Contributions to Set Game Theory

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Chapter 1

Set games: the modelling part

Set games are a rather new type of cooperative games. They were introduced in an internal report by Hoede [38] in 1992. The first thesis on the subject was written by Aarts [1] in 1994. Chapter 7 of that thesis gave a first account of set games, also published in two papers by Aarts, Funaki and Hoede [2] [3] in 1997 and 2000. As the thesis of Aarts is the only source with an introduction character in the literature, we decided to give a summary of his Chapter 7 in this first chapter, that rather strictly follows his account. In this way, we hope that the thesis is self-contained and provides the reader with everything known about set games sofar. The essentially new aspect of set games is that the worth of a coalition is expressed by a set instead of by a real number, as for TU-games. For this class of games we define the notion of value, being a solution concept and also several axioms, which are similar to those for TU-games (Transferable Utility games). Furthermore, we study values for set games that satisfy the so-called additivity axiom, which is the analog of the additivity axiom of the Shapley value for the classical cooperative games.

1.1 Introduction

Game theory deals with mathematical models of situations of conflict and cooperation. A conflictive and/or cooperative situation arises naturally when two or more individuals (players) interact. The interaction between the players leads to various payoffs over which each player has his own preferences. Any player tries to obtain his best payoff but the other players may also influence the resulting payoff. The theory of games attempts to put the conflict and cooperative situations into mathematical models and then analyzes the models. Roughly speaking the theory of games can be regarded as consisting of two parts, a modelling theory and a solution theory.

Concerning the modelling part, by the mathematical models both conflict and cooperative situations are described. Most of the models studied in the mathematical theory of games use, more or less, one of the following three abstract forms: the extensive (or tree) form, the normal (or strategic) form and the characteristic function (or coalitional) form. The mathematical theory of games has been developed by von Neumann since 1928, but the fundamental modelling approach of games theory was presented for the first time in the classical book "theory of games and economic behavior" by von Neumann and Morgenstern (1944). The mathematical models are described by means of e.g. the rules, the strategic possibilities of the players, the potential payoffs to the players and the preferences of the players over the set of potential payoffs. According to the rules, it is allowed or forbidden that the players communicate with each other and make binding agreements with respect to how they correlate their actions. The cooperative (noncooperative) game theory deals with the situations in which cooperation between the players is allowed (forbidden).

Concerning the solution part, the resulting payoffs to the players are determined according to certain solution concepts. The objectives of a solution theory can be different and hence it is not surprising that several distinct solution concepts have been developed on the same modelling theory.

For TU-games, the literature provides many methods, called solution concepts, that prescribe for each game how the worth of the grand coalition should be divided among the players. Some solution concepts, like the core (Gillies [35]), the kernel (Davis & Maschler [18]), or the bargaining set (Aumann & Maschler [5]) might suggest more than one allocation. Others, like the Shapley value (Shapley [64]), the nucleolus (Schmeidler [63]) and the τ-value (Tijs [74]) prescribe for each game, for which the concept is defined, exactly one allocation. Such single-valued solution concepts are called values.

This monograph is devoted to one type of cooperation in characteristic function form, called set game (Hoede [38]), and solution concepts. In the following section, the mathematical model of a set game in characteristic function form is described and some examples of set games are given. Various axioms for solution concepts of set games in characteristic function form are described as well.

1.2 Set games: definitions and examples

In cooperative game theory, situations are studied in which a group of agents can decide to work alone, cooperate in subgroups (called coalitions) or work all together in the grand coalition. The benefits that they can gain in all these cases is described by the so-called characteristic function, which assigns to each coalition a real number. A fundamental question to which much attention has been paid in cooperative game theory is the following. Supposing that all agents decide to work together in the grand coalition, how should the reward obtained by this cooperation be divided among them? Several proposals, called solution concepts, to this division problem have been made. Among others, we mention in this context the one-point solutions (called values) like the Shapley value, nucleolus and τ-value. Also several properties (called axioms) that solution concepts can have were introduced. Without going into details we mention here the standard axioms of symmetry, additivity, efficiency and the null player property. Among the axioms mentioned above, the Shapley value satisfies all four axioms (Shapley [64]) whereas the nucleolus and the τ-value possess the symmetry, efficiency and the null player property (Maschler & Peleg [45]; Tijs [74]). If a value turns out to be the unique value that satisfies a certain collection of axioms we say that this value is characterized by these axioms. The axioms of symmetry, additivity, efficiency and null player property in fact characterize the Shapley value (Shapley [64]).

In cooperative game theory triples $(N, v, \mathbb{R})$ are considered, where N is the finite set of players, $N := \{1, 2, \cdots, n\}$ $(n \geq 2)$, $\mathbb{R}$ is the set of real numbers and v is a *characteristic function* v: $2^N \to \mathbb{R}$, where $2^N = \{S | S \subseteq$

$N\}$ represents the collection of all subsets of the players set N. The triple $(N, v, \mathbb{R})$ is called a cooperative game in characteristic function form or TU-game. Usually one defines $v(\emptyset) := 0$. Subsets S of N are called *coalitions*, N is called the *grand* coalition. The real number $v(S)$ is called the *worth* of coalition S. *Allocations* are payoff vectors $x = (x_i)_{i \in N} \in \mathbb{R}^n$, where its i-th real component x_i ($i \in N$) is interpreted as the amount that is paid to player i if the division takes place according to allocation x. Of particular interest is the set $\{x \in \mathbb{R}^n | x(N) = v(N)\}$, where $x(N) = \sum_{i=1}^n x_i$, the elements of which are called *efficient* allocations. A solution concept f is a method that determines a set of allocations for any cooperative game $(N, v, \mathbb{R})$. In case $f(N, v, \mathbb{R})$ is a singleton for any cooperative game $(N, v, \mathbb{R})$, f is called a *value*.

The most well-known value for TU-games is the value f defined by

$$f_i(N, v) := \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{(n-t)!(t-1)!}{n!}[v(T) - v(T \setminus \{i\})] \quad \text{for all } i \in N,$$

where $t = |T|$ is the cardinality of the set T, which was introduced by Shapley [64]. The Shapley value was originally characterized by four axioms, called additivity, efficiency, null player property and equal treatment property.

Definition 1.2.1. A *set game*, is a triple $(N, v, \mathcal{U})$, where N is a finite player set, $\mathcal{U}$ denotes an abstract set, called *universe*, v is a mapping v: $2^N \to 2^{\mathcal{U}}$, so the worth $v(S)$ of a coalition S is a subset of the universe $\mathcal{U}$.

If no confusion arises, we call (N, v) or v itself a set game. We suppose that $v(\emptyset) := \emptyset$ (the empty coalition has no worth), analogous to $v(\emptyset) = 0$ for TU-games.

Possible interpretations of $\mathcal{U}$ and v are the following. $\mathcal{U}$ can represent a countable set of items and $v(S)$ the (sub)set of items that can be obtained (are needed, preferred, owned) by coalition S if its members cooperate. Also $\mathcal{U}$ could represent an (infinite) amount of some infinitely divisible item (like air or water), indicated with $\mathbb{R}$ and $v(S)$ some indication about the quantity of this item that can be obtained (is needed, preferred, owned) by coalition S.

To give examples of set games, however, it is more convenient to assume that the universe is a countable, even a finite set, which we will do in the following five examples.

Example 1.2.2. (Aarts, Funaki & Hoede [3]) Antony, Bill and Charles are three friends who often play with one another. If Antony and Bill meet they like to play chess, badminton, cards, fly a kite or solve a jig-saw puzzle. Antony and Charles most of times play badminton, monopoly, cards or go out fishing. Bill and Charles play badminton, monopoly, soccer or fly a kite and if they are all together they play cards, soccer or a game of goose. Individually, Antony likes to play solitaire, solve a jig-saw puzzle or read a book; Bill reads a book or goes out fishing; and Charles flies a kite, reads a book or solves a jig-saw. Suppose that the following set of eleven items is available to be used by the three boys: a game of chess, goose, monopoly, solitaire and cards, a badminton set, a football, a kite , a jig-saw puzzle, a reading book and a set of fishing gear. The problem is which boy should be allocated which items. This situation can be modelled into a set game where the worth of a coalition corresponds to the set of items that is preferred by that coalition.

Example 1.2.3. (Aarts, Funaki & Hoede [3]) N represents a group of employees in a certain company. Each subgroup has its own task in the company and in order to fulfill such a task a part of the subgroup should possess certain skills or knowledge (the size of this part could depend on the particular skill). Therefore the employees should attend courses corresponding to the skills that they should acquire. Let $\mathcal{U}$ denote the set of the courses and let $v(S)$ denote the courses that must be attended by (some of) the members of coalition S in order to make coalition S capable for her task. In this situation we must deal with the problem of allocating courses to employees: which employees should attend which courses to assure that all coalitions can fulfill their tasks in the company (and preferably in such a way that no employee has to attend superfluous courses). Notice that, in contradistinction with Example 1.2.1, courses can be allocated to more than one player, so an allocation is not necessarily a partition of the universe.

Example 1.2.4.(Aarts, Funaki & Hoede [3])
Gambarelli and Owen [33] introduced a model concerning a set of shareholders $N = \{1, 2, 3, ..., n\}$ and a set of firms $M = \{f_1, f_2, ..., f_m\}$. For each firm f_j

they define a simple game with a collection of winning coalitions V_j. $S \in V_j$ means that coalitions can control firm f_j. Combining these m simple games we can define the set game $(N, v, \mathcal{U})$ by $\mathcal{U} = M$ and $v(S) = \{f_j | S \in V_j\}$, i.e., the worth of coalition S equals the subset of all firms that can be controlled by S. Set games can therefore be used to analyze this model.

Example 1.2.5. Analogous to the treatment of the bankruptcy situation in the field of cooperative game theory, let us introduce its set game counterpart. With the given *claim sets* $D_i \subseteq \mathcal{U}$, $i \in N$, of the *creditors*, and the *estate set* $E \subseteq \mathcal{U}$ of the bankrupt firm, there is associated the *bankruptcy set game* (N, v) defined to be $v(S) := \left[\bigcup_{j \in S} D_j \right] \cap E$ for all $S \subseteq N$, $S \neq \emptyset$. In words, the worth of coalition S consists of those items that are claimed by at least one member of S, provided the item belongs to the estate set.

Example 1.2.6. Set games come forward in a natural way in cost sharing problems, in which basic units are considered. A standard example is the landing strip cost sharing problem. Let a landing strip consist of three parts a, b and c, the basic units, and let P_1, P_2 and P_3 be three planes using the landing strips. If P_1 only uses part a, P_2 only uses parts a and b and P_3, a large plane, has to use a, b and c, then the cost sharing problem is to allocate costs C_a, C_b and C_c to the three planes. This problem can be modelled as a set game problem. The worths of the coalitions are $v(\emptyset) = \emptyset$, $v(\{P_1\}) = \{a\}$, $v(\{P_2\}) = \{a, b\}$, $v(\{P_3\}) = \{a, b, c\}$, $v(\{P_1, P_2\}) = \{a, b\}$, $v(\{P_1, P_3\}) = \{a, b, c\}$, $v(\{P_2, P_3\}) = \{a, b, c\}$, $v(\{P_1, P_2, P_3\}) = \{a, b, c\}$. These worths simply express the parts of the landing strips needed by players of a coalition. Suppose now that we have an allocation value giving $A(P_1) = \{a\}$, $A(P_2) = \{a, b\}$, and $A(P_3) = \{a, b, c\}$. Then this solution for the set game gives a cost sharing rule by the argument that all three players should pay for a, P_2 and P_3 should pay for b and P_3 should pay for c. Note that this reasoning did not take into account the actual values C_a, C_b and C_c. By the solution described P_1 would have to pay $\frac{1}{3}C_a$, P_2 would have to pay $\frac{1}{3}C_a + \frac{1}{2}C_b$, P_3 would have to pay $\frac{1}{3}C_a + \frac{1}{2}C_b + C_c$.

The reader familiar with cooperative game theory is warned that several aspects of the theory of normal TU-games will be of a different nature in set game theory. We will only mention a few of the most remarkable differences here.

In normal cooperative game theory the partitioning of the worth $v(N)$ of the grand coalition over the players is a standard aspect. In set game theory allocations of elements to players need not have an empty intersection. This means that the interpretation of an allocation is to be given carefully. As an example let us consider the concept of preferences of players. Any player will probably prefer to receive as many items as possible if they represent rewards, or as few items as possible if they represent costs. That is his first consideration. He might also hope to share one particular item with as few persons as possible, or as many persons as possible, for the same reasons.

Compare the situation in which an allocation is determined without taking into account preferences and the situation in which preferences play a role in determining the allocation. When players share an element of the universe in their allocation, this may be an unwanted feature by the players, but may be an essential feature for somebody from the outside, who has to distribute costs over the players, like in Example 1.2.6. In the first situation, the preferences of the players may be taken into account in the choice of the solution concept. In the second situation the preferences of the players may be neglected in the choice of the solution concept. This difference between solution concepts, determined by the set of players, and solution concepts, imposed on the set of players, also comes forward in the concept of fairness. Especially cost sharing methods that have been developed in real life situations reflect fairness principles, as felt by the players.

It should be remarked that normal game theory and set game theory are not as far apart as one might think. Recently Bumb and Hoede[11] introduced the standard set game of a normal cooperative game. They showed that any normal cooperative game has a combinatorial structure that can be described by a set game.

In this thesis the accent lies on the more technical aspects coming forward in trying to derive results analogous to those in normal game theory.

In this monograph the so-called elementary set games and the unanimity set games (which are the analogs of the elementary TU-games and the unanimity TU-games) will play an important role.

Definition 1.2.7. Let $\mathcal{U}$ be the universe. For any $T \subseteq N$, $T \neq \emptyset$, the *elementary set game* $(N, E_T, \mathcal{U})$, or shortly E_T associated with coalition T, is defined by

$$E_T(S) = \begin{cases} \mathcal{U} & \text{if} \quad S = T, \\ \emptyset & \text{if} \quad S \neq T, \end{cases}$$

whereas the *unanimity set game* $(N, U_T, \mathcal{U})$, or shortly U_T associated with coalition T, is defined by

$$U_T(S) := \begin{cases} \mathcal{U} & \text{if} \quad S \supseteq T, \\ \emptyset & \text{if} \quad S \not\supseteq T. \end{cases}$$

Let, for any subset $A \subseteq \mathcal{U}$ of items, and any set game v, the A-restricted set game $A \cap v$ be given by $(A \cap v)(S) := A \cap v(S)$ for all $S \subseteq N$. Any set game v can be expressed in terms of elementary set games as follows:

$$v = \bigcup_{\substack{T \subseteq N \\ T \neq \emptyset}} (v(T) \cap E_T). \tag{1.2.1}$$

We will use this fact several times in this section and the next section. For TU-games one often uses the collection of unanimity games. The relationship between elementary TU-games and unanimity TU-games has been studied by Dubey [26]. Elementary set games can be expressed in terms of unanimity set games:

$$E_T = U_T - \bigcup_{Q \supset T} U_Q \tag{1.2.2}$$

Combining (1.2.1) and (1.2.2) we can obtain that each set game can be expressed in terms of unanimity set games. However, this fact is not as useful as in the TU-world. The reason for this is the following. If we know the expressions of a value satisfying additivity for all unanimity set games, we cannot apply (1.2.1) in order to get the expression of this value for the elementary set games. In order to do so a value also has to satisfy the subtraction(or separability) axiom, introduced below. Note that in discussing subtraction we almost always mean set subtraction. However, we will use both the minus sign "-", for subtracting coalition worth, and the backward slash "̈for subtracting one element set.

1.3 Axioms of values for set games

Like in TU-games, *allocation* $x = (x_i)_{i \in N} \in (2^{\mathcal{U}})^n$ $(n = |N|)$ can be considered in set games. Here the set $x_i \subseteq \mathcal{U}$ for all $i \in N$, i.e., each player is allocated a certain subset of the universe. The interpretation of an allocation x might be that player i gets (or can make use of) the items in the set x_i. Also for set games we can consider solution concepts which determine sets of allocations and again we will call a solution concept $f(N, v, \mathcal{U}) = (f_i(N, v, \mathcal{U}))_{i \in N} \in (2^{\mathcal{U}})^n$, or shortly $f(N, v)$, *value* if $f(N, v, \mathcal{U})$ is a singleton for all set games $(N, v, \mathcal{U})$. The main issue of this dissertation is to present some allocations for set games. In the following axioms of values for set games are introduced, which are similar to those for TU-games. For values $f(N, v)$ it seems reasonable to require the following partitioning condition.

$$\text{Partitioning Condition}: f_i(N, v) \cap f_j(N, v) = \emptyset \quad \text{for all } i, j \in N,\ i \neq j.$$

This condition states that an item cannot be allocated to more than one player (cf. Example 1.2.1). However, we will not study values that satisfy the partitioning condition in this monograph for reasons that we will explain below. Now, we consider the following potential axioms for any value $f(N, v)$:

Efficiency:	$\bigcup_{i \in N} f_i(N, v) = v(N)$,
Weak Efficiency:	if $v(N) = \mathcal{U}$, then $\bigcup_{i \in N} f_i(N, v) = \mathcal{U}$,
Global Efficiency:	$\bigcup_{i \in N} f_i(N, v) = \bigcup_{S \subseteq N} v(S)$.
Weak Global Efficiency:	if $v(S) = \mathcal{U}$ for some $S \subseteq N$,
	then $\bigcup_{i \in N} f_i(N, v) = \mathcal{U}$.

