fit market electricity prices into a framework using several parameters. In some cases (for example when log returns are assumed to be normally distributed) such models produce closed form solutions for European options and futures.

The third types of models are called hybrid models as they combine characteristics of fundamental and reduced-form models. The key issue with reduced-form models is a lack of sufficient data for fitting. This problem could be solved by hybrid models, which can incorporate large set of historical data available for fundamental models and flexibility and simplicity of reduced-form models. Good examples of hybrid models can be found in [1]. From now on we consider reduced-forms models only.

In order to find appropriate model for spot/futures electricity prices we first need to agree which properties of observed prices are the most important and have to be captured. For example, as soon as we would like to model electricity futures prices and not interested in capturing spot prices behavior beyond needs for futures modeling, we will not consider jumps which could capture the spikes of the spot prices, but not necessary for futures prices, based on *spot averages*.

Secondly, if we want to develop stochastic model for electricity prices we can consider existing financial models for stocks and other commodities, which are used for a few decades already and are well known. Choosing an appropriate stochastic process model we should try to capture empirical moments (empirical mean, standard deviation, skewness and kurtosis), known characteristics of the process (mean reversion, seasonality, spikes) and observed dynamics of the price process.

In the next sections we briefly introduce classical financial models which could be consider for use in calculating of electricity prices and futures prices. We consider an asset price $S(t)$ at time t. See for example German [16] for more details on the models we describe below.

4.2 Geometric Brownian Motion

The simplest, but very important model for stock returns is a Geometric Brownian Motion model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t),$$

here $W(t)$ is a standard Brownian motion, μ is the expected return per unit of time, also called *drift* and σ is the standard deviation of the return per unit of time, also called *volatility.* Both μ and σ are constant. Using Itô formula we can express this formula as follows:

$$d\ln S(t) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW(t).$$

This model was used by Black and Scholes [3] and Merton [25] to derive their celebrated options on stock pricing formula. Stock returns in this model are normally distributed. Also according to this model stock prices on average are growing over time (with the constant rate μ). For commodities however this is not true in general.

4.3 Mean reversion Ornstein-Uhlenbeck model

To avoid the growth of returns over time Vasicek introduced mean-reversion Ornstein-Uhlenbeck model capturing interest rates dynamics. Dynamics of interest rate $r(t)$ is presented as follows:

$$dr(t) = \kappa(\theta - r(t))dt + \sigma dW(t),$$

where κ, θ and σ are constant. Parameter κ is called *the speed of mean-reversion* or *the rate of mean-reversion,* θ is *the long-term mean* and σ is the volatility.

As we saw on the Figures 2.5 and 2.6 of Base prices from EEX and APX market in the Chapter 3, there is visible mean-reversion effect of the spot prices on both markets. Thus, to model energy we want to keep the geometric Brownian motion and introduce mean reversion, thus we can consider returns to revert to its mean. Thus we present the following modification of the model:

$$\begin{aligned} X(t) &= \ln S(t), \\ dX(t) &= \kappa(\theta - X(t)) + \sigma dW(t). \end{aligned}$$

This model can capture mean reversion effect documented for most energy commodities and electricity in particular. We model the logarithm of the spot price instead of modeling price levels themselves to make sure that the modeled prices are non-negative. Another modification which improves the possibility of the model to fit market data is an introduction of seasonality component into the model:

$$\begin{aligned}\ln S(t) &= h(t) + X(t),\\ dX(t) &= \kappa(\theta - X(t)) + \sigma dW(t),\end{aligned}$$

where $h(t)$ is a deterministic component used for taking into account seasonality of the prices. We present several possibilities to introduce seasonal components into the model later in Chapter 6.

4.4 Schwartz-Smith two-factor model

The way to combine two models presented above was shown in the Schwartz and Smith model (see [29]). In this model two sources of randomness are considered. Logarithm of electricity spot prices is expressed as a sum of two factors:

$$\ln(S(t)) = \chi(t) + \xi(t),$$

where $\chi(t)$ is a short-term deviation in price, $\xi(t)$ is the equilibrium price level. The short-term deviations ($\chi(t)$) are assumed to revert toward zero following an Ornstein-Uhlenbeck process:

$$d\chi(t) = -\kappa\chi(t)\,dt + \sigma_\chi\,dW_\chi(t),$$

and the equilibrium level ($\xi(t)$) is assumed to follow a Brownian motion process

$$d\xi(t) = \mu_\xi\,dt + \sigma_\xi\,dW_\xi(t).$$

4.5 Jump diffusion model

The primary drawback of log-normal models presented before is the lack of kurtosis in the tails of distribution which is observed on spot market (see Chapter 2 for data analysis of spot ad futures electricity prices). The kurtosis in the distribution of electricity spot prices comes from the prices spikes. To capture high kurtosis two possible modification can be done: we can introduce a stochastic volatility or add a jump component into the model. Both of them bring a second source of randomness into the model.

The first jump-diffusion model was introduced by Merton in 1976 (see [26]). He proposed the following model:

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dW(t) + J(t)dq(t)),$$

where $q(t)$ is a Poisson process with intensity λ which counts for number of jumps, $J(t)$ is real random variable (usually normally distributed), which presents distribution of jump size. Probability of jump over time interval dt is equal to $\mathbb{P}(dq(t) = 1) = \lambda dt$, corresponding probability of no jump during dt is $\mathbb{P}\,(dq(t) = 0) = 1 - \lambda dt$. Probability of more than one jump over time interval dt is zero. The Poisson process $q(t)$ and Wiener process $W(t)$ are independent of each other ($dq(t)dW(t) = 0$).

Although this jump diffusion model solves the problem of fat tails in the probability distribution of returns, spikes of spot prices are not really included in the model. Spike is just upward jump which dissipate over time at the price converges back to its mean. Jump diffusion models use jumps to create spikes and high mean reversion rates to force the price to return to lower level price. Empirically, when the spot price does not jump we can also see mean-reversion, but not of the same rate. Thus we need to filter out jumps before estimating mean-reversion rate, otherwise the mean-reversion rate will be to high to adequately characterize the times without jump. One way to solve the problem is to consider different regimes with different parameters to capture the different behavior of the prices in each regime.

4.6 Stochastic volatility models

We can incorporate an additional source of randomness without using a jump–diffusion model by making volatility stochastic. The general form for these models can be expressed as

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t) dW^1(t), \\
d\sigma(t) &= a(t, S(t), \sigma(t))dt + b(t, S(t), \sigma(t))dW^2(t) \qquad \text{and} \\
dW^1(t)dW^2(t) &= \rho dt.
\end{aligned}$$

Considering different functions $a(t, S(t), \sigma(t))$ and $b(t, S(t), \sigma(t))$ we obtain models with different volatility structures. We show a few examples of the stochastic volatility models below.

Hull and White model

Hull and White [23] specify the square of volatility following the exponential Brownian motion.

$$\begin{aligned}
\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t) dW^1(t), \\
d\sigma^2(t) &= \kappa \sigma^2(t) dt + \gamma \sigma^2(t) dW^2(t) \qquad \text{with} \\
dW^1(t)dW^2(t) &= \rho dt.
\end{aligned}$$

Heston model

Heston [20] considers mean-reverting square root volatility process:

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t)dW^1(t), \\ d\sigma^2(t) &= \kappa(\theta - \sigma^2(t))dt + \gamma\sigma(t)dW^2(t) \qquad \text{with} \\ dW^1(t)dW^2(t) &= \rho dt.\end{aligned}$$

Stein and Stein model

The volatility in Stein and Stein [31] follows an Ornstein-Uhlenbeck process

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu dt + \sigma(t)dW^1(t), \\ d\sigma(t) &= \kappa(\theta - \sigma(t))dt + \gamma\sigma(t)dW^2(t) \qquad \text{with} \\ dW^1(t)dW^2(t) &= 0.\end{aligned}$$

4.7 Affine jump diffusion models

Both jump diffusion models and stochastic volatility models belong the broader class of Affine Jump Diffusions (AJD) models, proposed by Duffie, Pan and Singleton [11].

They consider a multidimensional state vector $X(t)$ (say a n-dimensional real-valued vector) as affine jump diffusion:

$$dX(t) = \mu(X(t), t) + \sigma(X(t), t)dW(t) + dQ(t)$$

where $W(t)$ is n-dimensional standard Brownian motion and Q is a jump process with jumps with distribution ν on $\mathbb{R}^n$ with intensity vector $\lambda(X)$.

All of the functions (the drift vector $\mu(X, t)$, the covariance matrix $\Sigma(X, t)) = \sigma(X, t)(\sigma(X, t))^T$, the intensities $\lambda(X, t)$ and the interest rates $R(Xt))$ have *affine* (linear) dependence on the state vector X:

$$\begin{aligned}\mu(X(t), t) &= K_0(t) + K_1(t)X(t), && \text{for } K(t) = (K_0(t), K_1(t)) \in \mathbb{R}^n \times \mathbb{R}^{n\times n}, \\ \Sigma(X(t), t)_{ij} &= (H_0(t))_{ij} + (H_1(t))_{ij} X(t), && \text{for } H(t) = (H_0(t), H_1(t)) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n\times n}, \\ \lambda(X(t), t) &= l_0(t) + l_1(t)X(t), && \text{for } l(t) = (l_0(t), l_1(t)) \in \mathbb{R} \times \mathbb{R}^n, \\ R(X(t), t) &= \rho_0(t) + \rho_1(t)X(t), && \text{for } \rho(t) = (\rho_0(t), \rho_1(t)) \in \mathbb{R} \times \mathbb{R}^n.\end{aligned}$$

In this context, stochastic volatility model can be described by two-dimensional state vector $X(t) = (S(t), \sigma(S, t))$ with $\lambda = 0$ and appropriate σ, μ and ρ. Jump diffusion model of Merton can be described by one dimensional state vector $X(t) = S(t)$ and non-zero constants λ, μ, σ and ρ.

This model is very general and allows to capture dynamics of very different path-types. However, as pointed for example by Anderson [1], one of the disadvantages of these models is difficulties for the risk-neutral pricing of the derivatives such as futures and options. Risk-neutral pricing is based on existence of a hedging strategy which can replicate the dynamics of the derivative under consideration and thus the risk-neutral price of this derivative should be the same as the cost of the hedge strategy. The addition of the jumps leads to the problems in finding hedging strategies, since they make markets incomplete. One has to deal with the problem in one of two ways. The first way is to assume that jumps are incorporated with market factors. In this case, jumps are considered to be a non-systematic risk and can be ignored (the asset holder is not rewarded for bearing this risk). In this way only mean-reversion, seasonality and volatility are taken into account during hedging.

Another way to deal with the pricing of derivatives problem is to suggest that there exist enough traded instruments to hedge the jumps. As we have independent jumps of random sizes this suggests that there are infinite number of instruments for hedging these jumps.

4.8 Regime switching models

Alternative to AJD way to incorporate spikes into model is the so-called *regime-switching model*. In these models few price regimes are usually considered and spikes and mean-reversion of the spot prices are considered to belong to different regimes.

Huisman and Mahieu [21] presented a regime-switching model where the spot price is a sum of a deterministic component f and a stochastic component x:

$$S(t) = f(t) + x(t), \quad \text{where } t = 1, \ldots, T.$$

The stochastic component follows three regimes: a normal regime (regime 0), when prices follow "normal" electricity price dynamics, a jump regime (regime +1) that models the process when price of electricity exhibits spikes, and a reverting regime (regime −1) that describes how the electricity price reverts back to the normal regime:

$$\begin{aligned}
dx(t) &= -\alpha_0 x(t-1) + \sigma_0 \varepsilon(t), && \text{in regime } 0 \\
dx(t) &= \mu_1 + \sigma_1 \varepsilon(t), && \text{in regime } +1 \\
dx(t) &= -\alpha_{-1} x(t-1) + \sigma_{-1} \varepsilon(t) && \text{in regime } -1
\end{aligned}$$

with $\varepsilon(t) \sim N(0,1)$. The mechanism to describe how to move from one regime to another is represented by Markov transition matrix π:

$$\pi = (\pi_{i,j}) = \begin{array}{cccc} & \text{regime } 0 & \text{regime } +1 & \text{regime } -1 \\ \text{regime } 0 & \pi_{00} & 0 & 1 \\ \text{regime } +1 & \pi_{+10} = 1 - \pi_{00} & 0 & 0 \\ \text{regime } -1 & 0 & 1 & 0 \end{array}$$

Here $\pi_{i,j}$ is the probability that the electricity prices switches from regime j to regime in period t to regime i in period $t+1$. Note that in this model the spike and reverting regimes last only for one period (one day if daily data are used) ($\pi_{-1,+1} = \pi_{0,-1} = 1$ and $\pi_{+1,0} = \pi_{0,0} = \pi_{+1,-1} = \pi_{-1,-1} = 0$).

Other possibilities and regimes are possible. For example Ethier and Mount [13] consider regime switching model of Hamilton [19] with two regimes:

$$y(t) - \mu_{s(t)} = \phi\left(y(t-1) - \mu_{s(t-1)}\right) + \varepsilon(t)$$

with $s(t) = 1, 2$ describing the regime, $\mu_{s(t)}$ being mean value in regime $s(t)$ and $\varepsilon(t) \sim N(0, \sigma_{s(t)})$ with volatility $\sigma_{s(t)}$ also depending on regime $s(t)$. In this case P is a 2×2 transition matrix of probabilities to jump from regime to regime:

$$P = \begin{bmatrix} p_{11} & 1 - p_{22} \\ 1 - p_{11} & p_{22} \end{bmatrix} = \begin{bmatrix} p & 1 - q \\ 1 - p & q \end{bmatrix}.$$

Other authors (see for example [18], [24]) consider different regimes and processes to describe dynamics of the price in each regime.

4.9 Other models

4.9.1 Villaplana model with short-term and long-term sources of risk and jumps

Villaplana [34] extended the Schwartz and Smith model by including the jump with a non-constant intensity in the short-term factor. A seasonality factor is also included. Logarithm of the spot prices is expressed as some of two factors:

$$\begin{aligned} \ln S(t) &= f(t) + X(x) + Y(t) \\ dX(t) &= -\kappa_1 X(t)dt + \sigma_1 dW^1(t) + J(t)dq(t) \\ dY(t) &= \kappa_2\left(\beta - Y(t)\right)dt + \sigma_2 dW^2(t) \\ dW^1(t)\, dW^2(t) &= \rho dt, \end{aligned}$$

where $f(t)$ is deterministic function, $X(t)$ is a short-term factor and $Y(t)$ is a long-term factor, which could be either mean-reverting or a generalized Brownian motion.

4.9.2 Lévy process for spot pricing: CGMY model

A more general way to incorporate jumps into the model is to use a Lévy process. Carr, Geman, Madan and Yor [8] presented CGMY model in which pure jump Lévy process is proposed for modeling spot price. The stock price modeled by upward and downward jumps where Lévy density $k_{CGMY}(x)$ represents the probability of occurring the jumps of size x in a unit time interval:

$$k_{CGMY}(x) = \begin{cases} C\frac{e^{-G||x|}}{|x|^{1+Y}} & \text{for } x < 0 \\ C\frac{e^{-M||x|}}{|x|^{1+Y}} & \text{for } x > 0 \end{cases}$$

where $C > 0,\ G \geq 0,\ M \geq 0$ and $Y < 2$.

Though attractive for electricity spot pricing, as pointed by Geman, this model cannot capture dependencies in the increments of the electricity prices and stochastic volatility should be incorporated in the Lévy process.

4.9.3 Hyperbolic distribution of the spot prices

Of course one natural way to fit distribution of spot returns is to consider different from (log)-normal distribution for the price process. For example in Eberlein and Stahl [12] hyperbolic distribution was use in order to capture fat tails we observed in the data analysis.

Chapter 5

Factor model for futures pricing

5.1 Motivation

In this section we argue in favor of the Schwartz and Smith model for electricity futures pricing and give motivation for the modification of this model. One of the main requirements for a good model is to make sure that the model can capture all important characteristics of the process under consideration and does not include unnecessary complications which could make the model unusable in practice.

In order to choose an appropriate model we need to decide which process we actually would like to model or which products we want to price. We find it difficult to find one model that can capture appropriately both spot and futures prices, due to the big difference in the process characteristics, although it is definitely the ultimate goal of all model-builders to try to describe the whole market with just one model. Our choice was to derive an appropriate model for the pricing of futures and options on electricity and not for spot pricing. That is why we consider the jump diffusion components, which are added for example in Villaplana model, to be unnecessarily complicated. Spikes observed on the spot market are not visible on the futures market and can be viewed as high volatility of the market without imposing any extra factor. On the other hand we believe that futures prices in the near future (say a month) and in the long term differ significantly. That is why we turned our attention to the two-factor Schwartz and Smith model.

Another important characteristic of the model we would like to have is its ability to derive is a closed or semi-closed form solution for futures and European-style derivatives. To express future prices properly we need to derive the theoretical price of a futures contract and take into account the fact that futures have delivery over a period $[T_0, T]$ rather than at a specific point in time T. This is a drawback of the Schwartz and Smith model. To our knowledge existing models for the electricity futures do not distinguish futures price before starting of delivery period and during delivery period, considering maturity of the futures to be a single point in time, though prices of the futures during delivery period uses average of the spot price and **not** the spot price in specific time point as underlying. We consider the modification to Schwartz and Smith model which takes into account changes in futures price during delivery period to be our main contribution to existing model.

In this chapter we use all the notations and basic arguments of the Schwartz and Smith model two-factor model from [29].

5.2 Model setup

Before being able to price futures and options on electricity we need to present here the model for spot prices.

We consider a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. As usual, $\mathcal{F}_t$ represents an information available at time t and the filtration $\mathbb{F}$ represents information flow evolving with time. Thus we consider filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$ which satisfies 'usual conditions' (See for example [10]). From now on for any stochastic process $X(t)_{t=0}^{t=\infty}$ we use notation $\mathbb{E}_t(X(T))$ for expectation conditional on filtration $\mathcal{F}_t$, exactly $\mathbb{E}(X(T)|\mathcal{F}_t)$.