Efficiency is a standard axiom and is similar to the efficiency condition for TU-games. For the class of monotonic games the axioms efficiency and global efficiency coincide. However, if the game is not monotonic, global efficiency might be more plausible. Next, we consider the concept of symmetry. The following two axioms are similar to those for TU-games.

Equal Treatment: if $i \neq j$, for all $S \subseteq N \setminus \{i, j\}$,
$v(S \cup \{i\}) = v(S \cup \{j\})$, then $f_i(N, v) = f_j(N, v)$.

Anonymity: for all permutations $\sigma : N \to N$, $f_{\sigma(i)}(\sigma v) = f_i(N, v)$
for all $i \in N$, where the game (σv)
is defined by $(\sigma v)(\sigma S) = v(S))$ for all $S \subseteq N$.

Equal treatment is a weaker axiom than anonymity. Concerning the partitioning condition it is remarked that for TU-games many solution concepts have the equal treatment property and the efficiency property. However for set games we have

Proposition 1.3.1 (Aarts, Funaki & Hoede [3])
There is no value satisfying the axioms of equal treatment, weak-efficiency and the partitioning condition.

Although it is certainly worth studying values that satisfy the partitioning condition, we choose to maintain the equal treatment axiom and some efficiency axiom in this monograph . So, as a consequence of Proposition 1.3.1, we drop the partitioning condition, which means that items can be allocated to more than one player. So in the sequel, elements of $\mathcal{U}$ are considered as club goods (i.e., facilities that can be used by an arbitrary number of players).

For characterizing additive values, we have to consider some more axioms. Two games v and w are called *disjoint* if $v(S) \cap w(S) = \emptyset$ for all $S \subseteq N$. Furthermore the game $v \cup w$ is defined by $(v \cup w)(S) = v(S) \cup w(S)$ for all $S \subseteq N$. We will use the following condition as an axiom.

Additivity: if v and w are disjoint games ,
then $f_i(v \cup w) = f_i(N, v) \cup f_i(N, w)$ for all $i \in N$.

We can, of course, consider a stronger additivity condition without the restriction that games must be disjoint, but the word additive fits better for disjoint games and the weaker condition suffices for our purpose.

Of the four axioms used to characterize the Shapley value for TU-games, anonymity, efficiency, additivity and the null player property, we have not yet considered the analog of the null player property.

Null Player Property: $f_i(N, v) = \emptyset$ for each player $i \in N$ that satisfies $v(S \cup \{i\}) = v(S)$ for all $S \subseteq N \setminus \{i\}$.

The analogs of four axioms used to characterize the Shapley value for TU-games do not characterize any value in the world of set games as is shown in the next proposition. First we introduce the following monotonicity axiom and state a lemma. The relevant theorem is stated, but for its exhaustive proof we refer to the original paper.

Monotonicity: if $v(S) \subseteq w(S)$ for all $S \subseteq N$, then $f_i(N, v) \subseteq f_i(N, w)$ for all $i \in N$.

Lemma 1.3.2. (Aarts, Funaki & Hoede [3])
Additivity implies monotonicity.

Theorem 1.3.3. (Aarts, Funaki & Hoede [3])
There is no value that has the equal treatment property, the monotonicity property, the weak-efficiency property and the null player property.

Since additivity and efficiency are stronger than monotonicity and weak-efficiency respectively, it is impossible to find a value for set games that satisfies the analogs of the four axioms used to characterize the Shapley value. Our object in the following is to study values for set games that do satisfy additivity. Also we want to hold on to an efficiency axiom and to the equal treatment axiom (as we already mentioned above). In that case we must drop the axiom of the null player property, by Theorem 1.2.7. The axiom of the destructive player property is introduced, which is a modification of the null player property.

Destructive Player Property: $f_i(N, v) = \emptyset$ for each player $i \in N$ that satisfies $v(S) = \emptyset$ for all $S \ni i$.

The axiom states that a player which has the destructive property of making each coalition worthless as soon as he enters, should receive nothing. In the class of monotonic games, the destructive player property is weaker than the null player property. Next, we define the analog of scalar multiplication (linearity) for set game values.

Multiplication Property: $f_i(A \cap v) = A \cap f_i(N, v)$, for all $i \in N$ and all $A \subseteq \mathcal{U}$.

The interpretation of this axiom might be as follows. If the universe is restricted to a certain subset the value should be restricted (in the natural way) to that particular subset as well.

Let us consider set game values that satisfy the axiom of additivity, which is the analog of the most important axiom of the Shapley value in the TU-world. We will discuss the characterization of additive values for set games in phases. We start with the following theorem.

Theorem 1.3.4. (Aarts, Funaki & Hoede [3])
(i) A value f that satisfies the axioms of anonymity, additivity and multiplication must have the form, for all $i \in N$,

$$f_i(v) = \bigcup_{T \ni i} (A(t) \cap v(T)) \cup \bigcup_{\substack{T \not\ni i, \\ T \neq \emptyset}} (B(t) \cap v(T)),$$

where $A(t)(t = 1, 2, \cdots, n)$ and $B(t)(t = 1, 2, \cdots, n-1)$ are subsets of $\mathcal{U}$ depending on the number $t = |T|$.
(ii) Any value of this form satisfies the axioms of anonymity, additivity and multiplication.

The value in Theorem 1.3.4 describes allocation based on the sets $A(t)$ and $B(t)$, which can be chosen freely. The interpretation of the sets $A(t)$ and $B(t)$ might be that $A(t)$ determines a fraction of the items at the disposal of the members of a coalition T containing i and $B(t)$ determines a fraction of the items 'spilled' by coalition T not containing i. $B(t)$ puts items at i's disposal without him earning these due to membership of coalitions. In other words, if a player enters a game, the $A(t)$'s determine which items he might expect to get (these items might be considered as enjoyable objects) and the $B(t)$'s determine which unpleasant items (e.g. entree costs, garbage) he has to accept at the same time.

Theorem 1.3.5. (Aarts, Funaki & Hoede [3])
(i) A value f that satisfies the axioms of anonymity, additivity, multiplication and has the destructive player property must have the form, for all $i \in N$,

$$f_i(v) = \bigcup_{T \ni i} (A(t) \cap v(T)),$$

where $A(t)(t = 1, 2, \cdots, n)$ are subsets of $\mathcal{U}$ depending on the number $t = |T|$.
(ii) Any value of this form satisfies the axioms of anonymity, additivity, multiplication and has the destructive player property.

The expression has reduced to one in which the 'spilling' by a coalition to which a player does not belong (entree costs, garbage) has been eliminated. The $A(t)$ can still be chosen freely. A final axiom which can be added to achieve uniqueness is an efficiency condition, as is stated in the next theorem.

Theorem 1.3.6. (Aarts, Funaki & Hoede [3])
The value

$$f_i(N, v) = \bigcup_{T \ni i} v(T), \qquad \text{for all } i \in N,$$

is the unique value that satisfies the axioms of anonymity, additivity, multiplication and has the destructive player property as well as the weak-efficiency property.

Remark: It can be checked easily that the value f in Theorem 1.3.6 also satisfies the axioms of global efficiency and strong additivity (additivity without the restriction of disjoint games). The statement in the theorem still holds if anonymity is replaced by the weaker axiom of equal treatment. One can prove that this value can also be characterized by the axioms of equal treatment, additivity, destructive player and global efficiency, without using multiplication.

If one considers the axioms anonymity, additivity, multiplication, destructive player property and weak-efficiency as reasonable, Theorem 1.3.6 states that one should allocate each player all items that he can get in any coalition he belongs to. Intuitively, this is what one might expect, taking into consideration the axioms: weak-efficiency and multiplication, All items that appear in the game must be allocated among the players (reduce $\mathcal{U}$ to the effective items).

Applying this value to the situation in Example 1.2.1 yields that only Antony and Bill are allowed to play chess, and only Antony can play Solitaire. The other items are available for everyone. If the value is applied to the situation in Example 1.2.2, an employee must attend all courses necessary for the functioning of the coalitions that he is a member of and, in Example 1.2.3 each shareholder can control each firm that can be controlled by a coalition that be belongs to.

The additivity axiom cannot be considered as a real analog of the additivity axiom for TU-games. To explain this, let $\tilde{f}$ be an additive value for the class of TU-games and let f be an additive value for the class of set games. For any couple of TU-games $\tilde{v}$ and $\tilde{w}$ define the game $\tilde{v}-\tilde{w}$ by $(\tilde{v}-\tilde{w})(S) = \tilde{v}(S)-\tilde{w}(S)$ for all $S \subseteq N$, also define the set game $v - w$ in the same way. Then we have

$$\tilde{f}_i(\tilde{v}) = \tilde{f}_i((\tilde{v} - \tilde{w}) + \tilde{w}) = \tilde{f}_i(\tilde{v} - \tilde{w}) + \tilde{f}_i(\tilde{w}) \quad \text{for all } i \in N,$$

as well as

$$f_i(N, v) = f_i((v - w) \cup w) = f_i(v - w) \cup f_i(N, w), \quad \text{for all } i \in N.$$

In the TU-case this automatically implies that

$$\tilde{f}_i(\tilde{v} - \tilde{w}) = \tilde{f}_i(\tilde{v}) - \tilde{f}_i(\tilde{w}), \qquad \text{for all } i \in N.$$

However, in the world of set games the analogous conclusion only holds if $f_i(v - w) \cap f_i(N, w) = \emptyset$ for all $i \in N$. This encourages to consider the following axioms as possible properties for additive values for set games.

Subtraction: $f_i(v-w) = f_i(N,v) - f_i(N,w)$ for all $i \in N$ and all games v and w with $v(S) \supseteq w(S)$ for all $S \subseteq N$, where the game $v-w$ is defined by $(v-w)(S) = v(S) - w(S)$ for all $S \subseteq N$.

Separability: $f_i(N,v) \cap f_i(N,w) = \emptyset$ for all $i \in N$ and all disjoint games v and w.

Notice that the games v and w in the axiom of separability must be disjoint, which means *coalition wise* disjoint (i.e., $v(S) \cap w(S) = \emptyset$, for all $S \subseteq N$). So $v(S) \cap w(T)$ need not be empty if $S \neq T$.

First we state the following relationship concerning the axioms of subtraction, additivity and separability.

Proposition 1.3.7. (Aarts, Funaki & Hoede [3])
A value satisfies the axiom of subtraction if and only if it satisfies the axioms of additivity and separability.

With Proposition 1.3.7 in mind, we consider the set of axioms anonymity, additivity, multiplication and separability. Just like above we now gradually extend this set of axioms to obtain a characterization of a value.

Theorem 1.3.8 (Aarts, Funaki & Hoede [3])
(i) A value f that satisfies the axioms of anonymity, additivity, multiplication, and separability must have the form, for all $i \in N$,

$$f_i(v) = (A(1) \cap v(\{i\})) \cup (v(N) \cap A(n)) \cup (v(N \setminus \{i\}) \cap B(n-1)),$$

where $A(1), A(n)$ and $B(n-1)$ are mutually disjoint subsets of $\mathcal{U}$.
(ii) Any value of this form satisfies the axioms of anonymity, additivity, multiplication, and separability.

$A(1)$ and $A(n)$ determine what a one player coalition and the grand coalition might maximally contribute, provided that the one player and the grand coalition justify this allocation by their coalition worths. The spilling by the others is at most $B(n-1)$ and only contributes when justified by the coalition worth of 'the others'. In other words, the items that can be obtained by the players are divided into three disjoint subsets of the universe. The first determines the part of the allocation which is earned by a player himself, the second determines a part which is the same for all players, and which could be interpreted as a reward for joining the game, and the third subset determines the unpleasant part which each player has to accept for joining the game.

When we add the null player property or efficiency to anonymity, additivity, multiplication and separability the form of the value again becomes simpler as is shown in the next theorem.

Theorem 1.3.9: (AARTS, FUNAKI & HOEDE [3])
(*i*) A value f satisfies the axioms of anonymity, additivity, multiplication, separability and destructive player if and only if f has the form

$$f_i(v) = (A(1) \cap v(\{i\})) \cup (v(N) \cap A(n)), \qquad \text{for all } i \in N,$$

for some disjoint subsets $A(1)$and $A(n)$ of $\mathcal{U}$.
(*ii*) $f_i(N, v)$ satisfies the axioms of anonymity, additivity, multiplication, separability and has the null player property if and only if

$$f_i(v) = A(1) \cap v(\{i\}), \qquad \text{for all } i \in N,$$

for some subsets $A(1)$ of $\mathcal{U}$.
(*iii*) f satisfies the axioms of anonymity, additivity, multiplication, separability and weak-efficiency if and only if

$$f_i(v) = v(N), \qquad \text{for all } i \in N.$$

We see that the null player property removes the effects of $B(n-1)$ and $A(n)$. Weak-efficiency does lead to a value that allows all players to use all items available to the grand coalition.

Theorem 1.3.6 and 1.3.9(*iii*) give the values

$$f_i(N, v) = \bigcup_{T \ni i} v(T) \quad \text{and} \quad f_i(v) = v(N).$$

The first expression seems more natural for the interpretations that we have chosen as items that can be used by player i, depending on the items at the disposal of the coalitions to which player i belongs. The second expression is extreme in that it gives the same outcome for all players. Notice that both values coincide for monotonic set games. In case a set game is non-monotonic, these values may yield results that look somewhat strange at first sight. Consider for example the set game $(\{1,2,3\}, v, \{A,B,C\})$ given by $v(\{1\}) = \{A\}$, $v(\{2\}) = \{B\}$, $v(\{3\}) = v(\{1,3\}) = v(\{2,3\}) = \emptyset$, $v(\{1,2\}) = \{A,B,C\}$, and $v(\{1,2,3\}) = \{A,B\}$. We have $f(N,v) = (\{A,B,C\}, \{A,B,C\}, \{A,B,\})$ by Theorem 1.2.10 and $f(N,v) = (\{A,B,\}, \{A,B,\}, \{A,B,\})$ by Theorem 1.2.13(iii). Although player 3 seems a rather destructive player (since every coalition looses if he joins), still he gets the items A and B according to both values. If all items were desirable objects, this is not what one would expect. However, in this case the players will probably form the sub-games $(\{1,2\}, v, \mathcal{U})$ and $(\{3\}, v, \mathcal{U})$ since there is no use for players 1 and 2 to cooperate with 3 and form the grand coalition. Applying both values to these sub-games yields allocation $\{A,B,C\}$ to player 1 and 2 and $\emptyset$ to player 3, which is intuitively more acceptable. On the other hand, the allocations $f(N,v)$ mentioned in Theorems 1.2.10 and 1.2.13(iii) might be reasonable in the situation where item C is undesirable (e.g. entree costs). If player 3 joins coalition $\{1,2\}$ he helps these players to get rid of this item, which yields him the goods A and B as a reward. In this situation, the value $f(N,v)$ in Theorem 1.2.13(iii) seems more reasonable than in Theorem 1.2.10 since $f(N,v)$ in Theorem 1.2.13(iii) rewards not only 3 (with A and B) for the elimination of item C, but also players 1 and 2 for their cooperation with 3 (they are rid of C).

Sofar this summary of the first axiomatic results on set game values, which at the same time introduced some of the terminology needed for the results presented in the Chapters 2, 3, 4, and 5.

Chapter 2

Values for set games

An individually marginalistic value for monotonic set games, which is similar to the Shapley value for TU-games, was characterized by Aarts, *et al.* In this chapter, we propose several values for set games, which have some relationship with the marginalistic contribution with respect to any coalition containing the player, for set games. The characterization of the sub-coalitional marginalistic value, which coincides with the marginal value for monotonic set games, is shown using unanimity set games. A value analogous to the solidarity value (introduced by Nowak & Radzik) for set games, named co-marginalistic value, is proved to uniquely satisfy some axioms using elementary set games. Furthermore, we give a class of values (called semi-marginalistic values) which are generalizations. These values satisfy the axiom of global efficiency, have the equal treatment property and show some monotonicity. In order to characterize this class of values, we provide a rather different method, which is based on simple set games. In the last section, some properties of values for set games are presented.

2.1 Five examples of values for set games

Let $G^N(\mathcal{U})$ denote the space of set games with finite player set N. A *value* f on $G^N(\mathcal{U})$ is a mapping $f : G^N(\mathcal{U}) \to (2^{\mathcal{U}})^n$, which associates with any set game $(N, v) \in G^N(\mathcal{U})$ a set-valued vector $f(N, v, \mathcal{U}) = (f_i(N, v, \mathcal{U}))_{i \in N} \in (2^{\mathcal{U}})^n$, or shortly $f(N, v)$. Now we introduce five different values, studied throughout the solution theory for set games. Further, the relationships of these five values to the solution concepts called core and subcore are investigated.

Example 2.1.1. For every set game $(N, v) \in G^N(\mathcal{U})$, we say, on the one hand, that an item $x \in \mathcal{U}$ is *attainable* by player i through a certain coalition S containing i whenever the item belongs to the coalition's worth, that is $x \in v(S)$, on the other hand, we say a coalition T cannot *block* an item x whenever the item does not belong to the coalition's worth, that is $x \notin v(T)$.