Let us denote by $S(t)$ the spot price of electricity at time t. As in ([29]) we decompose spot prices into two stochastic factors as

$$\ln(S(t)) = \chi(t) + \xi(t) + h(t), \tag{5.1}$$

where $\chi(t)$ will be referred to as the short-term deviation in price, $\xi(t)$ is the equilibrium price level and $h(t)$ is a deterministic seasonality function.

The short-term deviations $\chi(t)$ are assumed to revert toward zero following an Ornstein-Uhlenbeck process:

$$d\chi(t) = -\kappa\chi(t)\,dt + \sigma_\chi\,dW_\chi(t), \tag{5.2}$$

and the equilibrium level $\xi(t)$ is assumed to follow a Brownian motion process:

$$d\xi(t) = \mu_\xi\,dt + \sigma_\xi\,dW_\xi(t).$$

Here W_χ and W_ξ are correlated standard Brownian motion processes under the real-world measure $\mathbb{P}$ with $dW_\chi(t)dW_\xi(t) = \rho_{\chi\xi}dt$. Spot price process is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ process.

We can write the analytical forms for the distributions of the state variables $\chi(t)$ and $\xi(t)$ as follows. Given $\chi(t)$ and $\xi(t)$ and using [29] we find that $\chi(T)$ and $\xi(T)$ are jointly normally distributed with conditional mean vector $\mathbb{E}_t\left[(\chi(T), \xi(T))\right]$ and covariance matrix $\mathbb{C}\mathrm{ov}_t\left[(\chi(T),\xi(T))\right]$:

$$\mathbb{E}_t\left[(\chi(T),\xi(T))\right] = \left[e^{-\kappa(T-t)}\chi(t), \xi(t)+\mu_\xi(T-t)\right], \qquad \text{and} \tag{5.3}$$

$$\mathbb{C}\mathrm{ov}_t\left[(\chi(T),\xi(T))\right] = \begin{bmatrix} \left(1-e^{-2\kappa(T-t)}\right)\dfrac{\sigma_\chi^2}{2\kappa} & \left(1-e^{-\kappa(T-t)}\right)\dfrac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} \\ \left(1-e^{-\kappa(T-t)}\right)\dfrac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa} & \sigma_\xi^2(T-t) \end{bmatrix} \tag{5.4}$$

Given $\chi(t)$ and $\xi(t)$, the logarithm of the future spot price at time T is normally distributed with conditional mean and variance as

$$\mathbb{E}_t\left[\ln S(T)\right] = \mathbb{E}_t\left[\chi(t)+\xi(T)\right] = e^{-\kappa(T-t)}\chi(t)+\xi(t)+\mu_\xi(T-t)\ +h(T) \tag{5.5}$$

$$\mathbb{V}\mathrm{ar}_t\left[\ln S(T)\right] = \left(1-e^{-2\kappa(T-t)}\right)\frac{\sigma_\chi^2}{2\kappa}+\sigma_\xi^2(T-t)+2\left(1-e^{-\kappa(T-t)}\right)\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}. \tag{5.6}$$

5.3 Geometric average versus arithmetic average

The futures contracts on electricity are based on the arithmetic averages of the spot prices over a delivery period. Let T_0 be the day before the first day of the delivery period for monthly futures and T be the last day of delivery. For example, $T-T_0$ could be a month, a quarter or a year. Now for $i=1,\ldots,n$ consider n averaging points $t_i=T_0+i\cdot\Delta t$ with $t_0=T_0$, $t_n=T$ and $\Delta t=(T-T_0)/n$.

In the market prices of electricity futures are based on arithmetic average $\frac{1}{n}\sum\limits_{i=1}^{n}S(t_i)$. Unfortunately the sum of lognormal random variables is not lognormal and we cannot derive a closed-form solution for the futures based on the arithmetic average. However, it is known, that sum of lognormal distributed variables is *approximately* lognormal. We can use the geometric average as an approximation for an arithmetic average.

Geometric average of n positive values is always smaller than or equal to the arithmetic average of these n values. The difference between these two values decreases with number of averaging points.

We will use the geometric average of n prices to estimate the arithmetic average over the time period of n days. So t_1 is a first day of delivery and $t_n=T$ is the last day of delivery. First suppose that $t<t_1$ and let G_n to be the geometric average of n

prices at times $t_1, t_2, \dots, t_n$[1].

$$G_n = \left[\prod_{i=1}^{n} S(t_i)\right]^{1/n}. \tag{5.7}$$

Note that every $S(t_i)/S(t_{i-1})$ is (conditionally) lognormally distributed and thus G_n is also (conditionally) lognormally distributed.

5.4 Spot process under the risk-neutral measure

In order to use the risk-neutral valuation of future price we need to express the model under the risk-neutral measure $\mathbb{Q}$. We assume that the risk-neutral stochastic process for short-run deviations $(\chi(t))$ and equilibrium levels $(\xi(t))$ are of the form

$$\begin{aligned} d\chi(t) &= (-\kappa\chi(t) - \lambda_\chi)\, dt + \sigma_\chi\, dW^*_\chi(t) \qquad \text{and} && (5.8)\\ d\xi(t) &= \left(\mu_\xi - \lambda_\xi\right)\, dt + \sigma_\xi dW^*_\xi(t), && (5.9)\end{aligned}$$

where W^*_χ and W^*_ξ are correlated standard Brownian motion processes under the risk-neutral measure Q with $dW^*_\chi(t)dW^*_\xi(t) = \rho_{\chi\xi}dt$.

Now the risk-neutral process for the short-term deviations $(\chi(t))$ is an Ornstein-Uhlenbeck process reverting to $-\lambda_\chi/\kappa$ (instead of 0 in the real-world process) and the risk-neutral process for equilibrium prices $(\xi(t))$ is again a geometric Brownian motion, but now it has a drift $\mu^*_\xi = \mu_\xi - \lambda_\xi$.

We can also express $\chi(T)$ and $\xi(T)$ in integral form

$$\begin{aligned} \chi(T) &= -\frac{\lambda_\chi}{\kappa} + e^{-\kappa(T-t)}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) + \sigma_\chi \int_t^T e^{-\kappa(T-u)} dW^*_\chi(u), && (5.10)\\ \xi(T) &= \xi(t) + \mu^*_\xi\,(T-t) + \sigma_\xi\left(W^*_\xi(T) - W^*_\xi(t)\right). && (5.11)\end{aligned}$$

Given $\chi(t)$ and $\xi(t)$ and following derivations similar to those for equations (5.3) and (5.4) we find that $\chi(T)$ and $\xi(T)$ are jointly normally distributed under the risk-neutral measure Q with conditional mean vector and covariance

$$\begin{aligned} \mathbb{E}^*_t\left[(\chi(T), \xi(T))\right] &= \left[-\frac{\lambda_\chi}{\kappa} + e^{-\kappa(T-t)}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right), \xi(t) + \mu^*_\xi(T-t)\right], && (5.12)\\ \mathbb{C}\mathrm{ov}^*_t\left[(\chi(T), \xi(T))\right] &= \mathbb{C}\mathrm{ov}_t\left[(\chi(T), \xi(T))\right]. && (5.13)\end{aligned}$$

Here (and below) we use asterisks to denote expectations, covariances and variances under the risk-neutral measure Q. Under the risk-neutral process the logarithm of

[1] Here $ln(G_n) = \frac{1}{n}\sum_{i=1}^{n}\ln(S(t_i)) \le \ln(\frac{1}{n}\sum_{i=1}^{n} S(t_i))$ and thus $G_n \le \frac{1}{n}\sum_{i=1}^{n} S(t_i)$.

future spot price $\ln S(T)$ conditioned on time t is normally distributed with

$$\mathbb{E}_t^*\left[\ln S(T)\right] = -\frac{\lambda_\chi}{\kappa} + e^{-\kappa(T-t)}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) + \xi(t) + \mu_\xi^*(T-t) + h(T) \quad (5.14)$$

and

$$\mathbb{V}\mathrm{ar}_t^*\left[\ln S(T)\right] = \mathbb{V}\mathrm{ar}_t\left[\ln S(T)\right]. \quad (5.15)$$

Comparing equations (4) and (9), we see that the risk premiums reduce the logarithm of the expected spot prices by $\left(\lambda_\chi\left(1-e^{-\kappa(T-t)}\right)/\kappa + \lambda_\xi(T-t)\right)$ and this premium depends on the time to maturity, but not on the state variables. The premium $\lambda_\chi\left(1-e^{-\kappa(T-t)}\right)/\kappa$ comes from mean-reverting process $\chi(t)$ and the premium $\lambda_\xi(T-t)$ comes from geometric Brownian motion $\xi(t)$.

5.5 Calculation of future prices

Now, to value the future contract on the average of the spot prices we need to find distributions of G_n under risk-neutral measure Q. Define $A_n = \ln G_n$; A_n is conditionally normally distributed.

Lets $F(t, T_0, T)$ be the futures price of electricity at time t with delivery period $(T_0, T]$. First we will consider the case where the future prices is calculated before the delivery period has started, thus we consider the case where $t < T_0$.

We can derive that the conditional mean and the conditional variance of A_n before the delivery period are given below. Derivations figure in the Appendix.

$$\mathbb{E}_t^*\left[A_n\right] = \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] - \frac{\lambda_\chi}{\kappa} + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T-t)}\varphi(T_0, T, n)$$
$$+\xi(t) + \left(T - t - \frac{(n-1)}{2}\Delta t\right)\mu_\xi^* =: m_A(t, T_0, T, n), \quad (5.16)$$

and

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}_t^*\left[A_n\right] = \frac{\sigma_\chi^2}{2\kappa}\left(e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}\right)\left(\varphi(T_0,T,n)\right)^2 \\
+\frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(\frac{1}{n} - \frac{2e^{-\kappa(T-T_0)}}{n}\varphi(T_0,T,n) + \frac{1}{n^2}\frac{1-e^{-2\kappa n\Delta t}}{e^{2\kappa\Delta t}-1}\right) \\
+\frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\left(e^{-\kappa(T-T_0)}e^{\kappa(T-T_0)/n} - e^{-\kappa(T-t)}\right)\varphi(T_0,T,n) + \frac{1}{n^2}\sum_{i=1}^{n-1} i\left(1-e^{-i\kappa\Delta t}\right)\right] \\
+\frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t) + ((n-1)^2 + (n-2)^2 + \cdots + 1)\Delta t\right] \\
=: \sigma_A^2(t,T_0,T,n), \qquad (5.17)
\end{aligned}
$$

where

$$
\varphi(T_0,T,n) = \left(\frac{e^{\kappa(T-T_0)}-1}{n\left(e^{\kappa\Delta t}-1\right)}\right).
$$

Now, G_n is lognormally distributed, thus

$$
\begin{aligned}
\mathbb{E}_t^*\left[G_n\right] &= \exp\left\{\mathbb{E}_t^*\left[\ln G_n\right] + \frac{1}{2}\mathbb{V}ar_t^*\left[\ln G_n\right]\right\} = \exp\left\{\mathbb{E}_t^*\left[A_n\right] + \frac{1}{2}\mathbb{V}ar_t^*\left[A_n\right]\right\} \\
&= \exp\left\{m_A(t,T_0,T,n) + \frac{1}{2}\sigma_A^2(t,T_0,T,n)\right\}. \qquad (5.18)
\end{aligned}
$$

Under the risk-neutral measure

$$
\begin{aligned}
F(t,T_0,T) &= \mathbb{E}_t^*\left[G_n\right] = \exp\left\{m_A(t,T_0,T,n) + \frac{1}{2}\sigma_A^2(t,T_0,T,n)\right\} \\
&= \exp\left(\chi(t)e^{-\kappa(T-t)}\varphi(T_0,T,n) + \xi(t) + B(t,T_0,T,n)\right), \qquad (5.19)
\end{aligned}
$$

where $m_A(t,T_0,T,n)$ and $\sigma_A^2(t,T_0,T,n)$ are defined in (5.16) and (5.17) and

$$
\begin{aligned}
B(t,T_0,T,n) &= \frac{\lambda_\chi}{\kappa}\left(e^{-\kappa(T-t)}\varphi(T_0,T,n) - 1\right) + \left(T - t - \frac{(n-1)}{2}\Delta t\right)\mu_\xi^* \\
&\quad + \frac{1}{2}\sigma_A^2(t,T_0,T,n) + \frac{1}{n}\sum_{i=1}^{n} h(t_i). \qquad (5.20)
\end{aligned}
$$

Now consider the case where we would like to price a futures contract during delivery period, i.e., $T_0 < t \le T$. Let i^* to be such that $t_{i^*-1} < t \le t_{i^*}$. Then

$$\begin{aligned}
\mathbb{E}_t^*\left[A_n\right] &= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T-t)} \varphi^*(T_0, T, n) \\
&\quad + \frac{n-i^*+1}{n}\left(\xi(t) - \frac{\lambda_\chi}{\kappa} + \left[T - t - \frac{n-i^*}{2}\Delta t\right]\mu_\xi^*\right) \\
&= : \bar{m}_A(t, T_0, T, n), \qquad (5.21)
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t^*\left[A_n\right] &= \frac{\sigma_\chi^2}{2\kappa}\left[\left(e^{-2\kappa(n-i^*+1)\Delta t} - e^{-2\kappa(T-t)}\right)\varphi^*(T_0, T, n)\right. \\
&\quad \left. + \frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(\frac{n-i^*+1}{n^2} - 2\frac{e^{-\kappa(n-i^*+1)\Delta t}}{n}\varphi^*(T_0, T, n)\right)\right] \\
&\quad + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\frac{n-i^*+1}{n}\varphi^*(T_0, T, n)\left(e^{-\kappa(n-i^*)\Delta t} - e^{-\kappa(T-t)}\right) + \frac{1}{n^2}\sum_{i=1}^{n-i^*} i(1-e^{-\kappa i\Delta t})\right] \\
&\quad + \frac{\sigma_\xi^2}{n^2}\left[(n-i^*+1)^2(t_{i^*}-t) + ((n-i^*)^2 + (n-i^*-1)^2 + \cdots + 1)\Delta t\right] \\
&= : \bar{\sigma}_A^2(t, T_0, T, n), \qquad (5.22)
\end{aligned}$$

where

$$\varphi^*(T_0, T, n) = \left(\frac{e^{\kappa(n-i^*-1)\Delta t} - 1}{n\left(e^{\kappa\Delta t} - 1\right)}\right).$$

And the price of the futures contract in this case is thus

$$\begin{aligned}
F(t, T_0, T) &= \mathbb{E}_t^*\left[G_n\right] = \exp\left\{\bar{m}_A(t, T_0, T, n) + \frac{1}{2}\bar{\sigma}_A^2(t, T_0, T, n)\right\}, \\
&= \exp\left(\chi(t)e^{-\kappa(T-t)}\varphi^*(T_0, T, n) + \frac{n-i^*+1}{n}\xi(t) + B^*(T_0, T, n)\right) \qquad (5.23)
\end{aligned}$$

where $\bar{m}_A(t, T_0, T, n)$ and $\bar{\sigma}_A^2(t, T_0, T, n)$ are as in (5.21) and (5.22) and

$$
\begin{aligned}
B^*(t,T_0,T,n) \;&=\; \frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\cdots+\ln S(t_{i^*-1})+h(t_{i^*})+\cdots+h(t_n)\right)\\
&+\frac{\lambda_\chi}{\kappa}\left(e^{-\kappa(T-t)}\varphi^*(T_0,T,n)-\frac{n-i^*+1}{n}\right)\\
&+\frac{n-i^*+1}{n}\left(T-t-\frac{(n-1)}{2}\Delta t\right)\mu_\xi^*+\frac{1}{2}\bar{\sigma}_A^2(t,T_0,T,n). \qquad (5.24)
\end{aligned}
$$

5.6 Risk premium

The *risk premium* (or *term premium*) is the amount which the buyer or seller of the contract (future) is ready to pay in order to avoid risk of price fluctuations. Thus usually the risk premium is defined by

$$\pi(t,T)=F(t,T)-\mathbb{E}_t(S(T))=\mathbb{E}_t^*(S(T))-\mathbb{E}_t(S(T)).$$

The second equality is valid because the price of the future is calculated as the *risk-neutral* expectation of the future spot price. We would like to note here that in spite of definition, the risk premium could be negative in case when the future price is higher than expected spot price. A buyer would be willing to pay more to lock in the prices, but a seller might also be willing to receive less that the expected price at expiry to secure a fixed selling price today. The sign of the term premium depends on the intersection between supply and demand. Because we do not have futures with the delivery point, but futures with delivery period, we define risk premium in our model as

$$\pi(t,T)=F(t,T)-\mathbb{E}_t(G_n), \qquad (5.25)$$

where

$$G_n=\left[\prod_{i=1}^{n}S(t_i)\right]^{1/n},$$

and which is used as an underlying for the future contracts.