(i) The *individually marginalistic* IM-value, named marginalistic value by Aarts, Funaki & Hoede [2], allocates those items that are attainable by player i, but cannot be blocked by the coalition consisting of the remaining members (different from player i). To be exact,

$$IM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \Big[v(S) - v(S \setminus \{i\}) \Big], \quad \text{for all } i \in N. \tag{2.1.1}$$

(ii) The *overall-individually marginalistic* OIM-value, as introduced by Aarts, Funaki & Hoede [2], allocates those items that are attainable by player i, but cannot be blocked by any sub-coalition with one player less. To be exact,

$$OIM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \Big[v(S) - \bigcup_{j \in S} v(S \setminus \{j\}) \Big], \quad \text{for all } i \in N. \tag{2.1.2}$$

(iii) The *sub-coalitionally marginalistic* SCM-value, named coalitional power value by Sun, Zhang & Li [69], allocates those items that are attainable by player i, but cannot be blocked by any proper sub-coalition. To be exact,

$$SCM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \Big[v(S) - \bigcup_{T \subset S} v(T) \Big], \quad \text{for all } i \in N. \tag{2.1.3}$$

(iv) The *overall-coalitionally marginalistic* OCM-value, as introduced by Driessen and Sun [24], allocates those items that are attainable by player i, but cannot be blocked by any coalition not containing i. To be exact,

$$OCM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \Big[v(S) - \bigcup_{T \subseteq N \setminus \{i\}} v(T) \Big], \quad \text{for all } i \in N. \tag{2.1.4}$$

(v) The *individually co-marginalistic* ICM-value, named co-marginalistic contribution value by Sun, Zhang, Li, Driessen & Hoede [70], allocates those items that are attainable by player i, but cannot be blocked by at least one sub-coalition with one player less. To be exact,

$$ICM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) - \bigcap_{j\in S} v(S\setminus\{j\}) \right], \quad \text{for all } i\in N. \tag{2.1.5}$$

For any set game $(N,v)\in G^N(\mathcal{U})$, among these five solutions, the ICM-value is the largest allocation in that the sequence of inclusions $ICM_i(N,v) \supseteq IM_i(N,v) \supseteq OIM_i(N,v) \supseteq SCM_i(N,v)$ holds for any player i and moreover, the inclusion $IM_i(N,v) \supseteq OCM_i(N,v)$ holds for any player i. In this setting, for any $S\subseteq N$, the underlying expressions $v(S)-v(S\setminus\{i\})$, $v(S) - \bigcup_{j\in S} v(S\setminus\{j\})$, $v(S) - \bigcup_{T\subset S} v(T)$, $v(S) - \bigcup_{T\subseteq N\setminus\{i\}} v(T)$ and $v(S) - \bigcap_{j\in S} v(S\setminus\{j\})$ respectively, are called the *marginalistic contribution of coalition* S induced by either one particular member, all members, or all sub-coalitions.

In the framework of TU-games, some solution rules in their core play an important role in the allocation problem. We are also interested in the solution concept of core for set games.

Definition 2.1.2. For any set game $(N,v)\in G^N(\mathcal{U})$, the *core* $C(N,v)$ is defined by

$$C(N,v) := \{x\in (2^{\mathcal{U}})^N \mid \bigcup_{i\in N} x_i = \bigcup_{T\subseteq N} v(T) \text{ and } \bigcup_{i\in S} x_i \supseteq \bigcup_{T\subseteq S} v(T) \quad \text{for all } S\subseteq N\}. \tag{2.1.6}$$

The core of any set game is always nonempty because it always contains the value that allocates those items that are attainable for any player. That is, $f(N,v)\in C(N,v)$ whenever $f_i(N,v) = \bigcup_{\substack{T\subseteq N\\ T\ni i}} v(T)$ for any $i\in N$ and any $(N,v)\in G^N(\mathcal{U})$. Aarts, Funaki & Hoede [2] strengthened this result by showing that the "smaller" IM-value for monotonic set games belongs to the core. As a further strengthening of these results, we claim in the following theorem that the "smaller" SCM-value is in the core. For that purpose, we prove the next preliminary lemma.

Lemma 2.1.3. For any set game $(N, v) \in G^N(\mathcal{U})$, we have

$$\bigcup_{S \subseteq R} \left[v(S) - \bigcup_{T \subset S} v(T) \right] = \bigcup_{S \subseteq R} v(S), \quad \text{for all } R \subseteq N. \tag{2.1.7}$$

Proof. Let us proceed by induction on the number $r = |R|$. If $r = 1$, the lemma clearly holds. For $r \geq 2$, by the induction hypothesis (2.1.7) applied to $r - 1$, we have

$$\begin{aligned}
&\bigcup_{S \subseteq R} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \\
= &\left[v(R) - \bigcup_{T \subset R} v(T) \right] \cup \left[\bigcup_{S \subset R} (v(S) - \bigcup_{T \subset S} v(T)) \right] \\
= &\left[v(R) - \bigcup_{i \in R} \bigcup_{T \subseteq R \setminus \{i\}} v(T) \right] \cup \left[\bigcup_{i \in R} \bigcup_{S \subseteq R \setminus \{i\}} (v(S) - \bigcup_{T \subset S} v(T)) \right] \\
= &\left[v(R) - \bigcup_{i \in R} \bigcup_{T \subseteq R \setminus \{i\}} v(T) \right] \cup \left[\bigcup_{i \in R} \bigcup_{S \subseteq R \setminus \{i\}} v(S) \right] = \bigcup_{S \subseteq R} v(S).
\end{aligned}$$

□

Theorem 2.1.4. The allocation rule given by the SCM-value for set games belongs to its core, *i.e.*, for all $(N, v) \in G^N(\mathcal{U})$,

$\bigcup_{i \in R} SCM_i(N, v) \supseteq \bigcup_{S \subseteq R} v(S)$, for all $R \subseteq N$, and $\bigcup_{i \in N} SCM_i(N, v) = \bigcup_{S \subseteq N} v(S)$.

Proof. For any $R \subseteq N$, we have

$$\begin{aligned}
\bigcup_{i \in R} SCM_i(N, v) &\overset{(2.1.3)}{=} \bigcup_{i \in R} \bigcup_{\substack{S \ni i \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \\
&= (\bigcup_{i \in R} \bigcup_{\substack{S \ni i \\ S \subseteq R}} \left[v(S) - \bigcup_{T \subset S} v(T) \right]) \cup (\bigcup_{i \in R} \bigcup_{\substack{S \ni i \\ S \not\subseteq R}} \left[v(S) - \bigcup_{T \subset S} v(T) \right]) \\
&= (\bigcup_{S \subseteq R} \left[v(S) - \bigcup_{T \subset S} v(T) \right]) \cup (\bigcup_{i \in R} \bigcup_{\substack{S \ni i \\ S \not\subseteq R}} \left[v(S) - \bigcup_{T \subset S} v(T) \right]) \\
&\overset{(2.1.7)}{=} (\bigcup_{S \subseteq R} v(S)) \cup (\bigcup_{i \in R} \bigcup_{\substack{S \ni i \\ S \not\subseteq R}} \left[v(S) - \bigcup_{T \subset S} v(T) \right]) \supseteq \bigcup_{S \subseteq R} v(S),
\end{aligned}$$

where the fourth equality follows from Lemma 2.1.3.

Furthermore,

$$\bigcup_{i\in N} SCM_i(N,v) = \bigcup_{i\in N} \bigcup_{\substack{S\subseteq N\\ S\ni i}} \Big[v(S) - \bigcup_{T\subset S} v(T)\Big] = \bigcup_{S\subseteq N} \Big[v(S) - \bigcup_{T\subset S} v(T)\Big] = \bigcup_{S\subseteq N} v(S),$$

where the third equality is due to taking $N = R$ in Lemma 2.1.3. □

Remark 2.1.5. It is an obvious consequence that the other three values, namely OIM-value, IM-value and ICM-value on $G^N(\mathcal{U})$ are in the core too. Recall the inclusions $SCM_i(N,v) \subseteq OIM_i(N,v) \subseteq IM_i(N,v) \subseteq ICM_i(N,v)$ for all $i \in N$. In view of this, it suffices to show that $SCM(N,v) \in C(N,v)$ in order to conclude that all four values belong to the core. Indeed, by Theorem 2.1.4, we read that $SCM(N,v) \in C(N,v)$.

Definition 2.1.6. Let $(N,v) \in G^N(\mathcal{U})$ be a set game.

(i) The *set of restricted globally efficient allocations* $X^*(N,v) \subseteq (2^{\mathcal{U}})^N$ of (N,v) is defined to be

$$X^*(N,v) := \{x = (x_i)_{i\in N} \in (2^{\mathcal{U}})^N \mid \bigcup_{i\in N} x_i = \Big[\bigcup_{S\subseteq N} v(S)\Big] - \Big[\bigcap_{i\in N} \bigcup_{T\subseteq N\setminus\{i\}} v(T)\Big]\}$$

(ii) The *subcore* $SUBC(N,v) \subseteq (2^{\mathcal{U}})^N$ of the set game (N,v) is defined to be

$$SUBC(N,v) := \{x \in X^*(N,v) \mid \bigcup_{i\in S} x_i \supseteq \bigcup_{T\subseteq S} v(T) \quad \text{for all } S \subset N\}.$$

Proposition 2.1.7. With every set game $(N,v) \in G^N(\mathcal{U})$ there is associated the *monotonic cover set game*$(N,w) \in G^N(\mathcal{U})$ defined to be $w(S) := \bigcup_{R\subseteq S} v(R)$ for all $S \subseteq N$. Then it holds that

(i) $v \subseteq w$, that is $v(S) \subseteq w(S)$ for all $S \subseteq N$.

(ii) (N,w) is a monotonic set game, that is $w(S) \subseteq w(T)$ for all $S \subseteq T \subseteq N$.

(iii) The OCM–value is invariant under the monotonic cover, that is $OCM(N,w) = OCM(N,v)$.

(iv) The equality $OCM(N,w) = IM(N,w)$ holds if and only if (N,w) is a weakly convex set game, that is $w(S) - w(S \setminus \{i\}) \subseteq w(N) - w(N \setminus \{i\})$, for all $S \subseteq N$ and all $i \in S$.

Proof of Proposition 2.1.7.
Let $(N,v) \in G^N(\mathcal{U})$. To prove the invariance of the OCM–value under the monotonic cover, we derive from (2.1.4) and the monotonicity of (N,w) the following:

$$OCM_i(N,v) \overset{(2.1.4)}{=} [\bigcup_{S \subseteq N} v(S)] - [\bigcup_{T \subseteq N \setminus \{i\}} v(T)] = w(N) - w(N \setminus \{i\}) \overset{(2.1.4)}{=} OCM_i(N,w),$$

for all $i \in N$. This proves part (iii). Moreover, for all $i \in N$, it holds that $IM_i(N,w) := \bigcup_{\substack{S \subseteq N, \\ S \ni i}} \Big[w(S) - w(S \setminus \{i\}) \Big] \supseteq w(N) - w(N \setminus \{i\}) = OCM_i(N,w)$.
So, the equality $IM_i(N,w) = OCM_i(N,w)$ holds if and only if $\bigcup_{\substack{S \subseteq N, \\ S \ni i}} \Big[w(S) - w(S \setminus \{i\}) \Big] \subseteq w(N) - w(N \setminus \{i\})$ or, equivalently, $w(S) - w(S \setminus \{i\}) \subseteq w(N) - w(N \setminus \{i\})$ for all $S \subseteq N$ with $i \in S$. This proves part (iv). □

Proposition 2.1.8

(i) For monotonic set games $(N,v) \in G^M(\mathcal{U})$, the OCM-value belongs to the subcore if and only if $v(S) \cap \Big[\cap_{i \in S}\, v(N \setminus \{i\}) \Big] = \emptyset$ for all $S \subset N$.

(ii) For arbitrary set games $(N,v) \in G^N(\mathcal{U})$, it holds that $OCM(N,v) \in SUBC(N,v)$ if and only if the *monotonic cover* $(N,w) \in G^M(\mathcal{U})$ satisfies $w(S) \cap \Big[\cap_{i \in S}\, w(N \setminus \{i\}) \Big] = \emptyset$ for all $S \subset N$.

Proof. Let $(N,v) \in G^M(\mathcal{U})$ be a monotonic set game. The monotonicity implies $OCM_i(N,v) = v(N) - v(N \setminus \{i\})$, for all $i \in N$, and so the subcore constraint for the OCM–value, induced by every coalition $S \subset N$, reduces as follows:

$$\bigcup_{i\in S} OCM_i(N,v) \supseteq \bigcup_{T\subseteq S} v(T) \iff \bigcup_{i\in S}\Big[v(N) - v(N\setminus\{i\})\Big] \supseteq v(S)$$

$$\iff v(S) \cap \Big[\cap_{i\in S} v(N\setminus\{i\})\Big] = \emptyset.$$

For arbitrary set games $(N,v) \in G^N(\mathcal{U})$, the monotonicity of its monotonic cover set game (N,w) yields the equalities $OCM(N,w) = OCM(N,v)$ and $SUBC(N,w) = SUBC(N,v)$. Hence, the equivalence $OCM(N,v) \in SUBC(N,v)$ $\iff OCM(N,w) \in SUBC(N,w)$ holds. Part (ii) is a direct consequence of part (i) applied to the monotonic cover (N,w). □

2.2 An individually marginalistic value for monotonic set games

In Section 1.2, we saw that additive values for set games have a structure completely different from that of the Shapley value for TU-games. The object of this section is to recall the characterization of a value for set games more similar to the Shapley value, *i.e.*, a value which is determined from the marginal contributions of the players corresponding to joining of coalitions. We will refer to it as the *individually marginalistic value* for set games. For its axiomatic characterization we use a modification of the strong monotonicity axiom of Young [75].

By Aarts, Funaki and Hoede [2], the *individually marginalistic* IM-value (named marginalistic value) is defined by

$$IM_i(N,v) = \bigcup_{\substack{T\subseteq N\\ T\ni i}} \Big[v(T) - v(T\setminus\{i\})\Big], \qquad \text{for all } i \in N. \tag{2.2.1}$$

For TU-games, Young [75] found a characterization for the Shapley value by the axioms of efficiency, equal treatment property and strong monotonicity, so without using the additivity axiom. Our characterization of the IM-value is a bit similar to this. Notice that the value IM(v) satisfies the axiom of global efficiency on the class of all set games (cf. Remark2.1.5). However, it does also satisfy efficiency on the class of monotonic set games (as we will show later). Therefore, we will restrict ourselves to the class of all monotonic set games with a fixed universe $\mathcal{U}$, denoted by $G^M(\mathcal{U})$ (although the axioms can be defined on the class of all set games).

The analog of the strong monotonicity axiom for set games is as follows:

Individually Marginalistic Monotonicity: if $v(S) - v(S \setminus \{i\}) \subseteq w(S) - w(S \setminus \{i\})$ for any (N, v), $(N, w) \in G^N(\mathcal{U})$, and any $S \subseteq N$ with $S \ni i$, then $f_i(N, v) \subseteq f_i(w)$.

Due to the following result, however, it is possible to characterize individually marginalistic value using a modified axiom of strong monotonicity. It turns out, by two different approaches, that another value, named *overall-individually marginalistic* OIM-value,

$$OIM_i(N, v) = \bigcup_{\substack{T \ni i, \\ T \subseteq N}} \left[v(T) - \bigcup_{j \in T} v(T \setminus \{j\}) \right] \quad \text{for all } i \in N \tag{2.2.2}$$

is, for monotonic set games, equal to the individually marginalistic value. This new expression is more useful to characterize the IM-value.

Theorem 2.2.1. (Aarts, Funaki & Hoede [2])
$IM(N, v) = OIM(N, v)$, for all $v \in G^M(\mathcal{U})$.

At first sight, this result is rather surprising since the marginal contribution in $OIM_i(v)$ seems to be smaller than in $IM_i(v)$. Because the proof of Theorem 2.2.1 showed by Aarts *et.al.*[2] is rather extensive, we will provide another proof of the theorem.

Proof of Theorem 2.2.1.
Let $(N, v) \in G^M(\mathcal{U})$ be a monotonic set game and $i \in N$. As noted earlier, the inclusion $IM_i(N, v) \supseteq OIM_i(N, v)$ is always valid, since $v(S \setminus \{i\}) \subseteq \bigcup_{j \in S} v(S \setminus \{j\})$ for all $S \subseteq N$ with $S \ni i$. In order to prove the inverse inclusion

$IM_i(N,v) \subseteq OIM_i(N,v)$, it suffices to show that $x \notin OIM_i(N,v)$ implies $x \notin IM_i(N,v)$.

Suppose $x \notin OIM_i(N,v)$. Let $S \subseteq N$ with $S \ni i$. We will show that $x \notin v(S) - v(S \setminus \{i\})$. We only distinguish two cases.

case 1. If $x \notin v(S)$, then $x \notin v(S) - v(S \setminus \{i\})$.

case 2. Let $x \in v(S)$. Under these circumstances we show $x \in v(S \setminus \{i\})$. Since $x \notin OIM_i(N,v) = \bigcup_{\substack{T \subseteq N \\ T \ni i}} \Big[v(T) - \bigcup_{j \in T} v(T \setminus \{j\}) \Big]$, it holds that $x \notin v(S) - \bigcup_{j \in S} v(S \setminus \{j\})$ and, together with the assumption $x \in v(S)$, we arrive at $x \in \bigcup_{j \in S} v(S \setminus \{j\})$. In summary, so far we conclude, from $x \in v(S)$ (where $i \in S$), that $x \in v(S \setminus \{j\})$ for some $j \in S$. By repeating the same procedure, step by step, there exists some $k \in S \setminus \{j\}$ such that $x \in v(S \setminus \{j,k\})$ and so on. Note that $x \notin v(\{i\})$ because of the assumption $x \notin OIM_i(N,v) = \bigcup_{\substack{T \subseteq N \\ T \ni i}} \Big[v(T) - \bigcup_{j \in T} v(T \setminus \{j\}) \Big]$. By repeatedly applying the same procedure, there exists a coalition $R \subseteq S$ not containing player i such that $x \in v(R)$. Finally, from $x \in v(R)$, $R \subseteq S \setminus \{i\}$ and the (tacitly assumed) monotonicity of the set game (N,v), we deduce that $x \in v(S \setminus \{i\})$ as was to be shown. □

The expression of the OIM-value gives rise to the axiom of overall-individually marginalistic monotonicity which is a modification of individually marginalistic monotonicity. This new axiom states that, if for each coalition containing player i the marginal contribution in game w includes the marginal contribution in game v, then in game w player i should get at least the items that he is allocated in game v.