Now for convenience of calculations, we define the term premium coefficient R as

$$R=\log\left(\frac{F(t,T_0,T)}{\mathbb{E}_t(G_t)}\right)=\log\left(\frac{\mathbb{E}_t^*\left[G_n\right]}{\mathbb{E}_t\left[G_n\right]}\right).$$

To calculate this we need to find distribution of A_n under the real-world measure $\mathbb{P}$. Using the derivation from the Appendix B we get:

$$
\begin{aligned}
\mathbb{E}_t\left[A_n\right] &= e^{-\kappa(T-t)}\varphi(T_0,T,n)\,\chi(t)+\xi(t)+\left(T-t-\frac{n-1}{2}\Delta t\right)\mu_\xi \\
&+\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \qquad (5.26)
\end{aligned}
$$

$$
\mathbb{V}\mathrm{ar}_t(A_n) = \mathbb{V}\mathrm{ar}^*_t(A_n)=\sigma^2(t,T_0,T,n). \qquad (5.27)
$$

Now $\mathbb{E}_t\left[G_n\right]$ can be calculated as

$$
\mathbb{E}_t\left[G_n\right]=\exp\left\{\mathbb{E}_t\left[A_n\right]+\frac{1}{2}\sigma_A^2(t,T_0,T,n)\right\}, \qquad (5.28)
$$

and thus the term premium coefficient can be expressed as

$$
\begin{aligned}
R &= \log\left(\frac{\mathbb{E}^*_t\left[G_n\right]}{\mathbb{E}_t\left[G_n\right]}\right)= \\
&= -\frac{\lambda_\chi}{\kappa}\left(1-e^{-\kappa(T-t)}\varphi(T_0,T,n)\right)-\lambda_\xi\left(T-t-\frac{(n-1)}{2}\Delta t\right). \qquad (5.29)
\end{aligned}
$$

For the case where $T_0 < t \leq T$ we get

$$
\begin{aligned}
\mathbb{E}_t\left[A_n\right] &= \frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\cdots+\ln S(t_{i^*-1})+h(t_{i^*})+\cdots+h(t_n)\right) \\
&+\frac{1}{n}e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\chi(t) \\
&+\frac{n-i^*+1}{n}\left(\xi(t)+\mu_\xi\left[(T_0-t)+\frac{(i^*+n)}{2}\right]\Delta t\right) \qquad (5.30)
\end{aligned}
$$

$$
\mathbb{V}ar_t(A_n) = \mathbb{V}ar^*_t(A_n)=\bar{\sigma}_A^2(t,T_0,T,n), \qquad (5.31)
$$

and thus term premium is:

$$
\begin{aligned}
R &= \log\left(\frac{\mathbb{E}^*_t\left[G_n\right]}{\mathbb{E}_t\left[G_n\right]}\right)= \\
&= -\frac{\lambda_\chi}{\kappa}\left(\frac{n-i^*+1}{n}-\frac{1}{n}e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right) \\
&\quad -\lambda_\xi\frac{n-i^*+1}{n}\left[(T_0-t)+\frac{(i^*+n)}{2}\Delta t\right]. \qquad (5.32)
\end{aligned}
$$

Note that if $n = 1$ there is no averaging over period $(T_0, T]$ and thus for $\forall t \in [0, T]$

$$R(T-t) = -\lambda_\chi \left(\frac{1 - e^{-\kappa(T-t)}}{\kappa} \right) - \lambda_\xi (T-t),$$

which the risk premium coefficient for the original Schwartz and Smith model [29]. As we can see risk premium consist of two time-factors. The first "short-term" factor $-\lambda_\chi \left(1 - e^{-\kappa(T-t)}\right)/k$ influences the risk premium curvature (few month to delivery) and converge to the limit value $-\lambda_\chi/\kappa$ as time to maturity $T-t$ tends to infinity. Value $-\lambda_\chi/\kappa$ can be both positive and negative, depending on the sign of the parameter λ_χ. The second factor $-\lambda_\xi (T-t)$ is decreasing linearly, because, as we will see in the Chapter 7, the long-term parameter λ_ξ is positive. This factor influences the risk premium more in the long run. We will see how different parameters change the shape of the term premium function in Chapter 7.

5.7 The option formula

Now using risk-neutral valuation we can derive the analytic formulae for pricing the European options on futures contracts. The value of a European option on a future is given by calculating the expected future value of the option using the risk-neutral measure and by discounting by risk-free rate r. By (5.19)

$$\ln(F(t, T_0, T)) = \chi(t) e^{-\kappa(T-t)} \varphi(T_0, T, n) + \xi(t) + B(t, T_0, T, n),$$

thus $\ln(F(t, T_0, T))$ is normally distributed with mean

$$\begin{aligned} m_F(t, T) &= \mathbb{E}_t^* \left[\chi(t) e^{-\kappa(T-t)} \varphi(T_0, T, n) + \xi(t) + B(t, T_0, T, n) \right] \\ &= \mathbb{E}_t^* \left[\chi(t) e^{-\kappa(T-t)} \varphi(T_0, T, n) + \xi(t) \right] + B(t, T_0, T, n) \\ &= \chi_0 e^{-\kappa T} \varphi(T_0, T, n) + \xi_0 + \mu_\xi^* t + B(t, T_0, T, n) \end{aligned}$$

and volatility

$$\begin{aligned} \sigma_F^2(t, T) &= \mathbb{V}\mathrm{ar}_t^* \left[\ln(F(t, T_0, T)) \right] = \mathbb{V}\mathrm{ar}_t^* \left[\chi(t) e^{-\kappa(T-t)} \varphi(T_0, T, n) + \xi(t) + B(t, T_0, T, n) \right] \\ &= e^{-2\kappa(T-t)} \varphi^2(T_0, T, n) \mathbb{V}\mathrm{ar}_t^* \left(\chi(t) \right) + \mathbb{V}\mathrm{ar}_t^* \left(\xi(t) \right) \end{aligned}$$

$$\begin{aligned} &+2e^{-2\kappa(T-t)} \varphi^2(T_0, T, n) \mathbb{C}\mathrm{ov}_t(\chi(t), \xi(t)) \\ &= e^{-2\kappa(T-t)} \varphi^2(T_0, T, n) \left(1 - e^{-2\kappa t}\right) \frac{\sigma_\chi^2}{2\kappa} + \sigma_\xi^2 t \\ &+ e^{-\kappa(T-t)} \varphi(T_0, T, n) \left(1 - e^{-\kappa t}\right) \frac{2\rho_{\chi\xi} \sigma_\chi \sigma_\xi}{\kappa}. \end{aligned} \tag{5.33}$$

It is clear that $\mathbb{E}_t^*\left[F(T_e,T_0,T)\right] = F(t,T_0,T)$. Knowing the fact that future futures prices are lognormally distributed we can use the Black-Scholes formula for calculating options using $e^{-r(T_e-t)}F(t,T_0,T)$ instead of usual stock price at time t.

Consider an European call option at time t on a futures contract with delivery between T_0 and T (and n averaging points) with strike price K and with time T_e as the option's expiry time.

The theoretical price of such option $C(t,T_e,F(t,T_0,T))$ is equal to the discounted (by risk-free rate) expected value of payoff function $\max(F(T_e,T_0,T)-K,0)$:

$$\begin{aligned} C(t,T_e,F(t,T_0,T)) &= e^{-r(T_e-t)}\mathbb{E}_t^*\left[\max(F(T_e,T_0,T)-K,0)\right] \\ &= e^{-r(T_e-t)}(\mathbb{E}_t^*\left[F(T_e,T_0,T)\right]N(d_1)-KN(d_2)) \\ &= e^{-r(T_e-t)}(F(t,T_0,T)N(d_1)-KN(d_2)), \end{aligned} \tag{5.34}$$

where

$$\begin{aligned} d_1 &= \frac{\ln(F(t,T_0,T)/K)}{\sigma_F(T_e,T)} + \frac{1}{2}\sigma_F(T_e,T) \\ d_2 &= \frac{\ln(F(t,T_0,T)/K)}{\sigma_F(T_e,T)} - \frac{1}{2}\sigma_F(T_e,T) \end{aligned}$$

and N denotes cumulative probability function for standard normal distribution.

We used here the Black-Scholes formula for valuing European call option with $e^{-r(T_e-t)}F(t,T_0,T)$ instead of usual stock price at present time (usually $S(t)$) and annualized volatility $\sigma_F(T_0,T)/\sqrt{t}$ instead of usual annualized volatility σ for stocks at time T_0 of option expiry.

Analogously, the price of corresponding put option $P(t,T_e,F(t,T_0,T))$ is equal to

$$\begin{aligned} P(t,T_e,F(t,T_0,T)) &= e^{-rt}\mathbb{E}_t^*\left[\max(K-F(T_e,T_0,T),0)\right] \\ &= e^{-r(T_e-t)}(-F(t,T_0,T)N(-d_1)+KN(-d_2)). \end{aligned} \tag{5.35}$$

Using the formulae for Call and Put options presented above we can calculate the prices of all options available in the market. Using formula 5.33 we can also calculate the implied volatility for the options. We will present the graphs of the options prices, calculated by formulae 5.34 and 5.35 using estimated parameters, and market options prices in the Chapter 7.

Chapter 6

Implementation

6.1 Estimation of the model parameters

In the two-factor model described above, short-term and long-term state variables $\xi(t)$ and $\chi(t)$ are not directly observed in the market. We observe only spot, futures and option prices. Options on electricity futures were introduced on the EEX at the end of 2004. However, since the option market still lacks liquidity, we prefer not to use this data. Thus we have the choice to estimate model parameters from the market spot or the futures market prices or both. Estimation of the parameters from the spot prices will obviously lead to a high value of the mean-reversion parameter because of the existence of spikes in the spot price, which are not included in our model. On the other hand, we have short-term and long-term futures which we can use for estimation. Spot prices can be used as the basis for the calculation of the futures prices during the delivery period because the spot prices for the days during the delivery period that have already passed are included in the futures price.

6.2 The Kalman filter

The short-term/long-term model of Schwartz and Smith described above allows us to use standard Kalman filtering and maximum likelihood techniques to fit the model future prices to the observed data and obtain the estimates of the model parameters and space variables: the short-term deviation in price $\chi(t)$ and the equilibrium price level $\xi(t)$. The Kalman filter is a recursive procedure for computing estimates of unobserved state variables ($\chi(t)$ and $\xi(t)$) based on observations ($\ln F(t, T_0, T)$) that depend on theses state variables.

To apply the Kalman filter for estimation purposes we need first to put our model in the state space form.

6.2.1 State space form representation of the model

We discretized Equations (5.3)-(5.4) describing the behavior of short-term factor ($\chi(t)$) and long term factor (or equilibrium level) ($\xi(t)$). Thus we consider linear time invariant discrete-time stochastic dynamic system. For each time point $t_k = k\Delta t$ for $k = 1, \ldots, N_T$, where Δt is a time step, we define the two-dimensional state vector $\mathbf{x}_k$, which represents unobserved values $\chi(t_k)$ and $\xi(t_k)$ and observations $\mathbf{y}_k$ of the logs of the prices of available future prices at time t_k. From Equations (5.3)-(5.4) the evolution of the state variables is described by the *transition equation*:

$$\mathbf{x}_k = \mathbf{c} + \mathbf{T}\mathbf{x}_{k-1} + \boldsymbol{\omega}_k, \qquad k = 1, ..., n_T, \tag{6.1}$$

where

$$\mathbf{x}_k := \begin{pmatrix} \chi(t_k) \\ \xi(t_k) \end{pmatrix}, \qquad \text{a } 2 \times 1 \text{ vector of state variables;}$$

$$\mathbf{c} := \begin{pmatrix} 0 \\ \mu_\xi \Delta t \end{pmatrix}, \qquad \text{a } 2 \times 1 \text{ vector;}$$

$$\mathbf{T} := \begin{pmatrix} e^{-\kappa\Delta t} & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{a } 2 \times 2 \text{ matrix;}$$

$\boldsymbol{\omega}_k$ is a 2×1 vector of serially uncorrelated, normally distributed disturbances with $\mathbb{E}(\boldsymbol{\omega}_k) = \mathbf{0}$ and $\mathbb{V}\mathrm{ar}(\boldsymbol{\omega}_k) = \mathbf{W} := \mathbb{C}\mathrm{ov}[(\chi_{\Delta t}, \xi_{\Delta t})]$, given from equation (5.4) by

$$\mathbb{C}\mathrm{ov}\left[(\chi_{\Delta t}, \xi_{\Delta t})'\right] = \begin{pmatrix} \left(1 - e^{-2\kappa\Delta t}\right)\dfrac{\sigma_\chi^2}{2\kappa} & \left(1 - e^{-\kappa\Delta t}\right)\dfrac{\rho_{\chi\xi}\ \sigma_\chi\sigma_\xi}{2\kappa} \\ \left(1 - e^{-\kappa\Delta t}\right)\dfrac{\rho_{\chi\xi}\ \sigma_\chi\sigma_\xi}{2\kappa} & \sigma_\xi^2\Delta t \end{pmatrix};$$

Δt is the length of the time steps;

n_T is the number of time periods in the data sets.

The *observation (or measurement) equation* describes the relationship between the state variables and observed prices. From Equations (5.19)-(5.23), this is

$$\mathbf{y}_k = \mathbf{d}_k + \mathbf{F}_k\mathbf{x}_k + \mathbf{v}_k, \qquad k = 1, ..., n_T, \tag{6.2}$$

where

$$\mathbf{y}_k := \begin{pmatrix} \ln F(t_k, T_{01}, T_1) \\ \cdots \\ \ln F(t_k, T_{0m}, T_m) \end{pmatrix} \qquad \text{is a } m \times 1 \text{ vector of observed (log) futures}$$

prices with expiries $T_1, T_2, \ldots, T_m$ and $T_{0i} = T_i - n\Delta t$, where n is equal to number days during delivery period (for example $n = 30$ for the monthly futures);

$$\mathbf{d}_k := \begin{pmatrix} B(t_k, T_{01}, T_1, n) \\ \ldots \\ B(t_k, T_{0m}, T_m, n) \end{pmatrix}$$

is a $m \times 1$ vector, with $B(t_k, T_{0i}, T_i, n)$ from (5.20);

$$\mathbf{F}_k := \begin{pmatrix} e^{-\kappa(T_1 - t_k)}\varphi(T_{01}, T_1, n) & 1 \\ \ldots & \ldots \\ e^{-\kappa(T_m - t_k)}\varphi(T_{0m}, T_m, n) & 1 \end{pmatrix}$$

is a $m \times 2$ matrix; and

$\mathbf{v}_k$ is a $m \times 1$ vector of serially uncorrelated, normally distributed disturbances with

$$\mathbb{E}\left[\mathbf{v}_k\right] = \mathbf{0}, \qquad \mathbb{C}\text{ov}\left(\mathbf{v}_k\right) = \mathbf{V}.$$

These observation errors ($\mathbf{v}_k$) can be interpreted as representing errors in reporting of prices or, alternatively, as errors of the model fit to observed prices.

The specification of the state space system is completed by introducing two further assumptions:

a) the initial state space vector $\mathbf{x}_0 = (\chi_0, \xi_0)'$ has a mean of $\hat{\mathbf{x}}_0$ and a covariance matrix $\mathbf{P}_0$:

$$\begin{aligned} \mathbb{E}\left[\mathbf{x}_0\right] &= \hat{\mathbf{x}}_0, \\ \mathbb{V}\text{ar}\left[\mathbf{x}_0\right] &= \mathbf{P}_0; \end{aligned}$$

b) the disturbances ω_k and $\mathbf{v}_j$ are uncorrelated with each other in all time periods, and uncorrelated with the initial state, that is

$$\mathbb{E}\left(\boldsymbol{\omega}_k \mathbf{v}_j'\right) = 0 \qquad \text{for all } k, j = 1, \ldots n_T,$$

and

$$\mathbb{E}\left(\boldsymbol{\omega}_k \mathbf{x}_0'\right) = 0, \qquad \mathbb{E}\left(\mathbf{v}_k \mathbf{x}_0'\right) = \mathbf{0}, \qquad \text{for } k = 1, \ldots n_T.$$

The observation and state equation matrices, $\mathbf{F}_k', \mathbf{d}_k, \mathbf{V}, \mathbf{T}, \mathbf{c}, \mathbf{W}$, depend on the unknown parameters of the model. The goal is to find estimates for these parameters. This can be achieved by maximizing the likelihood function with respect to the unknown parameters. Given the distribution of the initial value of state variables as in a) and likelihood function of the observations as a function of true values we run the Kalman filter recursively.

Let denote by $\mathcal{Y}_k = \{\mathbf{y}_k, \mathbf{y}_{k-1}, \dots, \mathbf{y}_1, \mathbf{y}_0\}$ the information set at time t_k. In each subsequent period, we use the observation $\mathbf{y}_k$ and the previous period's $k-1$ mean vector and covariance matrix to calculate the mean vector and covariance matrix for current state variables. Let denote by $\hat{\mathbf{x}}_{k\,|\,l}$ the optimal estimator of $\mathbf{x}_k$ based on information $\mathcal{Y}_l$. We call $\hat{\mathbf{x}}_{k\,|\,k-1} = \mathbb{E}\left[\mathbf{x}_k \,|\, \mathcal{Y}_{k-1}\right]$ the *a priori* state estimator and $\hat{\mathbf{x}}_{k\,|\,k} = \mathbb{E}\left[\mathbf{x}_k \,|\, \mathcal{Y}_k\right]$ the *a posteriori* state estimator. Let $\mathbf{P}_{k-1\,|\,k-1}$ denote the $m \times m$ covariance matrix of the *a posteriori* estimation error, i.e.,

$$\mathbf{P}_{k-1\,|\,k-1} := \mathbb{E}\left[\left(\mathbf{x}_{k-1} - \hat{\mathbf{x}}_{k-1\,|\,k-1}\right)\left(\mathbf{x}_{t-1} - \hat{\mathbf{x}}_{k-1\,|\,k-1}\right)'\right].$$

Given $\hat{\mathbf{x}}_{k-1\,|\,k-1}$ and $\mathbf{c}$, the optimal *a priori* estimator of $\mathbf{x}_k$ is given by

$$\hat{\mathbf{x}}_{k\,|\,k-1} = \mathbf{c} + \mathbf{T}\,\hat{\mathbf{x}}_{k-1\,|\,k-1} = \mathbb{E}\left[\mathbf{x}_k \mid \mathcal{Y}_{k-1}\right], \tag{6.3}$$

while the covariance matrix $\mathbf{P}_{k\,|\,k-1}$of *a priori* estimation error is

$$\mathbf{P}_{k\,|\,k-1} = \mathbf{T}\,\mathbf{P}_{k-1\,|\,k-1}\mathbf{T}' + \mathbf{W}, \qquad k = 1, \dots, n_T. \tag{6.4}$$

These two equations are known as the *prediction equation.* $\hat{\mathbf{x}}_{k\,|\,k-1}$ and $\mathbf{P}_{k\,|\,k-1}$ are the mean and covariance of $\mathbf{x}_k$ based on information know at time t_{k-1}.