Definition 2.2.2. Let f be a solution on the set game space $G^N(\mathcal{U})$. We say the solution f possesses the *overall-individually marginalistic monotonicity*, if

$$f_i(N,v) \subseteq f_i(N,w) \qquad \text{for all } (N,v),\ (N,w) \in G^N(\mathcal{U}), \text{ and all } i \in N, \quad (2.2.3)$$

satisfying $C_S^v \subseteq C_S^w$ for all $S \subseteq N$ with $i \in S$, where the overall-individually marginalistic contribution $C_S^v := v(S) - \bigcup_{j \in S} v(S \setminus \{j\})$. In words, with respect to two different set games, the larger the player's contributions in the game, the more items are allocated to the player.

Notice that, though the two values $IM(v)$ and $OIM(v)$ are equivalent, the corresponding two axioms of individually marginalistic monotonicity and overall-individually marginalistic monotonicity are not.

Theorem 2.2.3. (Aarts, Funaki & Hoede [2])
The individually marginalistic IM value is the unique value on $G^M(\mathcal{U})$ that satisfies the axioms of efficiency, equal treatment and overall-individually marginalistic monotonicity.

Finally, we list some interesting properties of the individually marginalistic IM-value.

Proposition 2.2.4. (Aarts [1]) The IM-value on $G^M(\mathcal{U})$ satisfies the following:

(*i*) $\bigcup_{i \in S} IM_i(N,v) \supseteq v(S)$ for all $S \subseteq N$,

(*ii*) If $v(R) = v(T)$ for all $T \supseteq R$, then $\bigcup_{i \in R} IM_i(N,v) = v(R)$,

(*iii*) If $v(T \cap R) = v(T)$ for all $T \subseteq N$, then $\bigcup_{i \in R} IM_i(N,v) = v(R)$.

2.3 A sub-coalitionally marginalistic value for set games

The aim of this section is to characterize the sub-coalitionally marginalistic value for set games by global efficiency, the equal treatment property and sub-coalitionally marginalistic monotonicity.

Definition 2.3.1. For any set game $(N,v) \in G^N(\mathcal{U})$, the *sub-coalitionally marginalistic* SCM-value is defined by

$$SCM_i(N,v) := \bigcup_{\substack{S \subseteq N \\ S \ni i}} \left[v(S) - \bigcup_{T \subset S} v(T) \right], \quad \text{for all } i \in N. \tag{2.3.1}$$

The interpretation of the SCM-value is that if a coalition obtains one item which is not obtained by any sub-coalition, then all players of the coalition can share the item. If for one item we have $x \in \{v(S) - \bigcup_{T \subset S} v(T)\}$, this implies that the coalition S is more powerful than any proper subset to obtain the item x. Notice that $SCM_i(N, v) = IM_i(N, v)$ for any monotonic set game $v \in G^M(\mathcal{U})$ and any $i \in N$, and $IM_i(N, v) \supseteq SCM_i(N, v)$ for any $i \in N$ and any set game $v \in G^N(\mathcal{U})$. We know that the core of set games is always nonempty. Then another allocation problem for set game arises, namely that of determining which allocation is more reasonable in the core. When the players choose the allocation for set games, they not only consider the allocation in the core, but they wish that fewer players shared an item with them. From this view, we think the SCM-value is a reasonable allocation within the solution theory for set games.

Definition 2.3.2. Let f be a value on the set game space $G^N(\mathcal{U})$. We say the value f possesses

(i) the *global efficiency* property, if the solution allocates all the attainable items to the players, that is

$$\bigcup_{i \in N} f_i(N, v) = \bigcup_{S \subseteq N} v(S), \qquad \text{for all } (N, v) \in G^N(\mathcal{U}). \tag{2.3.2}$$

(ii) the *equal treatment property*, if $f_i(N, v) = f_j(N, v)$ for any pair $i \in N$, $j \in N$, $i \neq j$, of *substitutes* in the set game $(N, v) \in G^N(\mathcal{U})$ (i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$). In words, two substitutes in a set game are allocated the same items.

(iii) the *sub-coalitionally marginalistic monotonicity* property, if

$$f_i(N, v) \subseteq f_i(N, w) \qquad \text{for all } (N, v), (N, w) \in G^N(\mathcal{U}), \text{ and all } i \in N, \tag{2.3.3}$$

satisfying $C_S^v \subseteq C_S^w$ for all $S \subseteq N$ with $i \in S$, where the sub-coalitionally marginalistic contribution $C_S^v := v(S) - \bigcup_{T \subset S} v(T)$. In words, with respect to two different set games, the larger the player's contributions in the game, the more items are allocated to the player.

The axiom of sub-coalitionally marginalistic monotonicity is a modification of axiom of strong monotonicity introduced by Aarts *et.al.* (AARTS, FUNAKI & HOEDE [3]). We will use the axiom of sub-coalitionally marginalistic monotonicity to characterize the SCM-value. For simplifying the formula in the sequel, let us define a new set game $(N, \widetilde{v})$ corresponding to any set game $(N, v) \in G^N(\mathcal{U})$.

Definition 2.3.3. For any set game $(N, v) \in G^N(\mathcal{U})$, the set game $(N, \widetilde{v})$ is defined by

$$\widetilde{v}(S) = v(S) - \bigcup_{T \subset S} v(T) \quad \text{for all } S \subseteq N. \tag{2.3.4}$$

Then the formula of the SCM-value reduces to

$$SCM_i(N, v) = \bigcup_{\substack{T \subseteq N \\ T \ni i}} \widetilde{v}(T) \quad \text{for all } i \in N. \tag{2.3.5}$$

Lemma 2.3.4. For any set game $(N, v) \in G^N(\mathcal{U})$, let the set game (N, w) be given by $w(S) = \bigcup_{T \subseteq S} v(T)$ for any $S \subseteq N$. Then we have

$$\widetilde{v}(S) = \widetilde{w}(S) \quad \text{for all } S \subseteq N. \tag{2.3.6}$$

Proof. By Definition 2.3.3, $\widetilde{w}(S) := w(S) - \bigcup_{T \subset S} w(T)$, for any $S \subseteq N$. Then we have that

$$\begin{aligned} \widetilde{w}(S) &= \bigcup_{R \subseteq S} v(R) - \bigcup_{T \subset S} \bigcup_{B \subseteq T} v(B) = \left(v(S) \cup \bigcup_{T \subset S} v(T) \right) - \bigcup_{B \subset S} v(B) \\ &= v(S) - \bigcup_{B \subset S} v(B) = \widetilde{v}(S) \quad \text{for all } S \subseteq N. \end{aligned}$$

□

Lemma 2.3.5. The SCM-value for set games satisfies the axioms of global efficiency, equal treatment and sub-coalitionally marginalistic monotonicity.

Proof. For global efficiency of the SCM-value, let us take $R = N$ in Lemma 2.1.3. Then we have

$$\bigcup_{i \in N} SCM_i(N, v) = \bigcup_{S \subseteq N} \left[v(S) - \bigcup_{T \subset S} v(T) \right] = \bigcup_{S \subseteq N} v(S).$$

The equal treatment property of the SCM-value is verified as follows. Let $i, j \in N$, and $i \neq j$ be such that $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$. Then we have

$$
\begin{aligned}
& SCM_i(N, v) \\
= & \bigcup_{\substack{S \ni i \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \\
= & \bigcup_{\substack{S \supseteq \{i,j\} \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \cup \bigcup_{S \subseteq N \setminus \{i,j\}} \left[v(S \cup \{i\}) - \bigcup_{T \subset S \cup \{i\}} v(T) \right] \\
= & \bigcup_{\substack{S \ni \{i,j\} \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \cup \bigcup_{S \subseteq N \setminus \{i,j\}} \Bigg[v(S \cup \{i\}) \\
& - \left(\bigcup_{T \subset S} [v(T \cup \{i\})] \cup \bigcup_{T \subseteq S} v(T) \right) \Bigg] \\
= & \bigcup_{\substack{S \supseteq \{i,j\} \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \cup \bigcup_{S \subseteq N \setminus \{i,j\}} \Bigg[v(S \cup \{j\}) \\
& - \left(\bigcup_{T \subset S} \left[v(T \cup \{j\}) \right] \cup \bigcup_{T \subseteq S} v(T) \right) \Bigg] \\
= & \bigcup_{\substack{S \subseteq \{i,j\} \\ S \subseteq N}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] \cup \bigcup_{S \subseteq N \setminus \{i,j\}} \left[v(S \cup \{j\}) - \bigcup_{T \subset S \cup \{j\}} v(T) \right] \\
= & \bigcup_{\substack{S \subseteq N \\ S \ni j}} \left[v(S) - \bigcup_{T \subset S} v(T) \right] = SCM_j(N, v).
\end{aligned}
$$

It is obvious that the SCM-value satisfies the axiom of sub-coalitionally marginalistic monotonicity. □

In order to complete the axiomatic characterization of the SCM-value, we consider an arbitrary, nonempty sub-collection $\mathcal{C}$ of 2^N. Let us define a new set game, according to

$$
\left(\bigcup_{R \in \mathcal{C}} (\widetilde{v}(R) \cap U_R) \right)(S) := \bigcup_{R \in \mathcal{C}} (\widetilde{v}(R) \cap U_R)(S) \quad \text{for all } \; S \subseteq N. \qquad (2.3.7)
$$

Lemma 2.3.6. For any set game $(N,v) \in G^N(\mathcal{U})$ and any $S \subseteq N$, we have

$$\left(\bigcup_{R\in\mathcal{C}} \widetilde{(\widetilde{v}(R)\cap U_R)}\right)(S) = \begin{cases} \widetilde{v}(S) & if \quad S \in \mathcal{C}, \\ \emptyset & if \quad S \notin \mathcal{C}. \end{cases}$$

Proof. For all $S \subseteq N$, it holds that

$$\begin{aligned} & \left(\bigcup_{R\in\mathcal{C}} \widetilde{(\widetilde{v}(R)\cap U_R)}\right)(S) \\ = & \left(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R)\cap U_R)\right)(S) - \bigcup_{T\subset S}\left(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R)\cap U_R)\right)(T) \\ = & \bigcup_{\substack{R\in\mathcal{C}\\R\subseteq S}} \widetilde{v}(R) - \bigcup_{T\subset S}\bigcup_{\substack{R\in\mathcal{C}\\R\subseteq T}} \widetilde{v}(R) = \bigcup_{\substack{R\in\mathcal{C}\\R\subseteq S}} \widetilde{v}(R) - \bigcup_{\substack{R\in\mathcal{C}\\R\subset S}} \widetilde{v}(R) \\ = & \begin{cases} \widetilde{v}(S), & if \;\; S \in \mathcal{C} \\ \emptyset, & if \;\; S \notin \mathcal{C}. \end{cases} \end{aligned}$$

If $R \subset S$, then $\widetilde{v}(S)\cap\widetilde{v}(R) = \emptyset$ and therefore we have $\widetilde{v}(S)\cap\left(\bigcup_{\substack{R\in\mathcal{C}\\R\subset S}} \widetilde{v}(R)\right) = \emptyset$.
□

Theorem 2.3.7. A value $f : G^N(\mathcal{U}) \rightarrow (2^{\mathcal{U}})^n$ satisfies the axioms of global efficiency, equal treatment and sub-coalitionally marginalistic monotonicity if and only if f is the sub-coalitionally marginalistic value, *i.e.*, $f(N,v) = SCM(N,v)$ for any set game $v \in G^N(\mathcal{U})$.

Proof. We only need to prove the uniqueness of the theorem. Suppose that $f(N,v)$ is a value satisfying the axioms of global efficiency, equal treatment and sub-coalitionally marginalistic monotonicity. Then, by global efficiency and sub-coalitionally marginalistic monotonicity, we have $f_i(N,v) = \emptyset$ if $v(S\cup\{i\}) = v(S)$ for all $S \subseteq N\backslash\{i\}$. For the set game $\widetilde{v}(R)\cap U_R$, any player $i \in N\backslash R$ satisfies

$$f_i(\widetilde{v}(R)\cap U_R) = \emptyset. \tag{2.3.8}$$

Furthermore, any players $i \in R$ and $j \in R$ are substitutes in the set game $\widetilde{v}(R)\cap U_R$, and hence, by the equal treatment property, we have

$$f_i(\widetilde{v}(R) \cap U_R) = f_j(\widetilde{v}(R) \cap U_R) \quad \text{for all} \quad i, j \in R.$$

And by global efficiency we have

$$\begin{aligned} \widetilde{v}(R) &= \bigcup_{T \subseteq N} (\widetilde{v}(R) \cap U_R)(T) = \bigcup_{i \in N} f_i(\widetilde{v}(R) \cap U_R) \\ &= f_i(\widetilde{v}(R) \cap U_R) \quad \text{for all } i \in R. \end{aligned} \tag{2.3.9}$$

We proceed by induction on the number $c = |\mathcal{C}|$ to show the following statement

$$f_i(\bigcup_{R \in \mathcal{C}} \Big(\widetilde{v}(R) \cap U_R\Big)) = \bigcup_{\substack{R \in \mathcal{C} \\ R \ni i}} \widetilde{v}(R), \quad \text{for all } i \in N. \tag{2.3.10}$$

For c=1, (2.3.10) clearly holds by (2.3.9). If $c \geq 2$, we will verify (2.3.10) in three cases.

Case 1. If $i \notin \bigcup_{R \in \mathcal{C}} R$ then, by (2.3.8), we have

$$f_i(\bigcup_{R \in \mathcal{C}} (\widetilde{v}(R) \cap U_R)) = \emptyset = \bigcup_{\substack{R \in \mathcal{C} \\ R \ni i}} \widetilde{v}(R).$$

Case 2. If $i \in \bigcap_{R \in \mathcal{C}} R$ then, applying Lemma 2.3.6 to $\mathcal{C}$ and any $R_1 \in \mathcal{C}$ respectively, we have

$$\Big(\widetilde{\bigcup_{R \in C} (\widetilde{v}(R) \cap U_R)}\Big)(S) = \begin{cases} \widetilde{v}(S) & if \;\; S \in C, \\ \emptyset & if \;\; S \notin C, \end{cases}$$

and

$$\Big(\widetilde{\widetilde{v}(R_1) \cap U_{R_1}}\Big)(S) = \begin{cases} \widetilde{v}(S) & if \;\; S = R_1, \\ \emptyset & if \;\; S \neq R_1. \end{cases}$$

This yields

$$\Big(\widetilde{\bigcup_{R \in \mathcal{C}} (\widetilde{v}(R) \cap U_R)}\Big)(S) \supseteq \Big(\widetilde{\widetilde{v}(R_1) \cap U_{R_1}}\Big)(S),$$

for all $S \subseteq N$ and $S \neq \emptyset$. By sub-coalitionally marginalistic monotonicity, we have $f_i(\bigcup_{R \in \mathcal{C}} (\widetilde{v}(R) \cap U_R)) \supseteq f_i(\widetilde{v}(R_1) \cap U_{R_1})$, for all $i \in N$ and all $R_1 \in \mathcal{C}$. Then, $f_i(\bigcup_{R \in \mathcal{C}} \widetilde{v}(R) \cap U_R) \supseteq \bigcup_{R_1 \in \mathcal{C}} f_i(\widetilde{v}(R_1) \cap U_{R_1})$ for all $i \in N$.

Because $i \in \bigcap_{R\in\mathcal{C}} R$, by (2.3.9), we have $f_i(\widetilde{v}(R_1) \cap U_{R_1}) = \widetilde{v}(R_1)$, for all $R_1 \in \mathcal{C}$. Then we have the conclusion that

$$f_i(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R)\cap U_R)) \supseteq \bigcup_{R_1\in\mathcal{C}} f_i(\widetilde{v}(R_1)\cap U_{R_1}) = \bigcup_{R_1\in\mathcal{C}} \widetilde{v}(R_1) = \bigcup_{R\in\mathcal{C}} \widetilde{v}(R), \quad (2.3.11)$$

and

$$\begin{aligned} f_i(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R) \cap U_R)) &\subseteq \bigcup_{j\in N} f_j(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R) \cap U_R)) \\ &= \bigcup_{S\subseteq N} (\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R) \cap U_R))(S) = \bigcup_{R\in\mathcal{C}} \widetilde{v}(R). \quad (2.3.12) \end{aligned}$$

From (2.3.11) and (2.3.12), one can easily find that

$$f_i(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R) \cap U_R)) = \bigcup_{R\in\mathcal{C}} \widetilde{v}(R) = \bigcup_{\substack{R\in\mathcal{C} \\ R\ni i}} \widetilde{v}(R).$$

Case 3. If $i \in \bigcup_{R\in\mathcal{C}} R$ but $i \notin \bigcap_{R\in\mathcal{C}} R$ then, for $\mathcal{C}_i := \{R | R \in \mathcal{C}, R \ni i\}$, by Lemma 2.3.6, we have

$$\Big(\bigcup_{R\in\mathcal{C}} \widetilde{(\widetilde{v}(R) \cap U_R)}\Big)(S) = \begin{cases} \widetilde{v}(S) & if \quad S \in \mathcal{C}, \\ \emptyset & if \quad S \notin \mathcal{C}, \end{cases}$$

and

$$\Big(\bigcup_{R\in\mathcal{C}_i} \widetilde{(\widetilde{v}(R) \cap U_R)}\Big)(S) = \begin{cases} \widetilde{v}(S) & if \quad S \in \mathcal{C}_i, \\ \emptyset & if \quad S \notin \mathcal{C}_i. \end{cases}$$

If $S \notin \mathcal{C}$ then $S \notin \mathcal{C}_i$ and if $S \in \mathcal{C}$, then $S \ni i$ implies $S \in \mathcal{C}_i$. So we have

$$\Big(\bigcup_{R\in\mathcal{C}} \widetilde{(\widetilde{v}(R) \cap U_R)}\Big)(S) = \Big(\bigcup_{R\in\mathcal{C}_i} \widetilde{(\widetilde{v}(R) \cap U_R)}\Big)(S), \quad \text{for all} \quad S \ni i.$$

By sub-coalitionally marginalistic monotonicity, it is easily seen that

$$f_i(\bigcup_{R\in\mathcal{C}} (\widetilde{v}(R) \cap U_R)) = f_i(\bigcup_{R\in\mathcal{C}_i} (\widetilde{v}(R) \cap U_R)).$$

By $|c_i| < |c|$ and the induction hypothesis, we have

$$\begin{aligned} f_i(\bigcup_{R\in\mathcal{C}}(\widetilde{v}(R)\cap U_R)) &= f_i(\bigcup_{R\in\mathcal{C}_i}(\widetilde{v}(R)\cap U_R)) \\ &= \bigcup_{\substack{R\in\mathcal{C}_i\\ R\ni i}}\widetilde{v}(R) = \bigcup_{\substack{R\in\mathcal{C}\\ R\ni i}}\widetilde{v}(R), \end{aligned}$$

which completes the induction. Let the sub-collection $\mathcal{C}$ be 2^N ($\emptyset \notin \mathcal{C}$), then we have

$$f_i(\bigcup_{\substack{R\subseteq N\\ R\neq\emptyset}}(\widetilde{v}(R)\cap U_R)) = \bigcup_{\substack{R\subseteq N\\ R\ni i}}\widetilde{v}(R), \quad \text{for all } \ i\in N. \tag{2.3.13}$$

Furthermore, we obtain $w = \bigcup_{\substack{R\subseteq N\\ R\neq\emptyset}}(\widetilde{v}(R)\cap U_R)$ since from Lemma 2.1.3 we derive

$$(\bigcup_{\substack{R\subseteq N\\ R\neq\emptyset}}(\widetilde{v}(R)\cap U_R))(S) = \bigcup_{\substack{R\subseteq S\\ R\neq\emptyset}}\widetilde{v}(R) = \bigcup_{R\subseteq S}v(R) = w(S), \quad \text{for all } S\subseteq N.$$

By Lemma 2.3.4, sub-coalitionally marginalistic monotonicity and (2.3.13) respectively, we have

$$f_i(N,v) = f_i(N,w) = f_i(\bigcup_{\substack{R\subseteq N\\ R\neq\emptyset}}(\widetilde{v}(R)\cap U_R)) = \bigcup_{\substack{R\subseteq N\\ R\ni i}}\widetilde{v}(R) = SCM_i(N,v),$$

for all $i\in N$, which completes the proof. □

2.4 An individually co-marginalistic value for set games

Value theory of TU-games started with the fundamental paper of Shapley [64] and now it has a central position in game theory and its application, especially in economics, social and political sciences.

Nowak and Radzik [55] have introduced and characterized a new value which reflects some social behavior of players in coalitions. To do this, they define, for any non-empty coalition T and any TU-game (N,v), the quantity

$$A^v(T) = \frac{1}{|T|}\sum_{k\in T}\Big[v(T) - v(T\setminus\{k\})\Big].$$

Clearly, $A^v(T)$ is the average marginal contribution of a member of the coalition T. Next, we define the payoff $f_i(N, v)$ to every player $i \in N$ in any TU-game v by

$$f_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{(n - |T|)!(|T| - 1)!}{n!} A^v(T). \tag{2.4.1}$$

(Recall that $n = |N|$) We call the mapping $f = (f_1, \cdots, f_n)$, given by the above expression, the *solidarity value* for TU-games. Its interpretation can be obtained by replacing the marginal contribution $\delta(T, i) = v(T) - v((T \setminus \{i\}))$ of player i in the well-known interpretation of the Shapley value by $A^v(T)$. To be more specified, if a player i becomes a member of some coalition T, then he/she obtains (as a payoff) the average marginal contribution $A^v(T)$ of a member of T. Thus, if $\delta(T, i) > A^v(T)$, player i offers some part of his/her marginal contribution $\delta(T, i)$ to the coalition T to support some "weaker" member of coalition T. If $\delta(T, i) < A^v(T)$, then player i benefits from the fact that he/she has been accepted to become a new member of the coalition T. Of course, $f_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{(n-|T|)!(|T|-1)!}{n!} A^v(T)$ is then the expected payoff to player i in the game v and (under the above interpretation) their name for the value $f(N, v)$ is justified.

Example 2.4.1. (Three Brothers): Players 1, 2, and 3 are brothers and they live together. Player 1 and 2 can make together a profit of one unit, *i.e.* $v(\{1, 2\}) = 1$. Player 3 is a disabled person and can contribute nothing to any coalition. Therefore, $v(\{1, 2, 3, \}) = 1$. Further, we have $v(\{1, 3\}) = v(\{2, 3\}) = 0$. Finally, we assume that $v(\{i\}) = 0$ for every player i. This is a classical unanimity game. The Shapley value of this game is

$(\frac{1}{2}, \frac{1}{2}, 0)$.

(Should the disabled brother leave his family?) If player 1 and 2 take the responsibility for their brother (player), then the solidarity value

$(\frac{7}{18}, \frac{7}{18}, \frac{4}{18})$.

seems to be "better" solution for the game (N, v) than its Shapley value. Of course, one can say that if some kind of solidarity of players 1 and 2 with player

3 is assumed then such a fact should be reflected by the characteristic function v itself. However, the question is then how to define the marginal contributions of player 3 to the grand coalition. The answer is not obvious. We do not want to say that the solidarity value is the only "right" solution concept ever for this example. We would rather like to point out that it could be used to take into account some (usually subjective and very difficult to measure) social or psychological aspects in a cooperative game. The characteristic function v might then be used to represent the underlying "pure economic" situations in the game only.

We note that the solidarity value does not belong to the core of v in the Example 2.3.1 and the Shapley value does. We now give another game v for which the solidarity value does belong to the core of v while the Shapley value does not.

Example 2.4.2. Consider a three person game v where $v(\{1\}) = v(\{2\}) = 0$, $v(\{3\}) = 1$, $v(\{1,2\}) = 3.5$, $v(\{1,3\}) = v(\{2,3\}) = 0$. Finally, $v(\{1,2,3\}) = 5$. The Shapley value of this game is

$$(\tfrac{25}{12}, \tfrac{25}{12}, \tfrac{10}{12}).$$

Note that the Shapley value of game v in Example 2.3.2 is not individually rational, and thus does not belong to the core of (N, v). The solidarity value of this game (N, v) is

$$(\tfrac{16}{9}, \tfrac{16}{9}, \tfrac{13}{9}),$$

and clearly is in the core of (N, v).

In this section, we will introduce a new value, called individually co-marginalistic value, which is the analog of the solidarity value for TU-games, for set games.

Definition 2.4.3. For any set game $(N, v) \in G^N(\mathcal{U})$, the *individually co-marginalistic* ICM-value, is defined by

$$\begin{aligned} ICM_i(N, v) &:= \bigcup_{\substack{T \subseteq N \\ T \ni i}} \bigcup_{j \in T} \Big[v(T) - v(T \setminus \{j\}) \Big] \\ &= \bigcup_{\substack{T \subseteq N \\ T \ni i}} \Big[v(T) - \bigcap_{j \in T} v(T \setminus \{j\}) \Big], \quad \text{for all } i \in N. \quad (2.4.2) \end{aligned}$$

The interpretation of the new value is that all members of a coalition can share the individually marginal contribution of every member of the coalition. The ICM-value on $G^N(\mathcal{U})$ reflects some social behavior of players within a coalition. In some cooperative situations, a coalition contains a disabled player who needs help. How do we allocate the items to the disabled player? The ICM-value for set games, which can be considered to be the analog of the solidarity value for TU-games, is beneficial for the disabled player. In the following, we will characterize the ICM-value for set games by some standard axioms.

Let us consider the following axioms.

Definition 2.4.5. Let f be a value on the set game space $G^N(\mathcal{U})$. We say the value f satisfies the axiom of

(i) *individually co-marginalistic monotonicity* , if

$$f_i(N, v) \subseteq f_i(N, w) \qquad \text{for all } (N, v), (N, w) \in G^N(\mathcal{U}), \text{ and all } i \in N, \quad (2.4.3)$$

satisfying $C_S^v \subseteq C_S^w$ for all $S \subseteq N$ with $i \in S$, where the individually co-marginalistic contribution $C_S^v := v(S) - \bigcap_{j \in S} v(S \setminus \{j\})$. In words, with respect to two different set games, the larger the player's contributions in the game, the more items allocated to the player.

Let us state several lemmas to prove the main result in this section

Lemma 2.4.6 Individually co-marginalistic monotonicity and global efficiency imply the destructive player property.

Proof. Let $w_1(S) = \emptyset$ for all $S \subseteq N$ and player i be a destructive player for a set game v, then

$$v(S) - \bigcap_{j \in S} v(S \setminus \{j\}) = \emptyset = w_1(S) - \bigcap_{j \in S} w_1(S \setminus \{j\}),$$

for all $S \ni i$. By individually co-marginalistic monotonicity, $f_i(N, v) = f_i(N, w_1)$. From global efficiency, we obtain $\bigcup_{k \in N} f_k(N, w_1) = \emptyset$. So, $f_i(N, v) = f_i(N, w_1) = \emptyset$. □

Lemma 2.4.7. The ICM-value for set games satisfies the axioms of global efficiency, equal treatment and individually co-marginalistic monotonicity.

Proof. In the following, the three properties for the ICM-value are satisfied respectively.

(i) It is obvious that the ICM-value for set game satisfies the axiom of global efficiency because the ICM-value is in the core of set games as shown in Remark 2.1.5.

(ii) For any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game $(N, v) \in G^N(\mathcal{U})$ (i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i, j\}$), we conclude that

$$\begin{aligned}
ICM_i(N, v) &= \bigcup_{\substack{T \subseteq N \\ T \ni i}} \bigcup_{k \in T} \Big[v(T) - v(T \setminus \{k\}) \Big] \\
&= \bigcup_{T \ni \{i,j\}} \bigcup_{k \in T} \Big[v(T) - v(T \setminus \{k\}) \Big] \\
&\quad \cup \bigcup_{T \subseteq N \setminus \{i,j\}} \bigcup_{k \in T \cup \{i\}} \Big[v(T \cup \{i\}) - v((T \cup \{i\}) \setminus \{k\}) \Big] \\
&= \bigcup_{T \subseteq N \setminus \{i,j\}} \bigcup_{k \in T \cup \{i,j\}} \Big[v(T \cup \{i,j\}) - v((T \cup \{i,j\}) \setminus \{k\}) \Big] \\
&\quad \cup \bigcup_{T \subseteq N \setminus \{i,j\}} \bigcup_{k \in T \cup \{i\}} \Big[v(T \cup \{i\}) - v((T \cup \{i\}) \setminus \{k\}) \Big]
\end{aligned}$$

$$= \bigcup_{T\subseteq N\setminus\{i,j\}} \bigcup_{k\in T\cup\{i,j\}} \Big[v(T\cup\{i,j\}) - v((T\cup\{i,j\})\setminus\{k\})\Big]$$
$$\cup \bigcup_{T\subseteq N\setminus\{i,j\}} \bigcup_{k\in T\cup\{j\}} \Big[v(T\cup\{j\}) - v((T\cup\{j\})\setminus\{k\})\Big]$$
$$= \bigcup_{\substack{T\subseteq N\\ T\ni j}} \bigcup_{k\in T} \Big[v(T) - v(T\setminus\{j\})\Big] = ICM_j(N,v),$$

where the fourth equality follows since i and j are substitutes.

(iii) Obviously, the ICM-value satisfies the axiom of individually co-marginalistic monotonicity. □

Definition 2.4.8. For any set game $(N, v) \in G^N(\mathcal{U})$, the corresponding set game $(N, \overline{v})$ is defined by

$$\overline{v}(S) = v(S) - \bigcap_{j\in S} v(S\setminus\{j\}), \qquad \text{for all } S\subseteq N. \tag{2.4.4}$$

Then the formula of the ICM-value reduces to

$$ICM_i(N,v) = \bigcup_{\substack{T\subseteq N\\ T\ni i}} \overline{v}(T) \qquad \text{for all } i\in N. \tag{2.4.5}$$

Lemma 2.4.9.
(i) For any $S\subseteq N$, $\bigcup_{T\subseteq S} \overline{v}(T) = \bigcup_{T\subseteq S} v(T)$.
(ii) $\overline{v}(S) = \overline{w}(S)$ for the set game $w(S) = v(S) - \bigcap_{j\in S} v(S\setminus\{j\})$ for all $S\subseteq N$, *i.e.*, $w(S) = \overline{v}(S)$ for all $S\subseteq N$.

Proof.
(i) We proceed by induction on the number $s = |S|$ as follows: For $s = 1$, it apparently holds. For $s \geq 2$, the induction hypothesis is

$$\bigcup_{T\subseteq R} \overline{v}(T) = \bigcup_{T\subseteq R} \Big[v(T) - \bigcap_{j\in T} v(T\setminus\{j\})\Big] = \bigcup_{T\subseteq R} v(T),$$

for all $R \subset S$. Then, we have that

$$\begin{aligned}
\bigcup_{T\subseteq S} \overline{v}(T) &= \bigcup_{T\subseteq S} \left[v(T) - \bigcap_{j\in T} v(T\setminus\{j\})\right] \\
&= \left[v(S) - \bigcap_{j\in S} v(S\setminus\{j\})\right] \cup \bigcup_{j\in S} \bigcup_{T\subseteq S\setminus\{j\}} \left[v(T) - \bigcap_{k\in T} v(T\setminus\{k\})\right] \\
&= \left[v(S) - \bigcap_{j\in S} v(S\setminus\{j\})\right] \cup \bigcup_{j\in S} \bigcup_{T\subseteq S\setminus\{j\}} v(T) = \bigcup_{T\subseteq S} v(T),
\end{aligned}$$

where the third equality follows from the induction hypothesis.
(ii)

$$\begin{aligned}
&\overline{w}(S) = w(S) - \bigcap_{j\in S} w(S\setminus\{j\}) \\
= &\left[v(S) - \bigcap_{j\in S} v(S\setminus\{j\})\right] - \{\bigcap_{j\in S}\left[v(S\setminus\{j\}) - \bigcap_{k\in S\setminus\{j\}} v(S\setminus\{j,k\})\right]\} \\
= &\ v(S) - \bigcap_{j\in S} v(S\setminus\{j\}) = \overline{v}(S), \qquad \text{for all } S\subseteq N.
\end{aligned}$$

□

Let $\mathcal{C}$ be an arbitrary non-empty sub-collection of 2^N. A set game is defined by

$$\left(\bigcup_{R\in\mathcal{C}} (\overline{v}(R)\cap E_R)\right)(S) = \bigcup_{R\in\mathcal{C}} (\overline{v}(R)\cap E_R)(S) \qquad \text{for all} \quad S\subseteq N, \quad (2.4.6)$$

where, for any $R\in\mathcal{C}$, the set game $(\overline{v}(R)\cap E_R)$ is defined by

$$(\overline{v}(R)\cap E_R)(S) = \begin{cases} \overline{v}(R) & \text{i}f \quad S = R, \\ \emptyset & \text{if} \quad S\neq R. \end{cases}$$

Lemma 2.4.10 For any set game $(N, v) \in G^N(\mathcal{U})$, any non-empty sub-collection $\mathcal{C}$ and all non-empty coalitions $T\subseteq N$, we have

$$\overline{\left(\bigcup_{R\in\mathcal{C}} (\overline{v}(R)\cap E_R)\right)}(T) = \begin{cases} \overline{v}(T) & \text{if} \quad T\in\mathcal{C}, \\ \emptyset & \text{if} \quad T\notin\mathcal{C}. \end{cases}$$

Proof. From the above definition, we obtain that

$$(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R))(S)=\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)(S)=\begin{cases}\overline{v}(S) & \text{if} \quad S\in\mathcal{C},\\ \emptyset & \text{if} \quad S\notin\mathcal{C},\end{cases}$$

and

$$\begin{aligned}&\Big(\overline{\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)}\Big)(T)\\ =\;&\Big[\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big](T)-\bigcap_{j\in T}\Big[\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big](T\setminus\{j\})\\ =\;&\begin{cases}\overline{v}(T) & \text{if } T\in\mathcal{C} \text{ but there exists}\\ & j\in T,\, T\setminus\{j\}\notin\mathcal{C},\\ \overline{v}(T)-\bigcap_{j\in T}\overline{v}(T\setminus\{j\}) & \text{if } T\in\mathcal{C} \text{ and } T\setminus\{j\}\in\mathcal{C},\\ & \text{for all } j\in T,\\ \emptyset & \text{if } T\notin\mathcal{C}.\end{cases}\end{aligned}$$

Furthermore, by Lemma 2.4.9(ii), we have that $\overline{v}(T)-\bigcap_{j\in T}\overline{v}(T\setminus\{j\})=\overline{v}(T)$. So,

$$\overline{\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)}(T)=\begin{cases}\overline{v}(T) & \text{if} \quad T\in\mathcal{C},\\ \emptyset & \text{if} \quad T\notin\mathcal{C}.\end{cases}$$

□

Theorem 2.4.11. A value $f:G^N(\mathcal{U})\to(2^{\mathcal{U}})^n$ satisfies the axioms of global efficiency, equal treatment and individually co-marginalistic monotonicity if and only if f is the individually co-marginalistic ICM-value, *i.e.*, $f(N,v)=ICM(N,v)$ for any set game $(N,v)\in G^N(\mathcal{M})$.