Once the new observation $\mathbf{y}_k$ become available, the estimator of $\mathbf{x}_t$ can be updated. The *updating equations* are

$$\mathbb{E}\left[\mathbf{x}_k \,|\, \mathcal{Y}_k\right] := \hat{\mathbf{x}}_{k\,|\,k} = \hat{\mathbf{x}}_{k\,|\,k-1} + \mathbf{A}_k \tilde{\mathbf{y}}_k, \tag{6.5}$$

and

$$\mathbf{P}_{k\,|\,k} = (\mathbf{I} - \mathbf{A}_k \mathbf{Q}_k)\mathbf{P}_{k\,|\,k-1}, \tag{6.6}$$

where $\tilde{\mathbf{y}}_k := \mathbf{y}_k - \mathbf{F}_k \hat{\mathbf{x}}_{k\,|\,k-1} - \mathbf{d}_k$ and $\mathbf{Q}_k := \mathbf{F}_k \mathbf{P}_{k\,|\,k-1} \mathbf{F}_k' + \mathbf{V}$ are *innovation residuals* and *innovation (or residual) covariance* correspondingly. The matrix $\mathbf{A}_k := \mathbf{P}_{k\,|\,k-1}\mathbf{F}_k'\mathbf{Q}_k^{-1}$ is called *the optimal Kalman gain.* By recursion, we can derive the estimates $\hat{\mathbf{x}}_{k\,|\,k}$ of state variables for each $k = 1, \dots, n_T$.

6.3 Maximum likelihood estimation

The Kalman filtering procedure infers the realizations of the (unobserved) state variables over time given particular parameters of the process. We denote by $\boldsymbol{\psi}$ the set of all unknown parameters of the model. The Kalman filter is also critical to the maximum likelihood estimation of the unknown parameters of the model. Maximum likelihood estimation is one of the common methods of estimation the parameters. Maximum likelihood estimator of parameters is a minimum variance estimator in the limit as the sample size increases. See [2] for more details on the maximum likelihood estimator.

The joint probability density function of information set $\mathcal{Y}_{n_T} = (\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...\mathbf{y}_{n_T})$ sampled at $k = 1, 2, \ldots, n_T$ and calculated with parameters set $\boldsymbol{\psi}$, the *likelihood function*, is given by

$$L(\mathbf{y}_{n_T}; \boldsymbol{\psi}) = \prod_{k=1}^{n_T} p\left(\mathbf{y}_k, \boldsymbol{\psi} | \mathcal{Y}_{k-1}\right), \tag{6.7}$$

where $p\left(\mathbf{y}_k, \boldsymbol{\psi} | \mathcal{Y}_{k-1}\right)$ denotes the distribution of $\mathbf{y}_k$ conditional on the information set $\mathcal{Y}_{k-1}$ available at time $k-1$ and calculated using parameters set $\boldsymbol{\psi}$. As soon as disturbances $\boldsymbol{\omega}_k$ and $\mathbf{v}_k$ and the initial state vector $\mathbf{x}_0$ have multivariate normal distributions, the distribution of $\mathbf{y}_k$ conditional on $\mathcal{Y}_{k-1}$ is itself normal. Furthermore, the mean and covariance matrix of this conditional distribution are given directly by the Kalman filter.

Remember, that state vector $\mathbf{x}_k$ conditional on $\mathcal{Y}_{k-1}$ is normally distributed with a mean $\hat{\mathbf{x}}_{k\,|\,k-1}$ and a covariance matrix of $\mathbf{P}_{k\,|\,k-1}$. If we write measurement equation as

$$\mathbf{y}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k\,|\,k-1} + \mathbf{F}_k \left(\mathbf{x}_k - \hat{\mathbf{x}}_{k\,|\,k-1}\right) + \mathbf{d}_k + \mathbf{v}_k$$

we can see that the conditional distribution of $\mathbf{y}_k$ is normal with mean

$$\mathbb{E}\left(\mathbf{y}_k, \boldsymbol{\psi} | \mathcal{Y}_{k-1}\right) = \mathbf{F}_k \hat{\mathbf{x}}_{k\,|\,k-1} + \mathbf{d}_k \tag{6.8}$$

and the covariance matrix, $\mathbf{Q}_k$, as was shown before,

$$\mathbf{Q}_k := \mathbf{F}_k \mathbf{P}_{k\,|\,k-1} \mathbf{F}_k^{'} + \mathbf{V}. \tag{6.9}$$

For the Gaussian model, therefore, the log-likelihood function is obtained by

$$\begin{aligned} \ln L(\mathbf{y}_{n_T}, \boldsymbol{\psi}) &= \sum_{k=1}^{n_T} \ln\left(p\left(\mathbf{y}_k, \boldsymbol{\psi} | \mathcal{Y}_{k-1}\right)\right) \\ &= -\frac{m\, n_T}{2} \ln 2\pi - \frac{1}{2} \sum_{k=1}^{n_T} \ln\left|\det \mathbf{Q}_k\right| - \frac{1}{2} \sum_{k=1}^{n_T} \boldsymbol{\nu}_k^{'} \mathbf{Q}_t^{-1} \boldsymbol{\nu}_k, \end{aligned} \tag{6.10}$$

where

$$\boldsymbol{\nu}_k = \mathbf{y}_k - \mathbb{E}\left(\mathbf{y}_k | \mathcal{Y}_{t-1}\right), \qquad k = 1, \ldots, n_T.$$

Vector $\boldsymbol{\nu}_k$ can be interpreted as a vector of prediction errors, because it is a minimum mean-square estimator for $\mathbf{y}_k$. Maximum likelihood function must be maximized with respect to the unknown parameters $\boldsymbol{\psi}$. There are seven model parameters (κ, μ_ξ, σ_χ, σ_ξ, $\rho_{\chi\xi}$, λ_χ, λ_ξ) plus the terms in the covariance matrix for the measurement errors ($\mathbf{V}$). There are $m(m+1)/2$ variables in the covariance matrix, where m is the number of futures contracts we use for calculations. To simplify, we assume that the matrix $\mathbf{V}$ is diagonal with diagonal elements $(s_1^2, \ldots, s_m^2)$. Thus we have to estimate $m + 7$ parameters by maximizing likelihood function from (6.10) with respect to these parameters .

It is possible to ensure that the model fits better particular contracts, to do that we can fix corresponding parameters in the matrix $\mathbf{V}$. For example, if we want to be sure that the model perfectly replicates short-term futures, we could choose the observation errors covariance matrix $\mathbf{V}$ with zero variances for short-term futures.

We use the simulated annealing algorithm in combination with sequential quadratic programming algorithm (SQP) for our optimization procedure. We describe these methods in the next two sections. For more details see, for example, [32].

6.4 Optimization procedure

6.4.1 Simulated Annealing

In our likelihood maximization procedure we use simulated annealing algorithm to find the global maximum of the likelihood function. This technique is suitable for the optimization problems of large scale, especially for the cases where the global extremum is hidden between many local extrema. The simulated annealing method originally comes from thermodynamics and statistical mechanics. The term *annealing* comes from analogies to the cooling of a liquid or metal. At high temperature molecules are very mobile, but as the temperature decrease, this thermal mobility is lost, atoms may line up and molecules may crystallize. This crystal is the minimum energy state. To achieve this, the liquid or metal should be cooled sufficiently slow, otherwise the substance ends up in amorphous state, having higher energy. The main principle of the annealing is *cooling slowly*, allowing atoms to find minimum energy state.

The probability of the system to be at the equilibrium energy state x and temperature T_a is expressed via Boltzmann probability distribution:

$$\mathbb{P}(x) = \alpha \exp\left(-\frac{x}{kT_a}\right),$$

where α is a normalizing constant, k is called Boltzmann constant and T_a is the temperature of the system. By this function, even at low temperature there is a small chance that the system is in a high energy state x. Therefore, there is a corresponding chance for the system to get out of the local energy minimum in order to find better, global one. Boltzmann constant k is the constant which relates temperature to energy.

In our optimization procedure we minimize the loss function called *loss* instead of energy state x.

6.4.2 Iteration procedure for maximum likelihood maximization algorithm

The main steps of estimation procedure of the model parameters are the following:

1. We start with some initial parameter vector $\boldsymbol{\psi}_0$ and initial temperature $T_a = T_{init}$. We set temperature iteration i_{temp} to 0. We set the current value of $\boldsymbol{\psi}$ to be equal to $\boldsymbol{\psi}_0$. We run Kalman filter over the whole period $[0, T]$ and evaluate the log-likelihood function $\ln L(\mathbf{y}, \boldsymbol{\psi})$ using matrix $\mathbf{Q}$ and vector $\mathbf{v}$ obtained from the Kalman filter procedure. Define $x = -\ln L(\mathbf{y}, \boldsymbol{\psi})$. Now we get a new random estimate of the parameters $\boldsymbol{\psi}_{new}$. More precisely we add standard normal random perturbation $\Delta\boldsymbol{\psi}$ (with mean 0 and standard derivation 1) to the current value of the parameters $\boldsymbol{\psi}$, thus $\boldsymbol{\psi}_{new} = \boldsymbol{\psi} + \boldsymbol{\Delta\psi}$.

2. We repeat the following iteration procedure.

   (a) For the new $\boldsymbol{\psi}_{new}$ we run the Kalman filter over the whole period $[0, T]$ and evaluate log-likelihood function $\ln L(\mathbf{y}, \boldsymbol{\psi}_{new})$ using matrix $\mathbf{Q}$ and vector $\mathbf{v}$ calculated during Kalman filter procedure. $x_{new} = -\ln L(\mathbf{y}, \boldsymbol{\psi}_{new})$.

   (b) We set τ to be some small positive number and we set the loss function to be $loss = x_{new} - x$. If $loss < \tau$, we accept $\boldsymbol{\psi}_{new}$, if $loss > \tau$ we accept new $\boldsymbol{\psi}_{new}$ only if $\Delta\boldsymbol{\psi} < \alpha \exp((-(loss - \tau))/kT_a)$, where α is a normalizing constant, k is the Boltzmann constant.

   (c) Now, if $\boldsymbol{\psi}_{new}$ is accepted we set current $\boldsymbol{\psi}$ to be $\boldsymbol{\psi}_{new}$, otherwise current $\boldsymbol{\psi}$ does not change.

3. We repeat the procedure above for a fixed number of iterations, say m, with fixed temperature T_a. Then we increase temperature iteration i_{temp} by 1 and decrease the temperature according to the formula: for each temperature iteration i_{temp} we set $T_a = T_{init}(1/ (1 + i_{temp} T_{init} S))$, where S is a scaling constant, which regulates the speed of cooling down the temperature.

4. The vector $\boldsymbol{\psi}$ obtained from the last iteration are desired parameter estimates.

   We used MATLAB 7.0 package to implement our procedure for estimation of model parameters. In addition to simulated annealing algorithm described above, we used the built-in MATLAB Sequential Quadratic Programming (SQP) algorithm. We used this algorithm using different starting points $\boldsymbol{\psi}_0$ for parameters in order to guarantee that the optimal parameters we found indeed represent the global maximum likelihood and not the local optimum of the likelihood function.

6.4.3 Sequential Quadratic Programming (SQP) algorithm

Sequential Quadratic Programming is a non-linear algorithm for optimization of non-linear function with constrains. This method is based on the work of Biggs, Han, and Powell [28] and the method allows us to closely mimic Newton's method for constrained optimization just as is usually done for unconstrained optimization. At each iteration Hessian matrix of second derivatives of minus log-likelihood function is updated using quasi-Newton method. This Hessian matrix is then used to generate a quadratic programming sub-problem. The solution of the sub-problem is used to determine a search direction for a line search procedure.

The MATLAB SQP implementation consists of three main stages:

- Updating of the Hessian matrix of the Lagrangian function.

  At each major iteration a positive definite quasi-Newton approximation of the Hessian of the Lagrangian function, H, is calculated using the Broyden, Fletcher, Goldfarb, and Shanno (BFGS) method (see [5], [15], [17] and [30]).

- Quadratic programming problem solution.

  At each new point we suggest, that our minus log-likelihood function can be approximated by quadratic function. We use approximation of the Hessian matrix H calculated before to minimize the following quadratic function

  $$\min_d d^t H d + cd,$$

  in order to find the search direction d.

- Line search

  The solution to the QP sub-problem produces a vector, which is used to form a new iterate

  $$\boldsymbol{\psi}_{new} = \boldsymbol{\psi}_{old} + \alpha d,$$

  where α is determined in order to produce a sufficient decrease in a merit function which uses an approximation of the gradient of the log-likelihood function. The merit function and penalty parameters used by Han and Powell can be found in [28].

6.4.4 Standard errors of parameters

The use of quasi-Newton method (in our case BFGS method) allows us to directly observe approximated standard errors of the covariance parameter estimates. These are the square roots of the diagonal elements of the observed Fisher information matrix, which equals H^{-1}, where H is the Hessian matrix. The H matrix consists of the second derivatives of the objective function with respect to the parameters. In our case objective function is $-\ln L(\mathbf{y}, \boldsymbol{\psi})$. The standard errors of the parameters are calculated as

$$SE(\boldsymbol{\psi}) = \sqrt{diag\,(H^{-1})}$$

where H is Hessian matrix derived from the BFGS algorithm.

The standard errors of parameters indirectly tell us how well the obtained parameters fit the data. If the standard error is small, it means that a small change in the parameter would produce the values that fit the data less well. Therefore, we say that we know the value of that parameter accurately. If the standard error is large, a relatively large variation of that parameter would not spoil the fit much; therefore, we do not really know the parameter's value well.

6.5 Seasonality

Before maximizing the likelihood function, we need to deseasonalize data. In this section we would like to describe different possibilities for modeling seasonality. There are many different ways to account for the seasonal effect in the electricity prices. We consider two ways to incorporate seasonality into the model: adding seasonality function in the spot price and adding seasonality function in the futures price.

6.5.1 Seasonality in the spot prices

We can estimate the deterministic seasonality function $exp(h(t))$, where $h(t)$ is the seasonal function used in the two-factor model. We estimated this seasonal function (with n seasons) by fitting the monthly averages of spot data with truncated Fourier series of order $m = n - 1$:

$$F(t) = a_0 + \sum_{k=1}^{m} \left(a_k \cos\left(\frac{2\pi kt}{m}\right) + b_k \sin\left(\frac{2\pi kt}{m}\right)\right).$$

For example we derived the seasonality function for EEX Base spot prices based on a Fourier series of order five and present this function in Figure 6.1.

On this graph the red dots denote the monthly averages of the EEX Base spot prices over three years. As we can see using just few different frequencies for Fourier series we can fit monthly average well. Nevertheless, for us it is not clear whether the Base spot prices exhibit seasonality and thus we prefer to remove seasonality based on futures data, which is easily observable if we plot the averaged prices, where we average the spot prices with specific delivery month (see Figure 6.2).

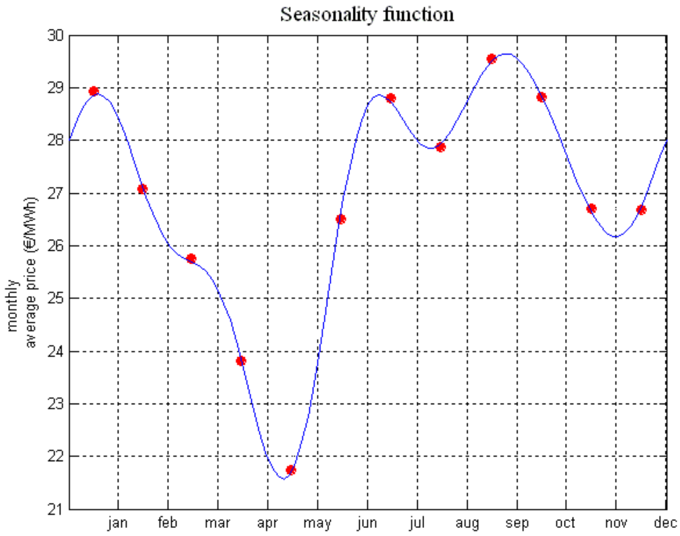

Figure 6.1: Seasonal function for EEX Base spot price (estimated from truncated Fourier series of order 6)

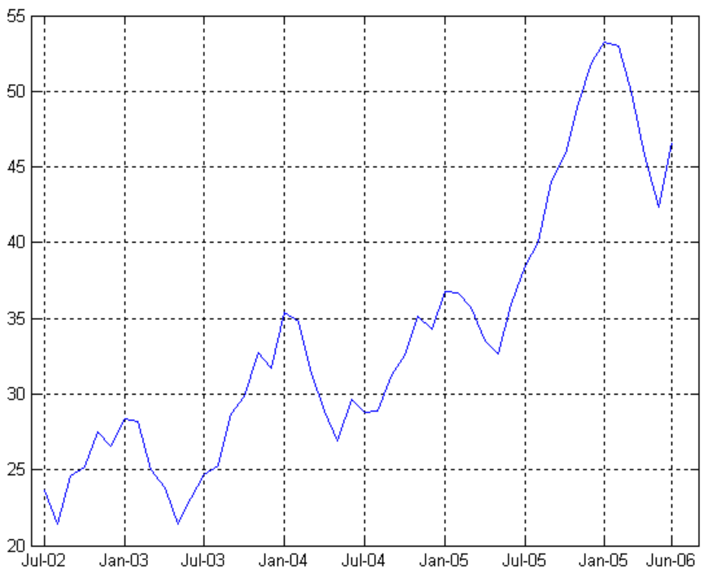

Figure 6.2: Averaged over delivery month EEX monthly Base futures prices.

6.5.2 Seasonality in the futures prices

To remove the seasonality from the futures prices we use the models of Geman and Borovkova ([4]) and Boogert and Dupont ([6]). We consider the monthly futures price $F(t, T_0, T)$ and suppose that the price depends on two factors: $a(t)$, called the common time factor, and $b(T)$, called the expiry effect, where at first the common time factor $a(t)$ is calculated as an average of all monthly futures traded at time t and the expiry effect factor $b(T)$ is calculated as an average of the differences between $F(t, T)$ and $a(t)$. Least-square techniques were used to actually find optimal values for $a(t)$ for t and $b(T)$ for all the delivery months T, which minimizes the sum of squared errors $e(t, T)$, where each error is defined by

$$e(t, T) = F(t, T) - a(t) - b(T),$$

taking into account the fact that function $b(T)$ should be seasonal, thus for example satisfy these conditions

$$\begin{aligned} b(0) &= 0, \\ \sum_{T=0}^{11} b(T) &= 0 \qquad \text{and} \\ b(T) &= b(T+12) \qquad \text{for all } T. \end{aligned}$$

The calculated functions $a(t)$ and $b(T)$ for both markets were already presented in the chapter 3 on the Figures 3.35 and 3.36.