Proof. The necessity of the condition of Theorem 2.4.11 has been proved in Lemma 2.4.7. We will only discuss the sufficiency of the condition. Suppose that a value f satisfies the axioms of global efficiency, equal treatment and individually co-marginalistic monotonicity then, by Lemma 2.4.5, f has the destructive player property as well. Let us apply the value f to the set game $\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)$ to show

$$f_i(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)) = \bigcup_{\substack{R\in\mathcal{C} \\ R\ni i}} \overline{v}(R), \tag{2.4.7}$$

for any $i \in N$ and any set game $(N, v) \in G^N(\mathcal{U})$, where $\mathcal{C}$ is a nonempty sub-collection of 2^N. For (2.4.7), we proceed by induction on the number $c = |\mathcal{C}|$.

(I) If $c = 1$ then (2.4.7) reads $f_i(\overline{v}(R) \cap E_R) = \overline{v}(R)$, for all $i \in R$, and $f_i(\overline{v}(R) \cap E_R) = \emptyset$ if $i \notin R$. For a set game $\overline{v}(R) \cap E_R$, if player $i \in N \setminus R$, then player i is a destructive player. By the destructive player property, $f_i(\overline{v}(R) \cap E_R) = \emptyset$ for all $i \in N \setminus R$. Furthermore, for any pair $\{i, j\} \subseteq R (i \neq j)$ and any $S \subseteq N \setminus \{i, j\}$, we have $(\overline{v}(R) \cap E_R)(S \cup \{i\}) = (\overline{v}(R) \cap E_R)(S \cup \{j\}) = \emptyset$. So, the equal treatment property implies $f_i(\overline{v}(R) \cap E_R) = f_j(\overline{v}(R) \cap E_R)$ for all $i, j \in R$. Therefore, by global efficiency, we obtain that

$$\begin{aligned}
\overline{v}(R) &= (\overline{v}(R) \cap E_R)(R) = \bigcup_{S\subseteq N} (\overline{v}(R) \cap E_R)(S) \\
&= \bigcup_{i\in N} f_i(\overline{v}(R) \cap E_R) \\
&= \left[\bigcup_{i\in N\setminus R} f_i(\overline{v}(R) \cap E_R)\right] \cup \left[\bigcup_{i\in R} f_i(\overline{v}(R) \cap E_R)\right] \\
&= \emptyset \cup \left[\bigcup_{i\in R} f_i(\overline{v}(R) \cap E_R)\right] = f_i(\overline{v}(R) \cap E_R),
\end{aligned}$$

for all $i \in R$. Then we have

$$f_i(\overline{v}(R) \cap E_R) = \begin{cases} \overline{v}(R) & \text{if} \quad i \in R, \\ \emptyset & \text{if} \quad i \notin R. \end{cases}$$

(II) For $c \geq 2$, the induction hypothesis is

$$f_i(\bigcup_{R\in\mathcal{C}^*} (\overline{v}(R) \cap E_R)) = \bigcup_{\substack{R\ni i \\ R\in\mathcal{C}^*}} \overline{v}(R),$$

for all $i \in N$, all $(N,v) \in G^N(\mathcal{U})$ and all sub-collections $\mathcal{C}^*$ with $c^* = |\mathcal{C}^*| < c = |\mathcal{C}|$. Under the hypothesis, we consider (2.4.7) in three cases.

(i) If $i \notin \bigcup_{R\in\mathcal{C}} R$, then player i is a destructive player for set game $\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)$. So, from the destructive player property we have

$$f_i(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)) = \emptyset = \bigcup_{\substack{R\in\mathcal{C}\\ R\ni i}} \overline{v}(R).$$

(ii) If $i \in \bigcap_{R\in\mathcal{C}} R$, then $i \in R^*$ for any $R^* \in \mathcal{C}$. Applying Lemma 2.4.10 to the sub-collection $\mathcal{C}$ and $\{R^*\}$ respectively, we obtain

$$\overline{\Big(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)\Big)}(T) = \begin{cases} \overline{v}(T) & \text{if} \quad T \in \mathcal{C}, \\ \emptyset & \text{if} \quad T \notin \mathcal{C}, \end{cases}$$

and

$$\overline{\Big(\overline{v}(R^*) \cap E_{R^*}\Big)}(T) = \begin{cases} \overline{v}(T) & \text{if} \quad T = R^*, \\ \emptyset & \text{if} \quad T \neq R^*. \end{cases}$$

This implies that

$$\overline{\Big(\bigcup_{R\in C} (\overline{v}(R) \cap E_R)\Big)}(T) \supseteq \overline{\Big(\overline{v}(R^*) \cap E_{R^*}\Big)}(T),$$

for all $T \subseteq N$ $(T \neq \emptyset)$. By individually co-marginalistic monotonicity of the value f, we get

$$f_i(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)) \supseteq f_i(\overline{v}(R^*) \cap E_{R^*}), \qquad \text{for all } i \in N, \text{ and all } R^* \in \mathcal{C}$$

and hence, it follows

$$f_i(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)) \supseteq \bigcup_{R^*\in\mathcal{C}} f_i(\overline{v}(R^*) \cap E_{R^*}), \qquad \text{for all } i \in N.$$

Moreover, since $i \in \bigcap_{R\in\mathcal{C}} R$, we have $f_i(\overline{v}(R) \cap E_R) = \overline{v}(R)$ for any $R \in \mathcal{C}$. Therefore

$$f_i(\bigcup_{R\in\mathcal{C}} (\overline{v}(R) \cap E_R)) \quad \supseteq \quad \bigcup_{R\in\mathcal{C}} f_i((\overline{v}(R) \cap E_R)) = \bigcup_{R\in\mathcal{C}} \overline{v}(R).$$

On the other hand,

$$f_i(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)) \subseteq \bigcup_{k\in N} f_k\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big) = \bigcup_{T\subseteq N}\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)(T) = \bigcup_{R\in\mathcal{C}}\overline{v}(R).$$

By the above induction, we can conclude that

$$f_i(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)) = \bigcup_{R\in\mathcal{C}}\overline{v}(R) = \bigcup_{\substack{R\in\mathcal{C}\\ R\ni i}}\overline{v}(R).$$

(iii) The remaining case is that $i \in \bigcup_{R\in\mathcal{C}} R$ but $i \notin \bigcap_{R\in\mathcal{C}} R$. For $\mathcal{C}_i := \{R|R \in \mathcal{C}, R \ni i\}$, we have $\mathcal{C}_i \subset \mathcal{C}$ and $\mathcal{C}_i \neq \emptyset$. By Lemma 2.4.10 we know that for any $T \subseteq N$, $T \neq \emptyset$,

$$\overline{\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)}(T) = \begin{cases} \overline{v}(T) & \text{if } T\in\mathcal{C}, \\ \emptyset & \text{if } T\notin\mathcal{C}, \end{cases}$$

and

$$\overline{\Big(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)\Big)}(T) = \begin{cases} \overline{v}(T) & \text{if } T\in\mathcal{C}_i, \\ \emptyset & \text{if } T\notin\mathcal{C}_i. \end{cases}$$

This yields

$$\overline{\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)}(T) \supseteq \overline{\Big(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)\Big)}(T), \quad \text{for all } T\subseteq N.$$

Conversely, if $T \notin \mathcal{C}$ then $T \notin \mathcal{C}_i$ and, if $T \in \mathcal{C}$, then $T \ni i$ implies $T \in \mathcal{C}_i$. So, we have

$$\overline{\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)}(T) \subseteq \overline{\Big(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)\Big)}(T),$$

for all $T \ni i$. Then it can be obtained that

$$\overline{\Big(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)\Big)}(T) = \overline{\Big(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)\Big)}(T),$$

for all $T \ni i$. Now, from individually co-marginalistic monotonicity, we can conclude that

$$f_i(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)) = f_i(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)).$$

By the induction hypothesis, it clearly holds that

$$f_i(\bigcup_{R\in\mathcal{C}}(\overline{v}(R)\cap E_R)) = f_i(\bigcup_{R\in\mathcal{C}_i}(\overline{v}(R)\cap E_R)) = \bigcup_{\substack{R\in\mathcal{C}\\ R\ni i}}\overline{v}(R).$$

This completes the proof of (2.4.7). Applying $\mathcal{C} = 2^N(\emptyset \notin \mathcal{C})$ to (2.4.7), we then obtain that

$$f_i(\bigcup_{R\subseteq N}(\overline{v}(R)\cap E_R)) = \bigcup_{R\ni i}\overline{v}(R), \qquad \text{for all } i \in N.$$

Let us note that $w = \bigcup_{R\subseteq N}(\overline{v}(R)\cap E_R)$ due to $w = \overline{v}$. From Lemma 2.4.9 (ii) and individually co-marginalistic monotonicity we can conclude that

$$f_i(N,v) = f_i(w) = f_i(\bigcup_{R\subseteq N}(\overline{v}(R)\cap E_R)) = \bigcup_{R\ni i}\overline{v}(R) = ICM_i(v),$$

for all $i \in N$, which completes the proof. □

2.5 Semi-marginalistic values for set games

Let us recall the five values for set games in Section 2.1. For any $S \subseteq N$ and any set game $(N,v) \in G^N(\mathcal{U})$, the underlying expressions $v(S) - v(S\backslash\{i\})$, $v(S) - \bigcup_{j\in S} v(S\backslash\{j\})$, $v(S) - \bigcup_{T\subset S} v(T)$, $v(S) - \bigcup_{T\subseteq N\backslash\{i\}} v(T)$ and $v(S) - \bigcup_{j\in S} v(S\backslash\{j\})$ respectively, are called the marginalistic contribution of coalition S induced by either one particular member, all members, or all sub-coalitions. In the section, we will present a class of values associating with the semi-marginalistic contributions of coalitions.

Definition 2.5.1. A *semi-marginalistic value*, or shortly SM-value on the set game space $G^N(\mathcal{U})$ is defined to be a member of the family of set games solutions of the following form:

$$SM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \Big[v(S) - \nabla^v_{S,i}\Big], \quad \text{for all } (N,v)\in G^N(\mathcal{U}) \text{ and all } i\in N \tag{2.5.1}$$

or, equivalently,

$$SM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} SMC^v_{S,i}, \tag{2.5.2}$$

where $SMC^v_{S,i} = v(S) - \nabla^v_{S,i}$. In words, for every player i, the *semi-marginalistic contribution* $SMC^v_{S,i} = v(S) - \nabla^v_{S,i}$ of every coalition S is determined by the set difference of the coalition's worth $v(S)$ and some (yet unspecified) expression $\nabla^v_{S,i}$ which is supposed to depend, to some weak or strong extent, upon the worths of a certain collection of coalitions, somehow determined by S and/or i (for instance, through the unions and/or intersections of a number of (sub)coalitions). By convention, $\nabla^v_{S,i} := \emptyset$ in the framework of one-person games.

By (2.1.1) and (2.1.4), the IM-value and OCM-value are semi-marginalistic values of the form (2.5.1), by choosing $\nabla^v_{S,i} = v(S\backslash\{i\})$ and $\nabla^v_{S,i} = \bigcup_{T\subseteq N\backslash\{i\}} v(T)$ respectively, the expression of which still depends upon player i. By (2.1.2), (2.1.3) and (2.1.5), the OIM-value, SCM-value and ICM-value are semi-marginalistic values by choosing $\nabla^v_{S,i} = \bigcup_{j\in S} v(S\backslash\{j\})$, $\nabla^v_{S,i} = \bigcup_{T\subset S} v(T)$ and $\nabla^v_{S,i} = \bigcap_{j\in S} v(S\backslash\{j\})$ respectively, the expression of which satisfies the so-called player's contribution independence.

Definition 2.5.2. We say a SM-value of the form (2.5.1) has the property of *player's contribution independence* whenever, for any coalition S, the associated expression $\nabla^v_{S,i}$ does not depend upon player i, that is $\nabla^v_{S,i} = \nabla^v_S$ is the same for all $i\in N$. Shortly, for all $(N,v)\in G^N(\mathcal{U})$ and all $i\in N$

$$SM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} SMC^v_{S,i} = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \Big[v(S) - \nabla^v_S\Big]. \tag{2.5.3}$$

For reasons that will be explained later: $\quad \nabla_N^v \subseteq \bigcup_{S\subset N} v(S). \qquad (2.5.4)$

For the class of monotonic set games $v \in G^M(\mathcal{U})$ (*i.e.*, $v(S) \subseteq v(T)$ for all $S \subseteq T \subseteq N$), it was shown in [2] that the IM-value and the OIM-value coincide. So, for every monotonic set game (N, v), it holds that $IM_i(N, v) = OIM_i(N, v) = SCM_i(N, v)$, whereas, by (2.1.4), $OCM_i(N, v) = v(N) - v(N \setminus \{i\})$ for all $i \in N$ and, consequently, $\bigcup_{i\in N} OCM_i(N, v) = v(N) - \bigcap_{i\in N} v(N \setminus \{i\})$ on the class of monotonic set games. According to the next lemma, the OCM-value differs from the remaining solutions in that another type of efficiency applies.

Lemma 2.5.3. Let the SM-value be of the form (2.5.3) assuming players' contribution independence, such that (2.5.4) holds. Then the SM-value satisfies the axiom of global efficiency, that is the value $SM(N, v)$ allocates all the attainable items to the players in that

$$\bigcup_{i\in N} SM_i(N, v) = \bigcup_{S\subseteq N} v(S), \quad \text{for all } (N, v) \in G^N(\mathcal{U}). \qquad (2.5.5)$$

Proof. Clearly, for the SM-value $SM(N, v)$ of the form (2.5.3), its global efficiency condition (2.5.5) is equivalent to the following condition: for all $(N, v) \in G^N(\mathcal{U})$

$$\bigcup_{S\subseteq N} SMC_S^v = \bigcup_{S\subseteq N} v(S). \qquad (2.5.6)$$

We prove (2.5.6) by induction on the number of players. The case $n = 1$ is trivial due to $\nabla_S^v := \emptyset$ in the framework of one-person set games. Let $(N, v) \in G^N(\mathcal{U})$ with $n \geq 2$. Then we obtain the following chain of equalities:

$$\begin{aligned}
&\bigcup_{S\subseteq N} SMC_S^v \\
&= SMC_N^v \bigcup \Big[\bigcup_{S\subset N} SMC_S^v\Big] = SMC_N^v \cup \Big[\bigcup_{k\in N}\Big[\bigcup_{S\subseteq N\setminus\{k\}} SMC_S^v\Big]\Big] \\
&= SMC_N^v \cup \Big[\bigcup_{k\in N}\Big[\bigcup_{S\subseteq N\setminus\{k\}} v(S)\Big]\Big] \\
&= SMC_N^v \cup \Big[\bigcup_{S\subset N} v(S)\Big] = \Big[v(N) - \nabla_N^v\Big] \cup \Big[\bigcup_{S\subset N} v(S)\Big] = \bigcup_{S\subseteq N} v(S),
\end{aligned}$$

where the third equality is obtained by the induction hypothesis. This completes the proof of the global efficiency (2.5.6) for the SM-value. □

In addition to the axiom of global efficiency, we study the axiom of equal treatment, in order to be able to provide, in the following, an axiomatization of any SM-value satisfying appropriately chosen semi-marginalistic contribution. Let us recall the substitution of a pair players and equal treatment property in Chapter 1.

Definition 2.5.4. (Substitutes in a set game and equal treatment property for a value)

(i) Two players $i, j \in N$, $i \neq j$, are said to be *substitutes* in the set game $(N, v) \in G^N(\mathcal{U})$ whenever $v(S \cup \{i\}) = v(S \cup \{j\})$ holds for all $S \in N \setminus \{i, j\}$.

(ii) We say a value f on the set game space $G^N(\mathcal{U})$ possesses the *equal treatment property* if $f_i(N, v) = f_j(N, v)$ for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set game $(N, v) \in G^N(\mathcal{U})$. In words, two substitutes in a set game are allocated the same items.

Lemma 2.5.5. Let f be an SM-value of the form (2.5.3) assuming players' contribution independence, such that the semi-marginalistic contribution concept inherits the role of substitutes, that is, for any pair $i \in N$, $j \in N$, $i \neq j$, of substitutes in the set games (N, v)

$$SMC^v_{S\cup\{i\}} = SMC^v_{S\cup\{j\}} \quad \text{or "equivalently"} \quad \nabla^v_{S,\{i\}} = \nabla^v_{S,\{j\}}, \qquad (2.5.7)$$

for all $S \subseteq N \setminus \{i, j\}$. Then the SM-value has the equal treatment property. property.