Of course there are a lot of different ways to deseasonalize the time series. We refer reader to Hylleberg [22] for more possibilities for seasonal adjustments.

6.5.3 Implementation constraints and data used

For model parameters estimation we used data from different electricity markets: European Energy Exchange (EEX) for spot prices and prices of futures and option based on EEX spot price, Amsterdam Energy Exchange for spot prices and Endex for futures prices on APX spot prices.

1. EEX

   For model implementation we use daily observations of prices of electricity spot and futures contracts on EEX market futures prices from 1 July 2002 till 30 December 2005. More precisely, there are EEX Base spot price observations, daily observations of 48 month futures contracts maturing in the months from July 2002 till December 2005, daily observations of 20 quarter contracts maturing in the quarters from April-June 2002 till July-September 2007 and also observations of

9 year futures maturing in the years 2002-2011. There are 77 futures contracts in total.

2. APX and Endex

   For model implementation we use daily observations of electricity APX Base spot prices and prices of the futures contracts on Endex market from 1st of January 2002 till 30th of December 2005. There are daily observations of 42 monthly futures maturing in the months from January 2003 till December 2005, 17 quarter contracts maturing in the quarters from April-May 2003 till April-June 2007 and 5 year futures maturing in the years 2004-2008. In total there are observations of the prices of 70 futures contracts.

To check how much parameters of the model will be influenced by the introducing trading in CO_2 emission permits, in addition to full data sets, we used separate data sets for futures prices before year 2005 (thus from July 2002 to December 2005 for EEX and from January 2003 to December 2005 for APX and Endex) and only for data from year 2005 for both markets. These data sets will be used in our parameters estimation and analyzed in the next chapter.

Chapter 7

Results

This chapter presents the results of the parameter estimation (which followed the procedure described in the previous chapter). Data from two markets were considered: spot and futures prices from EEX and APX were used for estimation of parameters. For each market, we estimate parameters on the full sample and reestimate the parameters for 2005 only by excluding this year from the sample to check the effect of implementing CO_2 emissions reduction. Risk premiums were calculated using estimated parameters for both markets and subsamples. Finally, the option prices for EEX option on Phelix futures were calculated. The results of option calculations and the comparison of the calculated prices with market quotes are presented in the last section of this chapter.

7.1 Parameters estimations and interpretation

For each set of parameters we present the point estimates of parameters, their standard errors, t-ratios[1] and p-values[2]. We also present the value of parameter λ_ξ, which is equal to the difference between real-world μ and risk-neutral μ^* drift parameters. As it was expected, for all markets and data sets the risk-neutral drift is close to zero.

First we compare results obtained using raw data or deseasonalized data on the EEX. Results when using raw data are presented in the Table 7.1. To deseasonalize prices of monthly futures contracts, we subtract $b(T)$ from the futures prices, where T is the expiry month and $b(T)$ is the expiry effect described in Section 6.5.2. To deseasonalize prices of quarterly futures contracts we subtract $b(T_1) + b(T_2) + b(T_3)$ where T_1, T_2 and T_3 are the months composing a given quarter and $b(T_i)$ is the monthly

[1] The t-ratio is the ratio of the point estimate of the parameter to the standard deviation of the estimate.

[2] The p-value of a statistical significance test represents the probability of obtaining values of the test statistic that are equal to or greater in magnitude than the observed test statistic. If the null hypothesis is true, the significance level is the probability that it will be erroneously rejected (Type I error). The smaller the p-value, the more strongly the test confirms the null hypothesis. Here because of large number of observations we calculate the two sided p-values using the formula p-value=$2(1 - \Phi(\text{T-ratio}))$, were $\Phi(x)$ is standard normal distribution function.

expiry effect as above. Results for deseasonalized time series are shown in the Table 7.2. The results of parameters estimates of Endex quotes for original and deseasonalized prices are presented in the Tables 7.3 and 7.4.

As we compare the estimated parameters in these tables, we can see that taking into account seasonality lowers, the speed of mean reversion κ and the volatility σ_χ, of the short-term factors as well as the associated price of risk λ_χ, while it slightly increases the parameters of the long-term factor (the drift μ_ξ, volatility σ_ξ and risk premium λ_ξ).

We also can see that parameter $\rho_{\chi\xi}$, which is correlation between short-term and long-term factors, shows higher standard errors and p-values for EEX data. We will test the hypothesis of $\rho_{\chi\xi}$ being zero later in this chapter.

If we compare EEX and Endex data we can see that speed of mean reversion κ and volatility parameters σ_χ and σ_ξ are higher on the Endex market, which can be explained by more volatile prices and more often and higher spikes on the Endex market.

We present graphs of risk premium coefficient in the Section 7.2.

In the Tables 7.5 and 7.6 we present parameters estimated for two separate data sets of futures prices. First table present results, where only deseasonalized futures prices from July 2002 till December 2004 were used, second Table present results of parameters estimates only for the deseasonalized prices in 2005. If we split original data sets into the futures prices before and after 1st of January 2005, when CO_2 emissions trading was introduced, we see that the drift term for the 2005 futures prices is at least twice higher than for the futures prices before year 2005. CO_2 emissions trading led to the constant increase in the the price levels in 2005 for both markets, which we already have seen from the Figures 2.5 and 2.6.

If we compare results for Endex data for years 2003-2004 and 2005, presented in the Tables 7.7 and 7.8, we can see that drift parameter μ_ξ is three time higher for the year 2005 than for years 2003-2004. CO_2 emissions trading influenced the level of Endex prices more than EEX. On the other side, speed of mean-reversion parameter, in opposition to EEX, decreases in the year 2005. The short-term risk premium parameter changed sign in 2005, which, as we see later, makes the shape of the risk premium similar to that of the EEX market. One of the consequences of higher mean-reversion parameter is the lower volatility for year 2005 on the Endex market.

In the Tables 7.5 and 7.6 we present parameters estimated for two separate data sets of futures prices. First table present results, where only deseasonalized futures prices from July 2002 till December 2004 were used, second Table present results of parameters estimates only for the deseasonalized prices in 2005. If we split original data sets into the futures prices before and after 1st of January 2005, when CO_2 emissions trading was introduced, we see that the drift term for the 2005 futures prices is at least twice higher than for the futures prices before year 2005. CO_2 emissions trading

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	2.168	0.744	0.296	0.071	0.178	0.022	0.156	−0.142
St.Error	0.077	0.062	0.018	0.006	0.022	0.001	N/A	0.104
T-ratio	28.189	11.984	16.704	12.909	8.211	23.781	N/A	−1.399
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.162

Table 7.1: Estimated parameters, standard errors and t-ratios of the parameters for EEX data from July 2002 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	1.491	0.472	0.189	0.078	0.181	0.025	0.156	−0.023
St.Error	0.024	0.028	0.005	0.003	0.019	0.001	N/A	0.029
T-ratio	63.330	16.584	35.276	25.231	9.460	33.318	N/A	−0.798
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.425

Table 7.2: Estimated parameters, standard errors and t-ratios of the parameters for deseasonalized EEX data from July 2002 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	3.810	−0.461	0.601	0.113	0.143	−0.038	0.181	−0.096
St.Error	0.018	0.002	0.003	0.003	0.023	0.002	N/A	0.010
T-ratio	207.73	−257.36	176.71	41.46	6.23	−20.51	N/A	−9.88
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.000

Table 7.3: Estimated parameters, standard errors and t-ratios of the parameters for Endex data from January 2003 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	3.237	−0.370	0.565	0.118	0.156	−0.032	0.188	−0.322
St.Error	0.002	0.005	0.006	0.006	0.001	0.002	N/A	0.007
T-ratio	1505.71	−77.72	89.69	20.26	166.65	−17.44	N/A	−49.41
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.000

Table 7.4: Estimated parameters, standard errors and t-ratios of the parameters for Endex deseasonalized data from January 2003 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	4.021	0.623	0.327	0.078	0.142	0.025	0.116	−0.190
St.Error	0.040	0.056	0.028	0.005	0.022	0.001	N/A	0.033
T-ratio	101.163	11.058	11.811	16.278	6.314	29.517	N/A	−5.804
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.000

Table 7.5: Estimated parameters, standard errors and t-ratios of the parameters for deseasonalized EEX data from July 2002 to December 2004

led to the constant increase in the price levels in 2005 for both markets, which we already have seen from the Figures 2.5 and 2.6. If we compare results for Endex data for years 2003-2004 and 2005, presented in the Tables 7.7 and 7.8, we can see that drift parameter μ_ξ is three time higher for the year 2005 than for years 2003-2004. CO_2 emissions trading influenced the level of Endex prices more than EEX. From the other side speed of mean-reversion parameter, in opposition to EEX, decreases in the year 2005. The short-term risk premium parameter changed sign in 2005, which, as we see later, makes the shape of the risk premium similar to that of the EEX market. One of the consequences of higher mean-reversion parameter is the lower volatility for year 2005 on the Endex market.

7.2 Risk premiums

Using the estimated parameters from the previous section we can calculate the risk premiums coefficients R for different data sets and different markets. Here we use equations (5.29) and (5.32). The graphs of for the risk premium coefficient R with respect to time to maturity (in days) for different markets and data sets are presented on the Figures 7.1-7.6.

As we can see from these graphs, for the EEX market, the risk premium coefficient is negative and decreasing. This mean that risk premium $\pi(t, T)$ computed in (5.25) is also negative and decreasing with respect to time to maturity. For the Endex market the situation is different for different subsamples. For all the data and for the sample excluding the year 2005, the risk premium is increasing during the first four months and then decreasing. Risk premium coefficient is positive for approximately seven months and negative for longer time to maturity. Interestingly, the risk premium behaves differently if we consider only data for Endex for the year 2005. In this case, the situation is similar to the EEX market.

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	1.321	0.623	0.210	0.070	0.296	0.027	0.269	0.588
St.Error	0.0264	0.0096	0.0189	0.0151	0.07838	0.0016	N/A	0.0091
T-ratio	49.983	64.566	11.096	4.597	3.777	16.322	N/A	64.787
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.000

Table 7.6: Estimated parameters, standard errors and t-ratios of the parameters for deseasonalized EEX data from January 2005 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	2.942	−0.487	0.658	0.114	0.175	−0.049	0.224	−1.000
St.Error	0.0001	0.00005	0.004	0.001	0.110	0.004	N/A	0.034
T-ratio	51.284	−12.732	43.127	12.505	4.747	−20.285	N/A	−18.425
P-value	0.000	0.000	0.000	0.000	0.111	0.000	N/A	0.000

Table 7.7: Estimated parameters, standard errors and t-ratios of the parameters for Endex data from January 2003 to December 2004

7.3 Alternative models testing

As we can see from the P-values presented above, all the parameters are significant except the correlation coefficient $\rho_{\chi\xi}$ for EEX futures estimation (with and without seasonality) and for Endex futures using only data from year 2005.

In order to compare our model with the model where the correlation parameter $\rho_{\chi\xi}$ is equal to zero and with one-factor mean-reverting model (all the parameters σ_ξ, μ_ξ, μ_ξ^* and $\rho_{\chi\xi}$ are equal to zero) we recalculate maximum log-likelihood values and parameter estimates. The results for the model with $\rho_{\chi\xi} = 0$ are presented in the Tables 7.9 and 7.10 and the results for one-factor model with only three parameters are presented in the Table 7.11.

Now if we would like to test the following two hypothesis:

- Hypothesis $\mathcal{H}_0^1 =$ (parameter $\rho_{\chi\xi} = 0$). We denote sets of estimated parameters for this hypothesis by $\tilde{\psi}$ (see Tables 7.9 and 7.10).

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	4.976	0.853	0.513	0.152	0.545	−0.034	0.579	0.234
St.Error	4.976	0.012	0.024	0.012	0.113	0.003	N/A	0.018
T-ratio	135.63	73.244	21.598	12.812	4.819	−9.986	N/A	12.666
P-value	0.000	0.000	0.000	0.000	0.000	0.000	N/A	0.000

Table 7.8: Estimated parameters, standard errors and t-ratios of the parameters for Endex data from January 2005 to December 2005

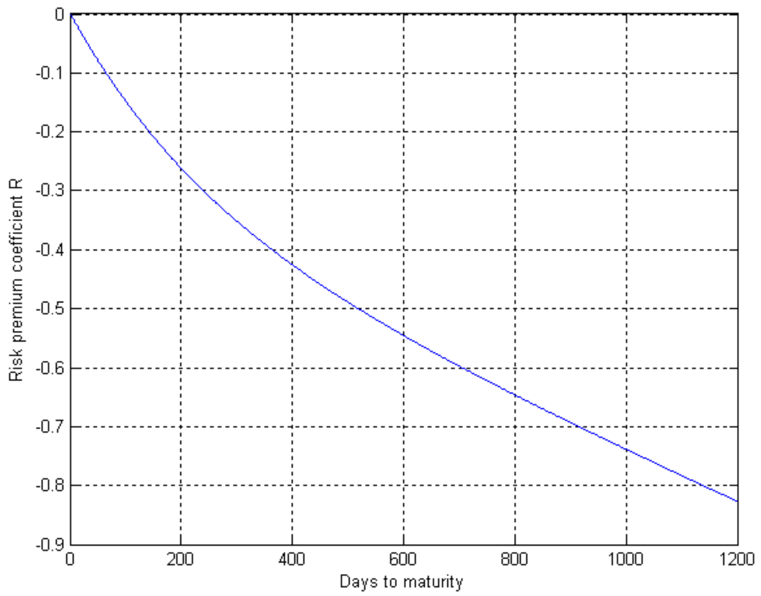

Figure 7.1: Risk premium coefficient R for deseasonalized EEX market, data from July 2002 to December 2005

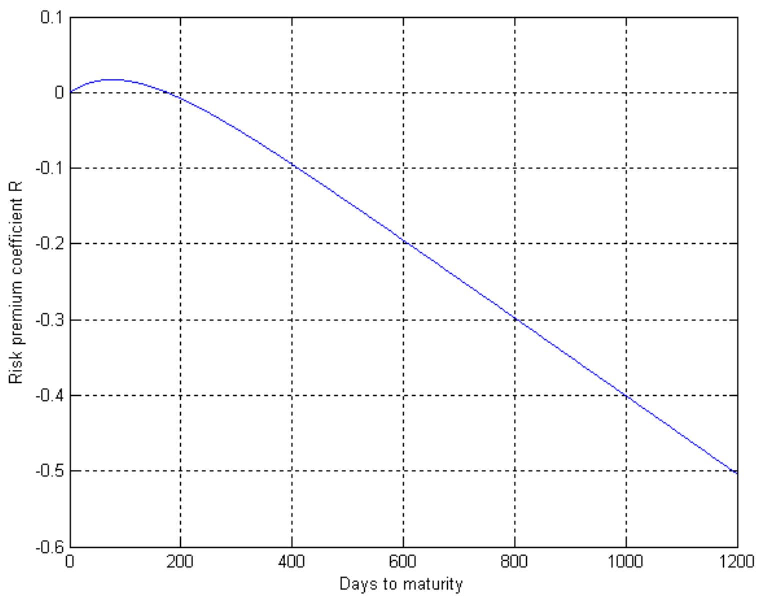

Figure 7.2: Risk premium coefficient R for deseasonalized Endex market, data from January 2003 to December 2005

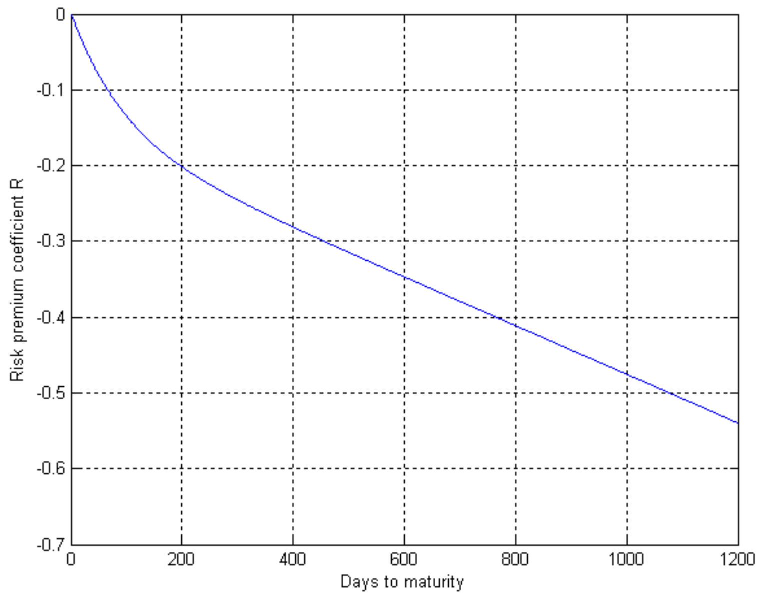

Figure 7.3: Risk premium coefficient R for EEX market, data from July 2002 to December 2004

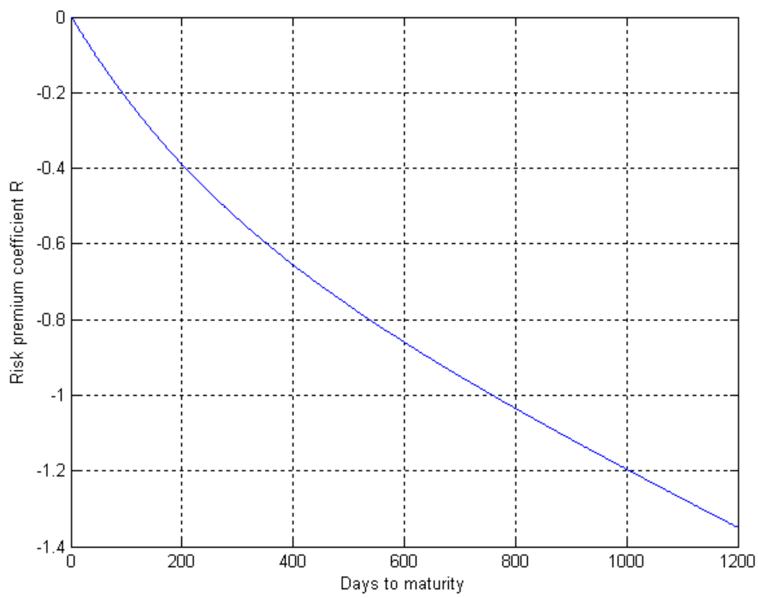

Figure 7.4: Risk premium coefficient R for deseasonalized EEX market, data from January 2005 to December 2005

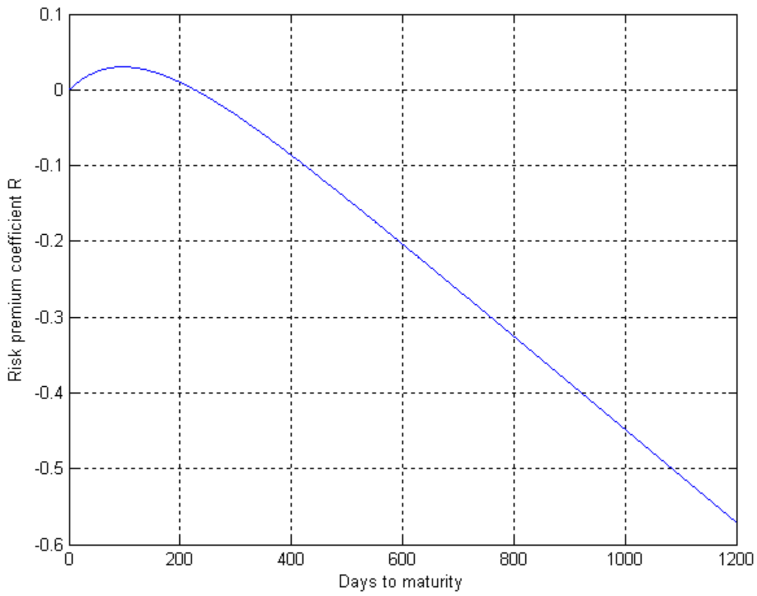

Figure 7.5: Risk premium coefficient R for deseasonalized Endex market, data from January 2003 to December 2004

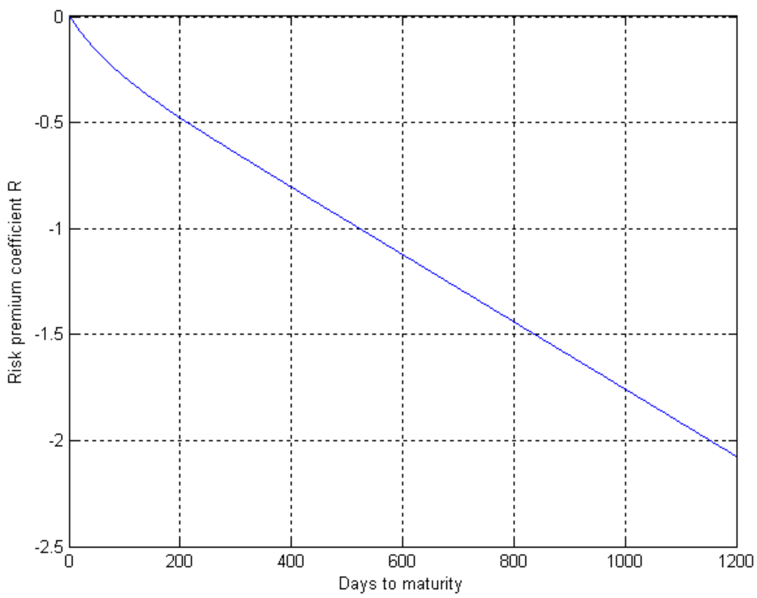

Figure 7.6: Risk premium coefficient R for Endex market, data from January 2005 to December 2005

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	1.493	0.473	0.188	0.078	0.182	0.025	0.156	0.000
St.Error	0.009	0.087	0.187	0.033	0.098	0.004	N/A	N/A
T-ratio	171.190	5.464	1.010	2.383	1.845	6.258	N/A	N/A
P-value	0.000	0.000	0.313	0.017	0.065	0.000	N/A	N/A

Table 7.9: Estimated parameters, standard errors, t-ratios and P-values of the parameters for EEX data for model with 6 parameters

	κ	λ_χ	σ_χ	σ_ξ	μ_ξ	μ_ξ^*	λ_ξ	$\rho_{\chi\xi}$
Estimate	3.279	-0.348	0.554	0.115	0.142	-0.032	0.174	0.000
St.Error	0.297	0.026	0.554	0.049	0.026	0.004	N/A	N/A
T-ratio	11.047	-13.413	0.999	2.367	5.536	-8.259	N/A	N/A
P-value	0.000	0.000	0.318	0.018	0.000	0.000	N/A	N/A

Table 7.10: Estimated parameters, standard errors, t-ratios and P-values of the parameters for Endex data for model with 6 parameters

	EEX				Endex			
	κ	λ_χ	σ_χ	μ_ξ	κ	λ_χ	σ_χ	μ_ξ
Estimate	0.262	0.162	0.149	0.093	0.325	0.338	0.228	0.194
St.Error	0.003	0.004	0.008	0.002	0.003	0.009	0.010	0.004
T-ratio	87.155	43.387	18.031	60.092	116.986	38.862	22.266	46.899
P-value	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Table 7.11: Estimated parameters, standard errors, t-ratios and P-values for one-factor model for both markets

- Hypothesis $\mathcal{H}_0^2 =$ (parameters $\sigma_\xi = \rho_{\chi\xi} = 0$ and $\mu_\xi = \mu_\xi^*$) We denote sets of estimated parameters for this hypothesis by $\bar{\boldsymbol{\psi}}$ (see Table 7.11).

We test these two null hypotheses against the hypothesis $\mathcal{H} =$ (parameters from two-factor model $\hat{\boldsymbol{\psi}}$). See Tables 7.1 and 7.3.

We calculate the likelihood ratios test by comparing likelihood ratios with χ^2 distribution with corresponding number of degrees of freedom by formulae:

$$\begin{aligned} 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\tilde{\boldsymbol{\psi}})\right] &\sim \chi^2(1) \qquad \text{for testing hypothesis } \mathcal{H}_0^1, \\ 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\bar{\boldsymbol{\psi}})\right] &\sim \chi^2(3) \qquad \text{for testing hypothesis } \mathcal{H}_0^2. \end{aligned}$$

For the first hypothesis $\mathcal{H}_0^1$ we get

$$\begin{aligned} 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\tilde{\boldsymbol{\psi}})\right] &= 0.0134 \qquad \text{for EEX and} \\ 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\tilde{\boldsymbol{\psi}})\right] &= 3.8454 \qquad \text{for Endex.} \end{aligned}$$

Because the probability that $\chi^2(1)$ variable exceeds 3.84 is equal to 0.05, we accept null hypothesis $\mathcal{H}_0^1$ that parameter $\rho_{\chi\xi} = 0$ at the 5% significance level for EEX market and reject this hypothesis for Endex market.

Now we consider second hypothesis $\mathcal{H}_0^1$ of the one-factor model. We get

$$\begin{aligned} 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\bar{\boldsymbol{\psi}})\right] &= 647.8269 \qquad \text{for EEX and} \\ 2\left[\ln L(\mathbf{y},\hat{\boldsymbol{\psi}}) - \ln L(\mathbf{y},\bar{\boldsymbol{\psi}})\right] &= 3451.9212 \qquad \text{for Endex.} \end{aligned}$$

Because the probability that $\chi^2(3)$ variable exceeds 7.81 is equal to 0.05, we reject null hypothesis $\mathcal{H}_0^2$ of one-factor model at the 5% significance level for both markets.

7.4 Short-term and long-term of the model

Here we present modeled state variables for original and deseasonalized data for both markets. Exponential of the short term ($\exp\chi(t)$) and long term ($\exp\xi(t)$) and spot price $S(t) = \exp(\chi(t)+\xi(t))$ for original and deseasonalized data for EEX are presented on the Figures 7.7 and 7.8. State variables for APX original and deseasonalized data are on the Figures 7.9 and 7.10. We can see only very small difference between the original and the deseasonalized prices in the short terms for both markets.

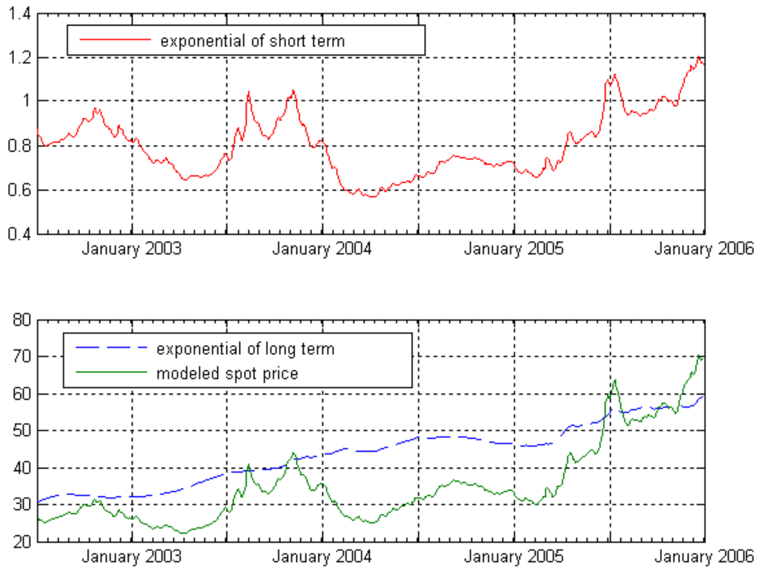

Figure 7.7: Exponentials of the state variables $exp(\chi(t))$ and $exp(\xi(t))$ and the modeled price $S(t) = exp(\chi(t) + \xi(t))$ for EEX data

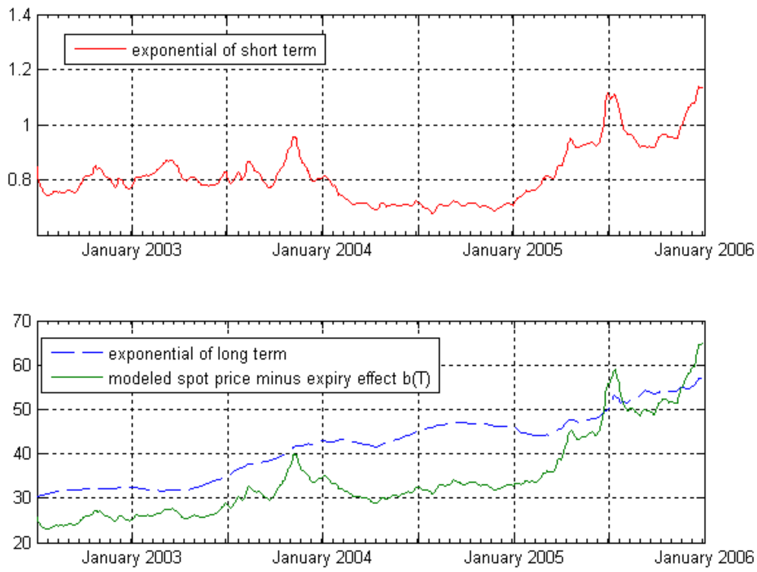

Figure 7.8: Exponentials of the state variables $exp(\chi(t))$ and $exp(\xi(t))$ and the modeled price $S(t) = exp(\chi(t) + \xi(t))$ for deseasonalized EEX data

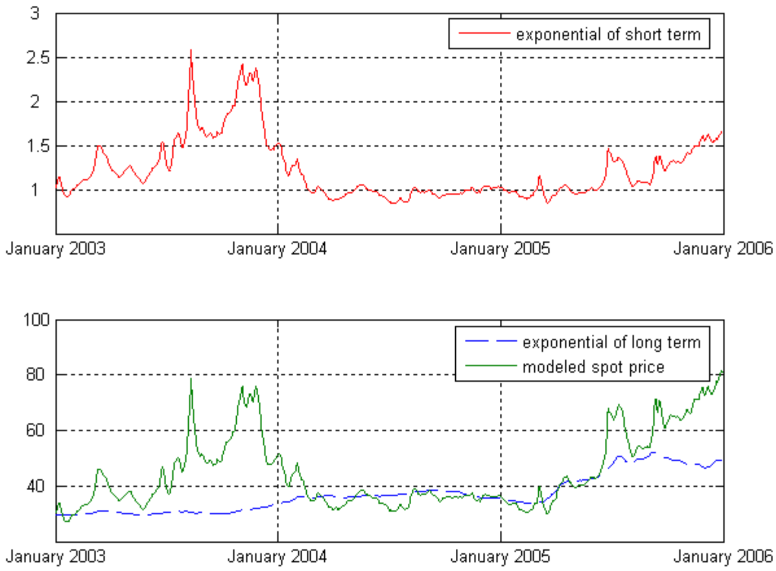

Figure 7.9: Exponentials of the state variables $exp(\chi(t))$ and $exp(\xi(t))$ and the modeled price $S(t) = exp(\chi(t) + \xi(t))$ for APX data

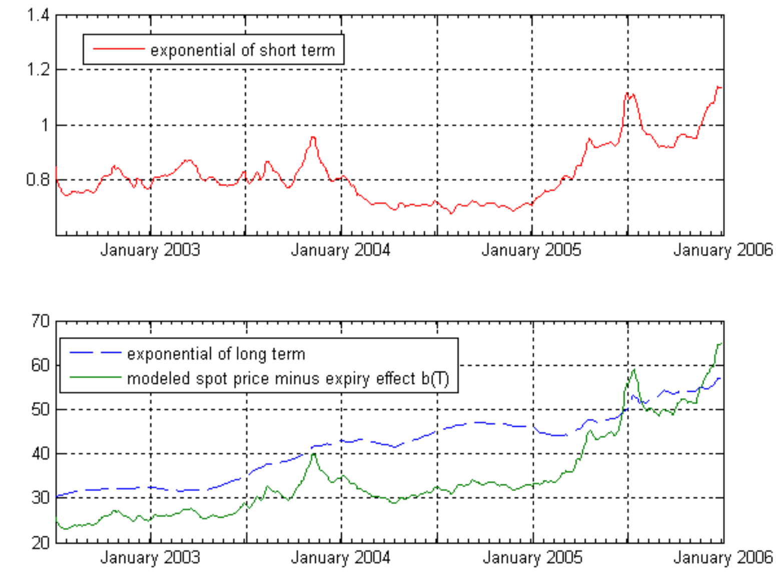

Figure 7.10: Exponentials of the state variables $exp(\chi(t))$ and $exp(\xi(t))$ and t and the modeled price $S(t) = exp(\chi(t) + \xi(t))$ for deseasonalized APX data

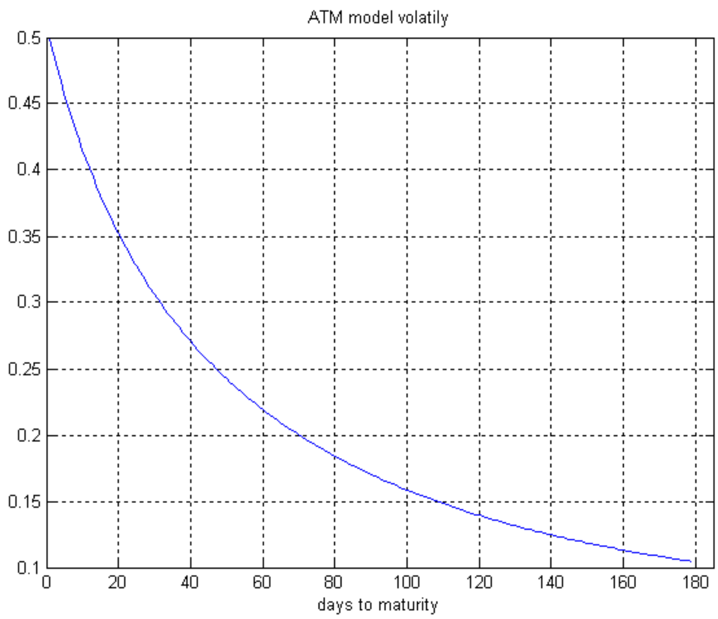

Figure 7.11: ATM model implied volatility for EEX monthly futures.

7.5 Options prices and implied volatilities

On the Figure 7.11 we present implied volatility, calculated from the estimated model parameters. Volatility is increasing as time to maturity decreases. We saw this property of the implied volatility on the Figures 3.13 in Section 3.3. The implied volatilities derived from the model parameters are lower than implied volatilities derived form the market options prices.

To compare the options prices we present plot the ATM Call options prices for the EEX market data (Figure 3.19) and ATM Call model options prices (Figure 3.20) calculated by formula (5.34) using estimated parameters from Table 7.1. As we can see from these figures, the model captures in general the behavior of the prices, although it shows much lower prices than the original data for the summer months 2005.

Chapter 8

Conclusions and future research

In this chapter we give the main conclusions, limitations of the model and directions for future research.

8.1 Conclusions

Energy commodity markets have been developing very rapidly in the past few years. Many new products on electricity have appeared and there is a need for a consistent and simple model to price electricity derivatives based on electricity prices and to manage financial risks.

In this thesis we addressed the issues of modeling electricity spot and futures prices and prices of options. In Chapter 2 we described the spot market and presented the derivatives traded on electricity markets and over-the-counter operations. We also pointed out the unique features of electricity, such as its non-storability, which makes the pricing and risk management of electricity derivatives more complicated than other financial products.

In Chapter 3 we presented an empirical analysis of the spot, futures and options prices available in the German and Dutch electricity markets. In Chapter 4 we presented some classes of existing financial models such as mean-reverting jump diffusion models, regime-witching models, models with short- and long-term factors which are the most used models for modeling electricity prices and prices of electricity derivatives.