Proof. For any arbitrary pair of players $i \in N$, $j \in N$, $i \neq j$, in a set game (N, v), we have

$$\begin{aligned} SM_i(N,v) &= \bigcup_{\substack{S\subseteq N\\ S\ni i}} SMC^v_S = \Bigg[\bigcup_{\substack{S\subseteq N\\ S\ni\{i,j\}}} SMC^v_S\Bigg] \cup \Bigg[\bigcup_{\substack{S\subseteq N\\ S\ni i, S\not\ni j}} SMC^v_S\Bigg] \\ &= \Bigg[\bigcup_{\substack{S\subseteq N\\ S\ni\{i,j\}}} SMC^v_S\Bigg] \cup \Bigg[\bigcup_{S\subseteq N\setminus\{i,j\}} SMC^v_{S\cup\{i\}}\Bigg] \end{aligned}$$

$$= \left[\bigcup_{\substack{S \subseteq N \\ S \ni \{i,j\}}} SMC_S^v \right] \cup \left[\bigcup_{S \subseteq N \setminus \{i,j\}} SMC_{S \cup \{j\}}^v \right]$$
$$= \left[\bigcup_{\substack{S \subseteq N \\ S \ni \{i,j\}}} SMC_S^v \right] \cup \left[\bigcup_{\substack{S \subseteq N \\ S \ni j, S \not\ni i}} SMC_S^v \right]$$
$$= \bigcup_{\substack{S \subseteq N \\ S \ni j}} SMC_S^v = SM_j(N, v),$$

where the fourth equality follows from $SMC_{S \cup i}^v = SMC_{S \cup j}^v$ for all $S \in N \setminus \{i, j\}$. So, the SM-value has the equal treatment property whenever (2.5.7) holds. □

The purpose of this section is to present an axiomatic characterization of any SM-value of the form (2.5.3). To be exact, we show that such a value is fully determined by global efficiency (2.5.5) and the (tacitly assumed) equal treatment property, as treated in Sections 2.3 and 2.4, together with a type of monotonicity property. One out of two proof techniques is based on the decomposition of any set game into a union of simple set games, in which the worth of any coalition equals either the empty set or a singleton consisting of one arbitrary, but fixed item.

Definition 2.5.6. Let an SM-value be of the form (2.5.3) assuming players' contribution independence. We say the value SM possesses the property of *semi-marginalistic contribution monotonicity* if for all $v, w \in G^N(\mathcal{U})$ and all $i \in N$,

$$SM_i(N, v) \subseteq SM_i(N, w) \qquad (2.5.8)$$

whenever $SMC_S^v \subseteq SMC_S^w$ for all $S \subseteq N$ with $S \ni i$, where the semi-marginalistic contribution SMC is associated with SM. In words, with respect to two different set games, the larger the player's semi-marginalistic contributions in the game, the more items are allocated to him.

Corollary 2.5.7. Any SM-value of the form (2.5.3) assuming players' contribution independence, such that both (2.5.4) and (2.5.7) hold, has the global efficiency, equal treatment and the semi-marginalistic contribution monotonicity properties.

Theorem 2.5.8. (Axiomatization)
Consider the setting of Definition 2.5.2, 2.5.4(ii) and 2.5.6. There exists a unique solution on the set game space $G^N(\mathcal{U})$ (with reference to a fixed player set N) satisfying the axioms of global efficiency, equal treatment and semi-marginalistic contribution monotonicity, and it is given by the SM-value of the form (2.5.3), based on players' contribution independence.

The proof of Theorem 2.5.8 proceeds in three steps. The first preliminary result provides another interpretation of any SM-value in that the value represents the maximal solution satisfying global efficiency and semi-marginalistic contribution monotonicity.

Proposition 2.5.9. If a value f on $G^N(\mathcal{U})$ has the global efficiency and semi-marginalistic contribution monotonicity properties, then the inclusion $f_i(N, v) \subseteq SM_i(N, v)$ holds for all $(N, v) \in G^N(\mathcal{U})$ and all $i \in N$.

Proof. Suppose a value f satisfies the axioms of global efficiency and marginalistic contribution monotonicity. Let $(N, v) \in G^N(\mathcal{U})$ and $i \in N$. In order to show the inclusion $f_i(N, v) \subseteq SM_i(N, v)$, let $x \in f_i(N, v)$ but $x \notin SM_i(N, v)$. Define a new set game (N, w) as follows: for any $S \subseteq N$

$$w(S) := \begin{cases} v(S) - \{x\}, & \text{if } \ x \in v(S), \\ v(S), & \text{otherwise.} \end{cases}$$

Notice that, for all $S \subseteq N$, $x \notin w(S)$. From this observation, together with the global efficiency (2.5.5) of f applied to the set game (N, w), we derive the following chain of inclusions:

$$f_i(N, w) \subseteq \bigcup_{j \in N} f_j(N, w) = \bigcup_{S \subseteq N} w(S) \subseteq \mathcal{U} - \{x\},$$

Particularly, $x \notin f_i(N, w)$. Next we claim $SMC_S^w = SMC_S^v$ for all $S \subseteq N$ and $S \ni i$ (where $SMC_S^v = v(S) - \nabla_S^v$). Consequently, $f_i(N, w) = f_i(N, v)$ by the marginalistic contribution monotonicity (2.5.8) of f, but this equality contradicts the facts that $x \in f_i(N, v)$ and $x \notin f_i(N, w)$. This contradiction completes the proof, provided we establish the claim above-mentioned.

Let $S \subseteq N$ with $i \in S$. We distinguish two cases. If $x \notin v(S)$, then $w(S) = v(S)$ and it holds that

$$SMC_S^w = w(S) - \nabla_S^w = v(S) - \nabla_S^w = v(S) - \nabla_S^v = SMC_S^v.$$

If $x \in v(S)$, then $w(S) = v(S) - \{x\}$ as well as $x \in \nabla_S^v$ (because of the assumption $x \notin SM_i(N, v)$) and thus it holds that

$$SMC_S^w = w(S) - \nabla_S^w = [v(S) - \{x\}] - \nabla_S^w = v(S) - \nabla_S^v = SMC_S^v.$$

This completes the proof of the remaining claim. Further, this proof indicates that the global efficiency may be replaced by any weak form of global efficiency, that is $\bigcup_{i \in N} f_i(N, v) \subseteq \bigcup_{S \subseteq N} v(S)$ for any set game (N, v). In addition, the definition of the expression ∇_S^w does not matter. □

The final part of the preliminary results, for the sake of a first proof technique of Theorem 2.5.8, deals with simple set games, which will be treated as the components of a decomposition for any arbitrary set game.

Definition 2.5.10. With every set game $(N, v) \in G^N(\mathcal{U})$ and every $x \in \mathcal{U}$, there is associated the *simple set game* (N, v_x) defined to be, for any $S \subseteq N$,

$$v_x(S) := \begin{cases} \{x\}, & \text{if } x \in v(S), \\ \emptyset, & \text{otherwise.} \end{cases}$$

The coalition $S \subseteq N$ is said to be winning in the simple set game (N, v_x) if $x \in v_x(S)$ or, equivalently, $x \in v(S)$.

Proposition 2.5.11. (Decomposition results for set games and semi-marginalistic values)
Let $(N, v) \in G^N(\mathcal{U})$ be a set game, $x \in \mathcal{U}$ and $S \subseteq N$. Recall $SMC_S^v := v(S) - \nabla_S^v$.

(i) $v = \bigcup_{y \in \mathcal{U}} v_y$ that is, for all $S \subseteq N$, $v(S) = \bigcup_{y \in \mathcal{U}} v_y(S)$. (2.5.9)

(ii) The following equivalence holds: $SMC_S^{v_x} = \{x\} \Leftrightarrow x \in SMC_S^v$.

$$ \tag{2.5.10} $$

$$(iii) \quad SM_i(N,v) = \bigcup_{y \in \mathcal{U}} SM_i(N, v_y). \tag{2.5.11}$$

For all $i \in N$ and every semi-marginalistic value $SM(N,v)$ of the form (2.5.3) assuming players' contribution independence.

(iv) If a solution $f(N,v)$ on $G^N(\mathcal{U})$ possesses the semi-marginalistic contribution monotonicity property, then it holds that $f_i(N,v) \supseteq f_i(N,v_x)$, for all $i \in N$ and all $x \in \mathcal{U}$.

Proof. The decomposition statement (2.5.9) of the set game (N,v) is trivial since $\mathcal{U} = v(T) \cup (\mathcal{U} - v(T))$ for all $T \subseteq N$. The decomposition statement (2.5.11) of the SM-value of the set game (N,v) is a direct consequence of the equivalence (2.5.10) because, for all $i \in N$, it holds that

$$\begin{aligned} \bigcup_{y \in \mathcal{U}} SM_i(N, v_y) &= \bigcup_{y \in \mathcal{U}} \bigcup_{\substack{S \subseteq N \\ S \ni i}} SMC_S^{v_y} = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \bigcup_{y \in \mathcal{U}} SMC_S^{v_y} \\ &= \bigcup_{\substack{S \subseteq N \\ S \ni i}} SMC_S^{v} = SM_i(N,v). \end{aligned}$$

The statement in part (iv) is a direct consequence of the equivalence (2.5.10) too due to the inclusion $SMC_S^{v_x} \subseteq SMC_S^{v}$ for all $S \subseteq N$ with $S \ni i$, and all $x \in \mathcal{U}$. It remains to prove, for every $S \subseteq N$, the equivalence (2.5.10) as follows.

$$\begin{aligned} SMC_S^{v_x} = \{x\} &\Leftrightarrow v_x(S) - \nabla_S^{v_x} = \{x\} \\ &\Leftrightarrow v_x(S) = \{x\} \quad and \quad \nabla_S^{v_x} = \emptyset \\ &\Leftrightarrow x \in v(S) \quad and \quad x \notin \nabla_S^{v_x} \\ &\Leftrightarrow x \in v(S) \quad and \quad x \notin \nabla_S^{v_y}, \quad \text{for all} \quad y \in \mathcal{U} \\ &\Leftrightarrow x \in v(S) \quad and \quad x \notin \nabla_S^{v} \\ &\Leftrightarrow x \in v(S) - \nabla_S^{v} \\ &\Leftrightarrow x \in SMC_S^{v} \end{aligned}$$

Concerning the fourth and fifth equivalence in the above chain, we make use of the relationships $\nabla_S^v = \nabla_S^{\bigcup_{y\in\mathcal{U}} v_y} = \bigcup_{y\in\mathcal{U}} \nabla_S^{v_y}$, while $v_y(T) \cap v_z(T) = \emptyset$ whenever $y \neq z$. □

Proof of the uniqueness part of Theorem 2.5.8.
Suppose a solution f on $G^N(\mathcal{U})$ satisfies the axioms of global efficiency, equal treatment and semi-marginalistic contribution monotonicity. Let (N, v) be a set game and $i \in N$. We show that $f_i(N, v) = SM_i(N, v)$ for all $i \in N$. By Propositions 2.5.9 and 2.5.11(iii)-(iv), the following relationships hold:

$$SM_i(N, v) = \bigcup_{y\in\mathcal{U}} SM_i(N, v_y)$$

and

$$SM_i(N, v) \supseteq f_i(N, v) \supseteq \bigcup_{y\in\mathcal{U}} f_i(N, v_y).$$

Fix the set game (N, v), player $i \in N$ and item $x \in \mathcal{U}$. It suffices to show that

$$SM_i(N, v_x) = f_i(N, v_x), \qquad \text{for every simple set game } (N, v_x). \tag{2.5.12}$$

The proof of (2.5.12) proceeds by induction on the number of winning coalitions in the semi-marginalistic contribution set games (N, SMC^{v_x}), defined to be $SMC^{v_x}(S) = SMC_S^{v_x} = v_x(S) - \nabla_S^{v_x}$ for all $S \subseteq N$. Coalition S is said to be winning in set game (N, SMC^{v_x}) if it holds that $SMC^{v_x}(S) = \{x\}$ or, equivalently, $x \in SMC_S^v$ (see (2.5.10)). We distinguish two cases, whether or not there exists a unique winning coalition.

Case one. Suppose there exists a unique winning coalition S_1 in the set game (N, SMC^{v_x}), that is $SMC^{v_x}(S_1) = \{x\}$ and $SMC^{v_x}(S) = \emptyset$, for all $S \neq S_1$. Our first claim is the following:

$$SM_j(N, v_x) = f_j(N, v_x), \qquad \text{for all } j \in N \setminus S_1. \tag{2.5.13}$$

Indeed, for all $j \in N \setminus S_1$, it holds, by definition of the set game, that $SMC_S^{v_x} = \emptyset$ for all $S \subseteq N$ with $S \ni j$. From this, together with Proposition 2.5.9 applied to the simple set game (N, v_x), we deduce the following chain of inclusions: for all $j \in N \setminus S_1$

$$f_j(N, v_x) \subseteq SM_j(N, v_x) = \bigcup_{\substack{S \subseteq N \\ S \ni j}} SMC_S^{v_x} = \emptyset,$$

so, for all $j \in N \setminus S_1$, $SM_j(N, v_x) = f_j(N, v_x)$.

Our second claim is the following: for all $S \subseteq N$

$$SMC_S^{v_x} = SMC_S^{SMC^{v_x}}$$

and thus

$$f_j(N, SMC^{v_x}) = f_j(N, v_x) \text{ and } SM_j(N, SMC^{v_x}) = SM_j(N, v_x), \text{ for all } j \in N. \tag{2.5.14}$$

Indeed, if $S \neq S_1$, then $SMC^{v_x}(S) = \emptyset$ and so $SMC_S^{SMC^{v_x}} = \emptyset$. Otherwise $SMC_{S_1}^{SMC^{v_x}} = SMC_{S_1}^{v_x} - \nabla_{S_1}^{SMC^{v_x}} = SMC_{S_1}^{v_x}$, since $\nabla_{S_1}^{SMC^{v_x}} = \emptyset$ due to $SMC^{v_x}(T) = \emptyset$, for all $T \subset S_1$. From $SMC_S^{v_x} = SMC_S^{SMC^{v_x}}$, for all $S \subseteq N$, together with the semi-marginalistic contribution monotonicity property for both f and SM, it follows immediately that, for all $j \in N$, $f_j(N, SMC^{v_x}) = f_j(N, v_x)$ and $SM_j(N, SMC^{v_x}) = SM_j(N, v_x)$.

The global efficiency (2.5.5) for both f and SM, applied to the set game (N, SMC^{v_x}), yields

$$\bigcup_{k \in N} f_k(N, SMC^{v_x}) = \bigcup_{k \in N} SM_k(N, SMC^{v_x}) = \{x\},$$

which equals $\{x\}$, or equivalently,

$$\bigcup_{k \in S_1} f_k(N, SMC^{v_x}) = \bigcup_{k \in S_1} SM_k(N, SMC^{v_x}) = \{x\},$$

since $f_j(N, SMC^{v_x}) = SM_j(N, SMC^{v_x}) = \emptyset$, for all $j \in N \setminus S_1$. Note that any pair of players in S_1 are substitutes in the set game (N, SMC^{v_x}) (since S_1 is the unique winning coalition). From the equal treatment property for both f and SM, applied to the game (N, SMC^{v_x}), we derive

$$f_j(N, SMC^{v_x}) = f_k(N, SMC^{v_x}),$$

as well as

$$SM_k(N, SMC^{v_x}) = SM_j(N, SMC^{v_x}),$$

for all $j, k \in S_1$. In summary, the latter efficiency equality simplifies to

$$f_i(N, SMC^{v_x}) = SM_i(N, SMC^{v_x}) = \{x\},$$

for all $i \in S_1$. From this and (2.5.13) and (2.5.14), we conclude that

$$SM_j(N, v_x) = f_j(N, v_x) = \emptyset$$

if $i \in N \setminus S_1$; and , if $i \in S_1$, it holds that

$$f_i(N, v_x) = f_i(N, SMC^{v_x}) = SM_i(N, SMC^{v_x}) = SM_i(N, v_x) = \{x\}.$$

This completes the proof of (2.5.12) if there exists one winning coalition in the game(N, SMC^{v_x}).