As compared with the other financial markets, basic electricity derivatives such as futures and options on futures are more complicated because these products are based not on the spot prices themselves but on the arithmetic averages of the spot prices during the delivery period. In Chapter 5 we extended the two-factor model of Schwartz and Smith [29] by including into the model the possibility to take the averaging of the spot price over the delivery period into account. We derived closed-form solutions for futures, options prices and risk premiums. These pricing formulae depend on the number of parameters; these parameters are the main price drivers and have clear interpretation. These parameters allow us to explain the movements of the prices in the electricity markets and calculate derivatives prices available in the market directly the moment parameters of the models are estimated.

We implemented this model for the pricing futures in the German and Dutch electricity markets in Chapter 6 using the Kalman filter and maximum likelihood technique in order to find the optimal parameters for both markets. We also tested the model with zero correlation between short- and long-term factor and one-factor model. In order to check the effect of introducing CO_2 emission permits on the electricity market in 2005, we also tested the model on the restricted information available before and after January 2005.

In Chapter 7 we presented the estimated parameters, risk premiums and options prices for different models for both markets. We illustrated how these parameters influence the futures price and risk premiums. We compared the results of one-factor model and model with zero correlation coefficient between short- and long-term factor with modified two-factor Schwartz and Smith model and concluded that the modified two-factor model with averaging over the delivery period performs well especially for capturing the futures prices. The averaging effect allows us to easily incorporate available spot prices in order to calculate the futures prices within the delivery period very precisely.

The model performs better and gives clear parameters interpretation when seasonal delivery effect is taken into account.

8.2 Limitations and directions for future research

The issue of liquidity risks, which is very large in such young markets, is considered to be the main limitation of the results presented in the thesis. The illiquidity of the options market leads to arbitrage prices or missing prices of the options. Because of data limitations the model described and analyzed in this thesis produces the lower options prices that the prices presented in the market. As the market for electricity derivatives develops we expect more quotes of option prices to be available, which will allow one to include these prices into the estimation of model parameters.

The simplicity of our model is its biggest advantage but it has also few obvious disadvantages. Although the two-factor model used in this thesis is based on the spot price modeling, the spot prices produced by the estimated parameters do not represent spikes and thus the spot price produced by the estimated parameters could be considered only as the averaged spot price. It is not possible to hedge the spot prices using the derivatives presented in the market, thus the market is incomplete and one needs to account for the spikes and use a different model (for example a regime-switching model) for spot prices.

Another limitation of the model is the absence of the natural drivers of the price movements such as load, demand and weather conditions in the model. One could develop a hybrid model, which could include both fundamental and financial drivers, although the development of such a model is restricted by the limitations of the data available for the analysis.

We could think of few promising extensions of the current model. One possibility is to include time varying parameters into the model. This extension will give us more parameters to estimate and possibility to capture even small movement in the futures market by incorporating appropriate functions for the parameters. Another possibility to extend the model is to make the volatility parameter stochastic. But this extension usually leads to the loss of a closed-form solution. Semi-closed solutions instead of closed-form solutions will complicate and reduce the speed of the parameter estimations dramatically.

Appendix A

Formulae 5.16 and 5.17 (case $t < T_0$)

Conditional mean and the conditional variance of geometric average A_n are equal to

$$
\begin{aligned}
\mathbb{E}_t^*[A_n] &= \mathbb{E}_t^*\left\{\ln\left[\left(S(t_1)\,S(t_2)\ldots S(t_n)\right)^{1/n}\right]\right\} \\
&= \mathbb{E}_t^*\left[\frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\cdots+\ln S(t_n)\right)\right] \\
&= \frac{1}{n}\left[-\frac{\lambda_\chi}{\kappa}+e^{-\kappa(t_1-t)}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)+\xi(t)+\mu_\xi^*(t_1-t)\right. \\
&\quad +-\frac{\lambda_\chi}{\kappa}+e^{-\kappa(t_2-t)}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)+\xi(t)+\mu_\xi^*(t_2-t)+\ldots \\
&\quad \left.+-\frac{\lambda_\chi}{\kappa}+e^{-\kappa(t_n-t)}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)+\xi(t)+\mu_\xi^*(t_n-t)\right] \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa}+\frac{1}{n}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)\left(e^{-\kappa(t_1-t)}+e^{-\kappa(t_2-t)}+\cdots+e^{-\kappa(t_n-t)}\right) \\
&\quad +\xi(t)+\frac{(t_1-t)+(t_2-t)+\cdots+(t_n-t)}{n}\mu_\xi^* \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)]
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{\lambda_\chi}{\kappa} + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right)\sum_{i=1}^{n} e^{-\kappa(t_i - t)} + \xi(t) \\
&\quad + \frac{(t_1 - t) + (t_1 + \Delta t - t) + \cdots + (t_n + (n-1)\Delta t - t)}{n}\mu_\xi^* \\
&\quad + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa} + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(t_1 - t)} \sum_{i=0}^{n-1} e^{-i\kappa\Delta t} + \xi(t) + \left(T_0 - t + \frac{1}{n}\sum_{i=1}^{n} i\Delta t\right)\mu_\xi^* \\
&\quad + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa} + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(t_1 - t)} \frac{1 - e^{-n\kappa\Delta t}}{1 - e^{-\kappa\Delta t}} + \xi(t) + \left(T_0 - t + \frac{(n+1)\Delta t}{2}\right)\mu_\xi^* \\
&\quad + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa} + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T - t)} e^{\kappa(T - T_0)} e^{-\kappa\Delta t} \frac{1 - e^{-\kappa(T - T_0)}}{1 - e^{-\kappa\Delta t}} \\
&\quad + \xi(t) + \left((T - t) - n\Delta t + \frac{(n+1)}{2}\Delta t\right)\mu_\xi^* + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa} + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T - t)} \frac{e^{\kappa(T - T_0)} - 1}{n\left(e^{\kappa(T - T_0)/n} - 1\right)} \\
&\quad + \xi(t) + \left(T - t - \frac{(n-1)}{2}\Delta t\right)\mu_\xi^* + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= -\frac{\lambda_\chi}{\kappa} + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T - t)} \varphi(T_0, T, n) + \xi(t) + \left(T - t - \frac{(n-1)}{2}\Delta t\right)\mu_\xi^* \\
&\quad + \frac{1}{n}[h(t_1) + h(t_2) + \cdots + h(t_n)] \\
&= : m_A(t, T_0, T, n),
\end{aligned}
$$

where

$$
\varphi(T_0, T, n) = \left(\frac{e^{\kappa(T - T_0)} - 1}{n\left(e^{\kappa\Delta t} - 1\right)}\right).
$$

The conditional variance of A_n is calculated as

$$\begin{aligned}
\mathbb{V}\mathrm{ar}_t^*[A_n] &= \mathbb{V}\mathrm{ar}_t^*\left[\ln\left[(S(t_1)\,S(t_2)\ldots S(t_n))^{1/n}\right]\right] \\
&= \mathbb{V}\mathrm{ar}_t^*\left[\frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\cdots+\ln S(t_n)\right)\right] \\
&= \frac{1}{n^2}\mathbb{V}\mathrm{ar}_t^*\left[\chi(t_1)+\chi(t_2)+\cdots+\chi(t_n)+\xi(t_1)+\xi(t_2)+\cdots+\xi(t_n)\right] \\
&= \frac{1}{n^2}\mathbb{V}\mathrm{ar}_t^*\left[\left(\chi(t_1)+e^{-\kappa(t_2-t_1)}\chi(t_1)+\sigma_\chi\int_{t_1}^{t_2}e^{-\kappa(t_2-u)}dW_\chi^*(u)+e^{-\kappa(t_3-t_1)}\chi(t_1)\right.\right. \\
&\quad \left.+\sigma_\chi\int_{t_1}^{t_3}e^{-\kappa(t_3-u)}dW_\chi^*(u)+\cdots+e^{-\kappa(t_n-t_1)}\chi(t_1)+\sigma_\chi\int_{t_1}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right)
\end{aligned}$$

$$+\xi(t_1)+\xi(t_1)+\sigma_\xi\left(W_\xi^*(t_2)-W_\xi^*(t_1)\right)+\xi(t_1)+\sigma_\xi\left(W_\xi^*(t_3)-W_\xi^*(t_1)\right)+\ldots$$

$$\left.+\xi(t_1)+\sigma_\xi\left(W_\xi^*(t_n)-W_\xi^*(t_1)\right)\right]$$

$$\begin{aligned}
= \ & \frac{1}{n^2}\mathbb{V}\mathrm{ar}_t^*\left[\chi(t_1)\left(1+e^{-\kappa\Delta t}+e^{-2\kappa\Delta t}+\ldots e^{-(n-1)\Delta t}\right)+n\xi(t_1)+\right. \\
& +\left(\sigma_\chi\int_{t_1}^{t_2}e^{-\kappa(t_2-u)}dW_\chi^*(u)+\sigma_\chi\int_{t_1}^{t_2}e^{-\kappa(t_3-u)}dW_\chi^*(u)+\ldots\right. \\
& \qquad \left.+\sigma_\chi\int_{t_1}^{t_2}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right)+(n-1)\,\sigma_\xi\left(W_\xi^*(t_2)-W_\xi^*(t_1)\right) \\
& +\left(\sigma_\chi\int_{t_2}^{t_3}e^{-\kappa(t_3-u)}dW_\chi^*(u)+\cdots+\sigma_\chi\int_{t_2}^{t_3}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right) \\
& +(n-2)\,\sigma_\xi\left(W_\xi^*(t_3)-W_\xi^*(t_2)\right)+\ldots \\
& \left.+\sigma_\chi\int_{t_{n-1}}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u)+\sigma_\xi\left(W_\xi^*(t_n)-W_\xi^*(t_{n-1})\right)\right].
\end{aligned}$$

Now we will use the fact that increments of Brownian motion are independent

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}_t^*\left[A_n\right] &= \frac{1}{n^2}\mathbb{V}\mathrm{ar}_t^*\left[\chi(t_1)\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}+n\xi(t_1)\right] \\
&\quad +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(\int_{t_1}^{t_2}\left[e^{-\kappa(t_2-u)}+e^{-\kappa(t_3-u)}+\cdots+e^{-\kappa(t_n-u)}\right]dW_\chi^*(u)\right)\right. \\
&\quad \left. +\,(n-1)\,\sigma_\xi\left(W_\xi^*(t_2)-W_\xi^*(t_1)\right)\right] \\
&\quad +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(\int_{t_2}^{t_3}\left[e^{-\kappa(t_3-u)}+\cdots+e^{-\kappa(t_n-u)}\right]dW_\chi^*(u)\right)\right. \\
&\quad \left. +\,(n-2)\,\sigma_\xi\left(W_\xi^*(t_3)-W_\xi^*(t_2)\right)\right]+\ldots \\
&\quad +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(\int_{t_{n-1}}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right)+\sigma_\xi\left(W_\xi^*(t_n)-W_\xi^*(t_{n-1})\right)\right] \\
&= \frac{1}{n^2}\left\{\mathbb{V}\mathrm{ar}_t^*\left[\chi(t_1)\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right]+2\mathbb{C}\mathrm{ov}\left[\chi(t_1)\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}},n\xi\left(t_1\right)\right]+\mathbb{V}\mathrm{ar}_t^*\left[n\xi(t_1)\right]\right. \\
&\quad +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(1+e^{-\kappa\Delta t}+\cdots+e^{-(n-2)\kappa\Delta t}\right)\left(\int_{t_1}^{t_2}e^{-\kappa(t_2-u)}dW_\chi^*(u)\right)\right. \\
&\quad \left. +\,(n-1)\,\sigma_\xi\left(W_\xi^*(t_2)-W_\xi^*(t_1)\right)\right] \\
&\quad +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(1+e^{-\kappa\Delta t}+\cdots+e^{-(n-3)\kappa\Delta t}\right)\left(\int_{t_2}^{t_3}e^{-\kappa(t_3-u)}dW_\chi^*(u)\right)\right. \\
&\quad \left. +\,(n-2)\,\sigma_\xi\left(W_\xi^*(t_3)-W_\xi^*(t_3)\right)\right]+\ldots \\
&\quad \left. +\mathbb{V}\mathrm{ar}_t^*\left[\sigma_\chi\left(\int_{t_{n-1}}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right)+\sigma_\xi\left(W_\xi^*(t_n)-W_\xi^*(t_{n-1})\right)\right]\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{n^2}\left\{\left(\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\left(\left(1-e^{-2\kappa(t_1-t)}\right)\frac{\sigma_\chi^2}{2\kappa}\right)\right.\\
&\qquad \left. +2n\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(1-e^{-\kappa(t_1-t)}\right)\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}+n^2\sigma_\xi^2(t_1-t)\right\}\\
& +\frac{1}{n^2}\left\{\sigma_\chi^2\left(\frac{1-e^{-(n-1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\mathbb{V}\mathrm{ar}_t^*\left(\int_{t_1}^{t_2}e^{-\kappa(t_2-u)}dW_\chi^*(u)\right)\right.\\
& +2\sigma_\chi\sigma_\xi\left(n-1\right)\left(\frac{1-e^{-(n-1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)\times\\
&\qquad \left.\times\mathbb{C}\mathrm{ov}_t^*\left(\int_{t_1}^{t_2}e^{-\kappa(t_2-u)}dW_\chi^*(u),\left(W_\xi^*(t_2)-W_\xi^*(t_1)\right)\right)+\left(n-1\right)^2\sigma_\xi^2(t_2-t_1)\right\}\\
&\qquad +\frac{1}{n^2}\left\{\sigma_\chi^2\left(\frac{1-e^{-(n-2)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\mathbb{V}\mathrm{ar}_t^*\left(\int_{t_2}^{t_3}e^{-\kappa(t_3-u)}dW_\chi^*(u)\right)\right.\\
&\qquad\qquad +2\sigma_\chi\sigma_\xi\left(n-2\right)\left(\frac{1-e^{-(n-2)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)\times\\
&\qquad\qquad\qquad \times\mathbb{C}\mathrm{ov}_t^*\left(\int_{t_2}^{t_3}e^{-\kappa(t_3-u)}dW_\chi^*(u),\left(W_\xi^*(t_3)-W_\xi^*(t_2)\right)\right)\\
&\qquad \left.+\left(n-2\right)^2\sigma_\xi^2(t_3-t_2)\right\}+\cdots+\frac{1}{n^2}\left\{\sigma_\chi^2\mathbb{V}\mathrm{ar}_t^*\left(\int_{t_{n-1}}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u)\right)\right.\\
&\qquad \left.+2\sigma_\chi\sigma_\xi\mathbb{C}\mathrm{ov}_t^*\left(\int_{t_{n-1}}^{t_n}e^{-\kappa(t_n-u)}dW_\chi^*(u),\left(W_\xi^*(t_n)-W_\xi^*(t_{n-1})\right)\right)+\sigma_\xi^2(t_n-t_{n-1})\right\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{n^2}\left\{\left(\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2 \left(\left(1-e^{-2\kappa(t_1-t)}\right)\frac{\sigma_\chi^2}{2\kappa}\right)\right. \\
& \qquad \left. +2n\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(1-e^{-\kappa(t_1-t)}\right)\frac{\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}+n^2\sigma_\xi^2(t_1-t)\right\} \\
& +\frac{1}{n^2}\left\{\sigma_\chi^2\left(\frac{1-e^{-(n-1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\frac{1-e^{-2\kappa\Delta t}}{2\kappa}\right. \\
& \qquad\qquad \left. +2\sigma_\chi\sigma_\xi\left(n-1\right)\left(\frac{1-e^{-(n-1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)\frac{\rho_{\chi\xi}}{k}\left(1-e^{-\kappa\Delta t}\right)+\left(n-1\right)^2\sigma_\xi^2\Delta t\right\} \\
& +\frac{1}{n^2}\left\{\sigma_\chi^2\left(\frac{1-e^{-(n-2)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\frac{1-e^{-2\kappa\Delta t}}{2\kappa}\right. \\
& \qquad +2\sigma_\chi\sigma_\xi\left(n-2\right)\left(\frac{1-e^{-(n-2)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)\frac{\rho_{\chi\xi}}{k}\left(1-e^{-\kappa\Delta t}\right)+\left(n-2\right)^2\sigma_\xi^2\Delta t+\dots \\
& +\frac{1}{n^2}\left\{\sigma_\chi^2\frac{1-e^{-2\kappa\Delta t}}{2\kappa}+2\sigma_\chi\sigma_\xi\frac{\rho_{\chi\xi}}{k}\left(1-e^{-\kappa\Delta t}\right)+\sigma_\xi^2\Delta t\right\} \\
= \quad & \frac{1}{n^2}\frac{\sigma_\chi^2}{2\kappa}\left[\left(1-e^{-2\kappa(t_1-t)}\right)\left(\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2\right. \\
& \qquad \left. +\left(1-e^{-2\kappa\Delta t}\right)\left(\frac{1-e^{-(n-1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2+\dots+\left(1-e^{-2\kappa\Delta t}\right)\right]+ \\
& +\frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa n^2}\left[n\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(1-e^{-\kappa(t_1-t)}\right)\right. \\
& \qquad \left. +\left(n-1\right)\left(1-e^{-(n-1)\kappa\Delta t}\right)+\dots+\left(1-e^{-\kappa\Delta t}\right)\right] \\
& +\frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t)+((n-1)^2+(n-2)^2+\dots+1)\Delta t\right] \\
= \quad & \frac{1}{n^2}\frac{\sigma_\chi^2}{2\kappa}\left[\left(e^{-2\kappa\Delta t}-e^{-2\kappa(t_1-t)}\right)\left(\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)^2+\frac{\left(1-e^{-2\kappa\Delta t}\right)}{\left(1-e^{-\kappa\Delta t}\right)^2}\sum_{i=1}^{n}(1-e^{-i\kappa\Delta t})^2\right] \\
& +\frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa n^2}\left[n\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(1-e^{-\kappa(T-t)}e^{\kappa(T-T_0)}e^{-\kappa(T-T_0)/n}\right)+\sum_{i=1}^{n-1}i\left(1-e^{-i\kappa\Delta t}\right)\right] \\
& +\frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t)+((n-1)^2+(n-2)^2+\dots+1)\Delta t\right]
\end{aligned}
$$

$$
\begin{aligned}
= \;& \frac{\sigma_\chi^2}{2\kappa}\left[\left(e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}\right)\left(\frac{e^{\kappa(T-T_0)}-1}{n\left(e^{\kappa\Delta t}-1\right)}\right)^2 \right.\\
&\qquad\qquad \left. + \frac{1}{n^2}\frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\sum_{i=1}^{n}(1-2e^{-i\kappa\Delta t}+e^{-2i\kappa\Delta t})\right]\\
& + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\frac{e^{\kappa(T-T_0)}-1}{n\left(e^{\kappa\Delta t}-1\right)}\left(e^{-\kappa(T-T_0)}e^{\kappa(T-T_0)/n} - e^{-\kappa(T-t)}\right) + \frac{1}{n^2}\sum_{i=1}^{n-1} i\left(1-e^{-i\kappa\Delta t}\right)\right]\\
& + \frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t) + ((n-1)^2 + (n-2)^2 + \cdots + 1)\Delta t\right]\\
= \;& \frac{\sigma_\chi^2}{2\kappa}\left[\left(e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}\right)(\varphi(T_0,T,n))^2 \right.\\
&\qquad \left. + \frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(\frac{1}{n} - \frac{2e^{-\kappa\Delta t}}{n}\frac{1-e^{-\kappa n\Delta t}}{n\left(1-e^{-\kappa\Delta t}\right)} + \frac{e^{-2\kappa\Delta t}}{n^2}\frac{1-e^{-2\kappa n\Delta t}}{1-e^{-2\kappa\Delta t}}\right)\right]\\
& + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\left(e^{-\kappa(T-T_0)}e^{\kappa(T-T_0)/n} - e^{-\kappa(T-t)}\right)\varphi(T_0,T,n) + \frac{1}{n^2}\sum_{i=1}^{n-1} i\left(1-e^{-i\kappa\Delta t}\right)\right]\\
& + \frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t) + ((n-1)^2 + (n-2)^2 + \cdots + 1)\Delta t\right]\\
= \;& \frac{\sigma_\chi^2}{2\kappa}\left[\left(e^{-2\kappa(T-T_0)} - e^{-2\kappa(T-t)}\right)(\varphi(T_0,T,n))^2 \right.\\
&\qquad \left. + \frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(\frac{1}{n} - \frac{2e^{-\kappa(T-T_0)}}{n}\varphi(T_0,T,n) + \frac{1}{n^2}\frac{1-e^{-2\kappa n\Delta t}}{e^{2\kappa\Delta t}-1}\right)\right]\\
& + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\left(e^{-\kappa(T-T_0)}e^{\kappa(T-T_0)/n} - e^{-\kappa(T-t)}\right)\varphi(T_0,T,n) + \frac{1}{n^2}\sum_{i=1}^{n-1} i\left(1-e^{-i\kappa\Delta t}\right)\right]\\
& + \frac{\sigma_\xi^2}{n^2}\left[n^2(t_1-t) + ((n-1)^2 + (n-2)^2 + \cdots + 1)\Delta t\right]\\
= \;& :\sigma_A^2(t,T_0,T,n).
\end{aligned}
$$

Formulae 5.21 and 5.22 (case $t \geq T_0$)

Now consider the case where we would like to price future during delivery period, i.e. $T_0 < t \le T$. Let i^* to be such that $t_{i^*-1} < t \le t_{i^*}$. Then

$$
\begin{aligned}
\mathbb{E}_t^*[A_n] &= \mathbb{E}_t^*\left\{\ln\left[(S(t_1)\,S(t_2)\ldots S(t_n))^{1/n}\right]\right\} \\
&= \mathbb{E}_t^*\left[\frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_n)\right)\right] \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1})\right) + \frac{1}{n}\left[\mathbb{E}_t^*[\ln S_{i^*}] + \cdots + \mathbb{E}_t^*[\ln S_n]\right] \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1})\right) + \frac{1}{n}\left(h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \frac{1}{n}\left[-\frac{\lambda_\chi}{\kappa} + e^{-\kappa(t_{i^*}-t)}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) + \xi(t) + \mu_\xi^*(t_{i^*} - t) + \ldots\right. \\
&\qquad \left. + \left(-\frac{\lambda_\chi}{\kappa} + e^{-\kappa(t_n-t)}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) + \xi(t) + \mu_\xi^*(t_n - t)\right)\right] \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right)\left(e^{-\kappa(t_{i^*}-t)} + \cdots + e^{-\kappa(t_n-t)}\right) \\
&\quad + \frac{n-i^*+1}{n}\left(-\frac{\lambda_\chi}{\kappa} + \xi(t)\right) + \frac{\mu_\xi^*}{n}\left((T_0 - t)\,(n - i^* + 1) + \sum_{i=i^*}^{n} i\Delta t\right) \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi \sigma_\chi}{\kappa}\right) e^{-\kappa(t_{i^*}-t)} \sum_{i=0}^{n-i^*} e^{-i\kappa\Delta t} + \frac{n-i^*+1}{n}\left(-\frac{\lambda_\chi \sigma_\chi}{\kappa} + \xi(t)\right) \\
&\quad + \frac{\mu_\xi^*}{n}\left((T_0 - t)\,(n - i^* + 1) + \frac{(i^*+n)(n-i^*+1)}{2}\Delta t\right) \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi \sigma_\chi}{\kappa}\right) e^{-\kappa(t_{i^*}-t)} \frac{1 - e^{-(n-i^*+1)\kappa\Delta t}}{1 - e^{-\kappa\Delta t}} \\
&\quad + \frac{n-i^*+1}{n}\left(-\frac{\lambda_\chi \sigma_\chi}{\kappa} + \xi(t) + \mu_\xi^*\left[(T_0 - t) + \frac{(i^*+n)}{2}\Delta t\right]\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \frac{1}{n}\left(\chi(t) + \frac{\lambda_\chi \sigma_\chi}{\kappa}\right) e^{-\kappa(t_{i^*}-t)} \frac{1 - e^{-(n-i^*+1)\kappa\Delta t}}{1 - e^{-\kappa\Delta t}} \\
&\quad + \frac{n-i^*+1}{n}\left(-\frac{\lambda_\chi \sigma_\chi}{\kappa} + \xi(t) + \mu_\xi^*\left[(T_0 - t) + \frac{(i^*+n)}{2}\Delta t\right]\right) \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1})\right) - \frac{n-i^*+1}{n}\frac{\lambda_\chi}{\kappa} \\
&\quad + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) \frac{e^{\kappa(n-i^*+1)\Delta t} - 1}{n(e^{\kappa\Delta t} - 1)} e^{-\kappa(T-t)} + \frac{n-i^*+1}{n}\xi(t) \\
&\quad + \frac{n-i^*+1}{n}\left[(T-t) - \frac{n-i^*}{2}\Delta t\right]\mu_\xi^* + \frac{1}{n}\left(h(t_{i^*}) + \cdots + h(t_n)\right) \\
&= \frac{1}{n}\left(\ln S(t_1) + \ln S(t_2) + \cdots + \ln S(t_{i^*-1}) + h(t_{i^*}) + \cdots + h(t_n)\right) \\
&\quad + \left(\chi(t) + \frac{\lambda_\chi}{\kappa}\right) e^{-\kappa(T-t)} \varphi^*(T_0, T, n) \\
&\quad + \frac{n-i^*+1}{n}\left(\xi(t) - \frac{\lambda_\chi}{\kappa} + \left[T - t - \frac{n-i^*}{2}\Delta t\right]\mu_\xi^*\right) \\
&= : \bar{m}_A(t, T_0, T, n),
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}_t^*\left[A_n\right] &= \frac{1}{n^2}\frac{\sigma_\chi^2}{2\kappa}\left[\left(1 - e^{-2\kappa(t_{i^*}-t)}\right)\left(\frac{1 - e^{-(n-i^*+1)\kappa\Delta t}}{1 - e^{-\kappa\Delta t}}\right)^2\right. \\
&\qquad \left. + \left(1 - e^{-2\kappa\Delta t}\right)\left(\frac{1 - e^{-(n-i^*)\kappa\Delta t}}{1 - e^{-\kappa\Delta t}}\right)^2 + \cdots + \left(1 - e^{-2\kappa\Delta t}\right)\right] \\
&\quad + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa n^2}\left[(n-i^*+1)\frac{1 - e^{-(n-i^*+1)\kappa\Delta t}}{1 - e^{-\kappa\Delta t}}\left(1 - e^{-\kappa(t_{i^*}-t)}\right)\right. \\
&\qquad \left. + (n-i^*)\left(1 - e^{-(n-i^*)\kappa\Delta t}\right) + \cdots + \left(1 - e^{-\kappa\Delta t}\right)\right] \\
&\quad + \frac{\sigma_\xi^2}{n^2}\left[(n-i^*+1)^2(t_{i^*} - t) + ((n-i^*)^2 + (n-i^*-1)^2 + \cdots + 1)\Delta t\right]
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{\sigma_\chi^2}{2\kappa}\Big[\left(e^{-2\kappa(n-i^*+1)\Delta t} - e^{-2\kappa(T-t)}\right)\varphi^*(T_0,T,n) \\
& \qquad\qquad + \frac{1+e^{-\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\left(\frac{n-i^*+1}{n^2} - 2\frac{e^{-\kappa(n-i^*+1)\Delta t}}{n}\varphi^*(T_0,T,n)\right)\Big] \\
& + \frac{2\rho_{\chi\xi}\sigma_\chi\sigma_\xi}{\kappa}\left[\frac{n-i^*+1}{n}\varphi^*(T_0,T,n)\left(e^{-\kappa(n-i^*)\Delta t} - e^{-\kappa(T-t)}\right) + \frac{1}{n^2}\sum_{i=1}^{n-i^*} i(1-e^{-\kappa i\Delta t})\right] \\
& + \frac{\sigma_\xi^2}{n^2}\left[(n-i^*+1)^2(t_{i^*}-t) + ((n-i^*)^2 + (n-i^*-1)^2 + \cdots + 1)\Delta t\right] \\
= \quad & :\bar{\sigma}_A^2(t,T_0,T,n),
\end{aligned}
$$

where

$$
\varphi^*(T_0,T,n) = \left(\frac{e^{\kappa(n-i^*-1)\Delta t} - 1}{n\left(e^{\kappa\Delta t}-1\right)}\right).
$$

Appendix B

Formulae 5.26, 5.27 and 5.29(case $t < T_0$)

Using (5.5) and (5.6) we get:

$$\begin{aligned}
\mathbb{E}_t\left[A_n\right] &= \mathbb{E}_t\left[\frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\cdots+\ln S(t_n)\right)\right] \\
&= \frac{1}{n}\Big\{\big(e^{-\kappa(t_1-t)}\chi(t)+\xi(t)+\mu_\xi(t_1-t)\big)+\big(e^{-\kappa(t_2-t)}\chi(t)+\xi(t)+\mu_\xi(t_2-t)\big)+\dots \\
&\qquad +\big(e^{-\kappa(t_n-t)}\chi(t)+\xi(t)+\mu_\xi(t_n-t)\big)\Big\}+\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \\
&= \frac{1}{n}\sum_{i=1}^{n} e^{-\kappa(t_i-t)}\chi(t)+\xi(t)+\frac{(t_1-t)+(t_1+\Delta t-t)+\cdots+(t_n+(n-1)\Delta t-t)}{n}\mu_\xi \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \\
&= \frac{1}{n}e^{-\kappa(t_1-t)}\sum_{i=0}^{n-1} e^{-i\kappa\Delta t}\chi(t)+\xi(t)+\left(T_0-t+\frac{1}{n}\sum_{i=1}^{n} i\Delta t\right)\mu_\xi \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \\
&= \frac{1}{n}e^{-\kappa(t_1-t)}\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\chi(t)+\xi(t)+\left(T_0-t+\frac{n+1}{2}\Delta t\right)\mu_\xi \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)]
\end{aligned}$$

$$
\begin{aligned}
&= e^{-\kappa(T-t)} \frac{e^{n\kappa\Delta t}-1}{n\left(e^{\kappa\Delta t}-1\right)}\chi(t)+\xi(t)+\left(T-t-\frac{n-1}{2}\Delta t\right)\mu_{\xi} \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)] \\
&= e^{-\kappa(T-t)}\varphi(T_0,T,n)\,\chi(t)+\xi(t)+\left(T-t-\frac{n-1}{2}\Delta t\right)\mu_{\xi} \\
&\quad +\frac{1}{n}[h(t_1)+h(t_2)+\cdots+h(t_n)]
\end{aligned}
$$

$$
\mathbb{V}\mathrm{ar}_t(A_n) = \mathbb{V}\mathrm{ar}^*_t(A_n) = \sigma^2(t,T_0,T,n),
$$

and $\mathbb{E}_t\left[G_n\right]$ can be calculated as before

$$
\mathbb{E}_t\left[G_n\right] = \exp\left\{\mathbb{E}_t\left[A_n\right]+\frac{1}{2}\sigma^2_A(t,T_0,T,n)\right\}.
$$

And thus term premium coefficient can be expressed as

$$
\begin{aligned}
R &= \log\left(\frac{\mathbb{E}^*_t\left[G_n\right]}{\mathbb{E}_t\left[G_n\right]}\right) = \log\left(\frac{\exp\left\{m_A(t,T_0,T,n)+\frac{1}{2}\sigma^2_A(t,T_0,T,n)\right\}}{\exp\left\{\mathbb{E}_t\left[A_n\right]+\frac{1}{2}\sigma^2_A(t,T_0,T,n)\right\}}\right) \\
&= m_A(t,T_0,T,n)-\mathbb{E}_t\left[A_n\right] \\
&= \left[-\frac{\lambda_\chi}{\kappa}+\frac{1}{n}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)e^{-\kappa(t_1-t)}\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right. \\
&\quad \left.+\xi(t)+\left(T_0-t+\frac{(n+1)}{2}\Delta t\right)\mu^*_{\xi}\right] \\
&\quad -\left[\frac{1}{n}e^{-\kappa(t_1-t)}\frac{1-e^{-n\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\chi(t)+\xi(t)+\left(T_0-t+\frac{n+1}{2}\Delta t\right)\mu_{\xi}\right] \\
&= -\frac{\lambda_\chi}{\kappa}\left(1-e^{-\kappa(T-t)}\varphi(T_0,T,n)\right)-\lambda_{\xi}\left(T-t-\frac{(n-1)}{2}\Delta t\right).
\end{aligned}
$$

Formulae 5.30, 5.31 and 5.32(case $t \geq T_0$)

For the case where $T_0 < t \leq T$ the calculations are almost the same as in previous section, we can write

$$
\begin{aligned}
\mathbb{E}_t\left[A_n\right] &= \frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\dots+\ln S(t_{i^*-1})+h(t_{i^*})+\dots+h(t_n)\right)\\
&+\frac{n-i^*+1}{n}\left[e^{-\kappa(t_{i^*}-t)}\chi(t)+\xi(t)+\mu_\xi(t_{i^*}-t)\right]\\
&+\frac{n-i^*}{n}\left[e^{-\kappa(t_{i^*+1}-t)}\chi(t)+\xi(t)+\mu_\xi(t_{i^*+1}-t)\right]+\dots\\
&+\frac{1}{n}\left[e^{-\kappa(t_n-t)}\chi(t)+\xi(t)+\mu_\xi(t_n-t)\right]\\
&= \frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\dots+\ln S(t_{i^*-1})+h(t_{i^*})+\dots+h(t_n)\right)\\
&+\frac{1}{n}e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\chi(t)\\
&+\frac{n-i^*+1}{n}\left(\xi(t)+\mu_\xi\left[(T_0-t)+\frac{(i^*+n)}{2}\right]\Delta t\right)\\
\mathbb{V}ar_t(A_n) &= \mathbb{V}ar_t^*(A_n)=\bar{\sigma}_A^2(t,T_0,T,n),
\end{aligned}
$$

and thus term premium coefficient in this case is

$$
\begin{aligned}
R &= \log\left(\frac{\mathbb{E}_t^*\left[G_n\right]}{\mathbb{E}_t\left[G_n\right]}\right)=\log\left(\frac{\exp\left\{\bar{m}(t,T_0,T,n)+\frac{1}{2}\bar{\sigma}_A^2(t,T_0,T,n)\right\}}{\exp\left\{\mathbb{E}_t\left[A_n\right]+\frac{1}{2}\bar{\sigma}_A^2(t,T_0,T,n)\right\}}\right)\\
&= \bar{m}_A(t,T_0,T,n)-\mathbb{E}_t\left[A_n\right]=\left[\frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\dots+\ln S(t_{i^*-1})\right)\right.\\
&\qquad\left.+\frac{n-i^*+1}{n}\left(-\frac{\lambda_\chi}{\kappa}+\xi(t)+\mu_\xi^*\left[(T_0-t)+\frac{(i^*+n)}{2}\Delta t\right]\right)\right]\\
&\quad-\frac{1}{n}\left(\ln S(t_1)+\ln S(t_2)+\dots+\ln S(t_{i^*-1})\right)-\frac{1}{n}e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\chi(t)\\
&\quad+\frac{1}{n}\left(\chi(t)+\frac{\lambda_\chi}{\kappa}\right)e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\\
&\quad-\frac{n-i^*+1}{n}\left(\xi(t)+\mu_\xi\left[(T_0-t)+\frac{(i^*+n)}{2}\right]\Delta t\right)
\end{aligned}
$$

$$= \quad -\frac{\lambda_\chi}{\kappa}\left(\frac{n-i^*+1}{n} - \frac{1}{n}e^{-\kappa(t_{i^*}-t)}\frac{1-e^{-(n-i^*+1)\kappa\Delta t}}{1-e^{-\kappa\Delta t}}\right)$$

$$-\lambda_\xi \frac{n-i^*+1}{n}\left[(T_0-t) + \frac{(i^*+n)}{2}\Delta t\right].$$

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