Case two. Suppose there are at least two winning coalitions in the set games (N, SMC^{v_x}), say, among others, coalition S_1. Particularly, it holds that $SMC^{v_x}(S_1) = \{x\}$ or equivalently, $x \in SMC^{v_x}_{S_1}$. Define two new set games (N, v_1) and (N, v_2), arising from the semi-marginalistic contribution game (N, SMC^v) such that v_1 is almost the semi-marginalistic contribution set game (N, SMC^v) and v_2 almost the empty set game. To be exact,

$$v_1(S) := \begin{cases} SMC^v_S, & \text{if} \quad S \neq S_1, \\ \emptyset, & \text{if} \quad S = S_1, \end{cases} \tag{2.5.15}$$

$$v_2(S) := \begin{cases} SMC^v_S, & \text{if} \quad S = S_1, \\ \emptyset, & \text{if} \quad S \neq S_1 \end{cases} \tag{2.5.16}$$

From the descriptions (2.5.15)-(2.5.16) of both set games, together with the equivalence (2.5.10), we obtain that their associated simple set games $(N, (v_1)_x)$ and $(N, (v_2)_x)$ are given by

$$(v_1)_x(S) := \begin{cases} SMC^{v_x}_S, & \text{if} \quad S \neq S_1, \\ \emptyset, & \text{if} \quad S = S_1, \end{cases} \tag{2.5.17}$$

$$(v_2)_x(S) := \begin{cases} \emptyset, & \text{if} \quad S \neq S_1, \\ SMC_S^{v_x}, & \text{if} \quad S = S_1. \end{cases} \tag{2.5.18}$$

Note that, for all $S \subseteq N$, the inclusions $(v_1)_x(S) \subseteq (v)_x(S)$ and $(v_2)_x(S) \subseteq (v)_x(S)$ hold. Concerning the semi-marginalistic contribution in both simple set games, as given by (2.5.17)-(2.5.18), we claim the following:

$$SMC_{S_1}^{(v_1)_x} = \emptyset \quad \text{and} \quad SMC_S^{(v_1)_x} = SMC_S^{v_x}, \quad \text{for all} \quad S \neq S_1 \tag{2.5.19}$$

$$SMC_{S_1}^{(v_2)_x} = SMC_{S_1}^{v_x} \quad \text{and} \quad SMC_S^{(v_2)_x} = \emptyset, \quad \text{for all} \quad S \neq S_1. \tag{2.5.20}$$

In order to verify (2.5.19), for all $S \neq S_1$, the following chain of equalities holds:

$$\begin{aligned} SMC_S^{(v_1)_x} &= (v_1)_x(S) - \nabla_S^{(v_1)_x} = SMC_S^{v_x} - \nabla_S^{(v_1)_x} \\ &= \left[v_x(S) - \nabla_S^{(v)_x} \right] - \nabla_S^{(v_1)_x} \\ &= \left[v_x(S) - \nabla_S^{(v)_x} \right] = SMC_S^{v_x}, \end{aligned}$$

where the second equality follows from the definition of the set game $(N, (v_1)_x)$ and the fourth from $\nabla_S^{(v_1)_x} \subseteq \nabla_S^{(v)_x}$, because of $T \subset S$, $(v_1)_x(T) \subseteq v_x(T)$. So, (2.5.19) holds. In order to verify (2.5.20), the following chain of equalities holds:

$$SMC_{S_1}^{(v_2)_x} = (v_2)_x(S_1) - \nabla_{S_1}^{(v_2)_x} = SMC_{S_1}^{v_x} - \nabla_{S_1}^{(v_2)_x} = SMC_{S_1}^{v_x},$$

due to the equality $\nabla_{S_1}^{(v_2)_x} = \emptyset$, because of $(v_2)_x(T) = \emptyset$, for all $T \subset S_1$. So, (2.5.20) holds too. Clearly, it concerns a disjoint union so that $SMC_S^{v_x} = SMC_S^{(v_1)_x} \cup SMC_S^{(v_2)_x}$ for all $S \subseteq N$. From this we deduce the following chain of equalities:

$$\begin{aligned} SM_i(N, v_x) &= \bigcup_{\substack{S \subseteq N \\ S \ni i}} SMC_S^{v_x} = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \left[SMC_S^{(v_1)_x} \cap SMC_S^{(v_2)_x} \right] \\ &= \left[\bigcup_{\substack{S \subseteq N \\ S \ni i}} SMC_S^{(v_1)_x} \right] \cup \left[\bigcup_{\substack{S \subseteq N \\ S \ni i}} SMC_S^{(v_2)_x} \right] \\ &= SM_i(N, (v_1)_x) \cup SM_i(N, (v_2)_x). \end{aligned}$$

By (2.5.20), the semi-marginalistic contribution set game $(N, SMC^{(v_2)_x})$ has a unique winning coalition S_1, whereas, by (2.5.19), the collection of winning coalitions in the semi-marginalistic contribution set game $(N, SMC^{(v_1)_x})$ is identical to the one in the initial semi-marginalistic contribution set game (N, SMC^{v_x}), except for coalition S_1. The induction hypothesis (2.5.12) applied to both set games $(N, (v_2)_x$ and $(N, (v_1)_x)$ yields

$$SM_i(N, (v_2)_x) = f_i(N, (v_2)_x)$$

as well as

$$SM_i(N, (v_1)_x) = f_i(N, (v_1)_x).$$

Further, from the inclusion $SMC_S^{(v_1)_x}) \subseteq SMC_S^{v_x}$ for all $S \subseteq N$, together with the semi-marginalistic contribution monotonicity for f, we derive the inclusion $f_i(N, (v_1)_x) \subseteq f_i(N, v_x)$ and, similarly, $f_i(N, (v_2)_x) \subseteq f_i(N, v_x)$. Finally, we conclude that the following chain of inclusion holds:

$$\begin{aligned} SM_i(N, v_x) &= SM_i(N, (v_1)_x) \cup SM_i(N, (v_2)_x) \\ &= f_i(N, (v_1)_x) \cup f_i(N, (v_2)_x) \\ &\subseteq f_i(N, v_x) \subseteq SM_i(N, v_x). \end{aligned}$$

We arrive at the equality $SM_i(N, v_x) = f_i(N, v_x)$. This completes both the inductive proof of (2.5.12) and the full proof of Theorem 2.5.8. □

Remark 2.5.12. Throughout the above Theorem 2.5.8, for any set game $(N, v) \in G^N(\mathcal{U})$ and any coalition $S \subseteq N$, the associated expression ∇_S^v is supposed to possess the following minor property:

$$\nabla_S^w \subseteq \nabla_S^v \quad \text{when } w(S) \subseteq v(S), \text{ for all } T \subset S. \tag{2.5.21}$$

In the context of the empty set game, (2.5.21) is meant to be read as $\nabla_S^w = \emptyset$, whenever $w(S) = \emptyset$, for all $T \subset S$.

In the last part of this section, we will give another proof of Theorem 2.5.8. The second proof technique is based on the decomposition of any set game into a union of elementary set games. Recall that for elementary set games, the worth of all coalitions, except one, equals the empty set. In fact, the decomposition technique is mainly applied to the semi-marginalistic contribution, as given by (2.5.3), the concept of which is treated as a new set game arising from an initial set game.

Definition 2.5.13. (the semi-marginalistic contribution set game and elementary set games)

(i) With every set game $(N, v) \in G^N(\mathcal{U})$, there is associated the *semi-marginalistic contribution set game* $(N, SMC^v) \in G^N(\mathcal{U})$,

$$SMC^v(S) := SMC^v_S = v(S) - \nabla^v_S, \quad \text{for all } S \subseteq N. \tag{2.5.22}$$

(ii) With every set game $(N, v) \in G^N(\mathcal{U})$, and every coalition $T \subseteq N$ with $T \neq \emptyset$, there is associated the *elementary set game* $(N, SMC^v_T \cap E_T) \in G^N(\mathcal{U})$

$$(SMC^v_T \cap E_T)(S) := \begin{cases} \emptyset, & \text{if } S \neq T, \\ SMC^v_T, & \text{if } S = T. \end{cases} \tag{2.5.23}$$

Proposition 2.5.14. Consider the setting of Definition 2.5.13. It is tacitly assumed that, for all $T \subset S$, (2.5.21) holds, that is $\nabla^w_S \subseteq \nabla^v_S$ whenever $w(T) \subseteq v(T)$ for all $T \subset S$.

(i) For all $S \subset N$,

$$SMC^v = \bigcup_{T \subseteq N} (SMC^v_T \cap E_T), \quad \text{that is} \quad SMC^v(S) = \bigcup_{T \subseteq N} (SMC^v_T \cap E_T)(S). \tag{2.5.24}$$

(ii)

$$SMC^{SMC^v}_S = SMC^v_S, \quad \text{for all } S \subset N. \tag{2.5.25}$$

(iii) If a solution f on $G^N(\mathcal{U})$ possesses the semi-marginalistic contribution monotonicity property, then it holds that $f_i(N, v) = f_i(N, SMC^v)$, for all $i \in N$.

Proof of Proposition 2.5.14.
By (2.5.22)-(2.5.23), the decomposition statement (2.5.24) of the semi-marginalistic

contribution set game (N, SMC^v) is trivial. To prove (2.5.25), we claim, for all $S \subset N$, the following chain of equalities:

$$\begin{aligned} SMC_S^{SMC^v} &= SMC_S^v - \nabla_S^{SMC^v} = \left[v(S) - \nabla_S^v \right] - \nabla_S^{SMC^v} \\ &= v(S) - \nabla_S^v = SMC_S^v, \end{aligned}$$

where the third equality holds because of the inclusion $\nabla_S^{SMC^v} \subseteq \nabla_S^v$. The latter inclusion is due to the fact that $SMC^v(T) \subseteq v(T)$, for all $T \subset S$ and all $S \subseteq N$. So, (2.5.25) holds. The statement in part (iii) is a direct consequence of (2.5.25) and the semi-marginalistic contribution monotonicity of the solution f. □

Lemma 2.5.15. Let $\mathcal{C}$ be an arbitrary, non-empty sub-collection of coalitions not including the empty set. With every set game $(N, v) \in G^N(\mathcal{U})$, let there be associated a partially semi-marginalistic contribution set game $(N, w_\mathcal{C})$ defined by $w_\mathcal{C} := \bigcup_{T \in \mathcal{C}} (SMC_T^v \cap E_T)$, that is, for all $S \subseteq N$

$$w_\mathcal{C}(S) := \begin{cases} \emptyset, & S \notin \mathcal{C}, \\ SMC_S^v, & S \in \mathcal{C}. \end{cases} \tag{2.5.26}$$

(i) Then the game $(N, w_\mathcal{C})$ is invariant under the SMC-concept, that is $SMC_S^{w_\mathcal{C}} = w_\mathcal{C}(S)$, for all $S \subseteq N$. Or, equivalently, for all $S \subseteq N$

$$SMC_S^{w_\mathcal{C}} := \begin{cases} \emptyset, & S \notin \mathcal{C}, \\ SMC_S^v, & S \in \mathcal{C}. \end{cases} \tag{2.5.27}$$

(ii) If a solution f on $G^N(\mathcal{U})$ possesses the properties of global efficiency, equal treatment and semi-marginalistic contribution monotonicity, then it holds that $f_i(N, w_\mathcal{C}) = \bigcup_{\substack{T \in \mathcal{C} \\ T \ni i}} SMC_T^v$, for all $(N, v) \in G^N(\mathcal{U})$ and all $i \in N$.

Before we prove Lemma 2.5.15, we claim that both Proposition 2.5.14(iii) and Lemma 2.5.15(ii), applied to the trivial collection $\mathcal{C} = 2^N$ without the empty set, complete the alternative proof of the main Theorem 2.5.8. Indeed, by

(2.5.22) and (2.5.26), the choice $\mathcal{C} = 2^N$ yields $SMC^v = w_{\mathcal{C}}$ and so, for every set game $(N, v) \in G^N(\mathcal{U})$, it holds that, for all $i \in N$,

$$\begin{aligned} f_i(N,v) &= f_i(N, SMC^v) = f_i(N, w_{\mathcal{C}}) \\ &= \bigcup_{\substack{T\in\mathcal{C}\\ T\ni i}} SMC^v_T = \bigcup_{\substack{T\subseteq N\\ T\ni i}} SMC^v_T = SM_i(N,v). \end{aligned}$$

We conclude that $f(N, v) = SM(N, v)$ whenever the value f on $G^N(\mathcal{U})$ possesses the properties of global efficiency, equal treatment and semi-marginalistic contribution monotonicity.

Obviously, any SM-value of the form (2.5.1) satisfies the destructive player property, whereas, for any solution f on $G^N(\mathcal{U})$ the destructive player property arises from the semi-marginalistic contribution monotonicity property of f (with reference to the SMC-concept induced by $SM(N, v)$) and the global efficiency of f (applied to the empty set game).

Proof of Lemma 2.5.15.
Let $S \subseteq N$. If $S \notin \mathcal{C}$, then $w_{\mathcal{C}}(S) = \emptyset$ and so, $SMC^{w_{\mathcal{C}}}_S = \emptyset$. In case $S \in \mathcal{C}$, then we claim the following chain of equalities:

$$\begin{aligned} SMC^{w_{\mathcal{C}}}_S &= w_{\mathcal{C}}(S) - \nabla^{w_{\mathcal{C}}}_S = SMC^v_S - \nabla^{w_{\mathcal{C}}}_S \\ &= \left[v(S) - \nabla^v_S \right] - \nabla^{w_{\mathcal{C}}}_S = v(S) - \nabla^v_S = SMC^v_S, \end{aligned}$$

where the fourth equality holds because of the inclusion $\nabla^{w_{\mathcal{C}}}_S \subseteq \nabla^v_S$. The latter inclusion is due to the fact that $w_{\mathcal{C}}(T) \subseteq v(T)$, for all $T \subset S$ and all $S \subseteq N$. So, (2.5.27) holds.
(ii) Suppose a solution f on $G^N(\mathcal{U})$ satisfies the axioms of global efficiency, equal treatment, and semi-marginalistic contribution monotonicity. Fix the set game (N, v) and player i. We show, by induction on the number $|C|$ of coalitions in the collection $\mathcal{C}$, that it holds that, for all sub-collection $\mathcal{C}$ of 2^N,

$$f_i(N, w_{\mathcal{C}}) = \bigcup_{\substack{T\in\mathcal{C}\\ T\ni i}} SMC^v_T. \tag{2.5.28}$$

Let $c = |\mathcal{C}|$.

Case one. Suppose, for the moment, $|C| = 1$, say $\mathcal{C} = \{T\}$. By (2.5.26), $w_{\mathcal{C}}(T) = SMC^v_T$ and $w_{\mathcal{C}}(S) = \emptyset$, for all $S \subseteq N$ with $S \neq T$. The global efficiency of f yields $\bigcup\limits_{j \in N} f_j(N, w_{\mathcal{C}}) = \bigcup\limits_{S \subseteq N} w_{\mathcal{C}}(S) = SMC^v_T$. On the one hand, any pair of members of T are substitutes in the game $(N, w_{\mathcal{C}})$ and, consequently, the equal treatment property of f yields $f_j(N, w_{\mathcal{C}}) = f_k(N, w_{\mathcal{C}})$, for all $j, k \in T$. On the other hand, non-members of T are destructive players in the game $(N, w_{\mathcal{C}})$ and consequently, the destructive player property of f yields $f_l(N, w_{\mathcal{C}}) = \emptyset$, for all $l \in N \setminus T$. Notice that the destructive player property follows immediately from semi-marginalistic contribution monotonicity together with global efficiency (applied to the empty set game). So far, in case $\mathcal{C} = \{T\}$, we conclude that

$$f_i(N, w_{\mathcal{C}}) = \begin{cases} \emptyset, & \text{if } \ i \in N \setminus T, \\ SMC^v_T, & \text{if } \ \imath \in T. \end{cases}$$

So, (2.5.28) holds.

Case two. From now on, we may suppose $c \geq 2$. We distinguish three subcases.

Subcase one. Suppose $i \in \left[\bigcup\limits_{S \in \mathcal{C}} S\right] - \left[\bigcap\limits_{S \in \mathcal{C}} S\right]$. Define the collection $\mathcal{C}_i := \{S \in \mathcal{C} | i \in S\}$. By assumption, the strict inclusion $\mathcal{C}_i \subset \mathcal{C}$ holds and so, the induction hypothesis applies to the new collection $\mathcal{C}_i$, yielding

$$f_i(N, w_{\mathcal{C}_i}) = \bigcup_{\substack{T \in \mathcal{C}_i \\ T \ni i}} SMC^v_T = \bigcup_{\substack{T \in \mathcal{C} \\ T \ni i}} SMC^v_T.$$

Thus, it remains to show the equality $f_i(N, w_{\mathcal{C}_i}) = f_i(N, w_{\mathcal{C}})$ and for that purpose it suffices, by semi-marginalistic contribution monotonicity of f, to show that $SMC^{w_{\mathcal{C}}}_S = SMC^{w_{\mathcal{C}_i}}_S$, for all $S \subseteq N$ with $S \ni i$. The latter equality is a direct consequence of Lemma 2.5.15(i) (applied to both collections $\mathcal{C}$ and $\mathcal{C}_i$ respectively), by taking into account that, for every $S \subseteq N$ with $S \ni i$, the equivalence $S \in \mathcal{C}_i \Leftrightarrow S \in \mathcal{C}$ is valid. So (2.5.28) holds.

Subcase two. Suppose $i \in \bigcap\limits_{S \in \mathcal{C}} S$, that is $i \in S$ for all $S \in \mathcal{C}$. Take any $S^* \in \mathcal{C}$. By applying Lemma 2.5.15(i) to both collections $\mathcal{C}$ and the one-element collection $\mathcal{C}_* := \{S^*\}$, we obtain the inclusion $SMC^{w_{\mathcal{C}_*}}_S \subseteq SMC^{w_{\mathcal{C}}}_S$, for all $S \subseteq N$. Now semi-marginalistic contribution monotonicity of f yields $f_i(N, w_{\mathcal{C}_*}) \subseteq f_i(N, w_{\mathcal{C}})$ for all $\mathcal{C}_* = \{S^*\}$, where $S^* \in \mathcal{C}$. On the one hand, for the one-element collection $\mathcal{C}_* = \{S^*\}$, it holds, as shown in case one, that $f_i(N, w_{\mathcal{C}_*}) = SMC^v_{S^*}$ (since $i \in S^*$ for all $S^* \in \mathcal{C}$). So far, we conclude the following chain of inclusions:

$$\bigcup_{S \in \mathcal{C}} SMC^v_S = \bigcup_{S^* \in \mathcal{C}} SMC^v_{S^*} = \bigcup_{S^* \in \mathcal{C}} f_i(N, w_{\mathcal{C}_*}) \subseteq f_i(N, w_{\mathcal{C}}).$$

On the other hand, the global efficiency of f yields another chain of inclusions:

$$f_i(N, w_{\mathcal{C}}) \subseteq \bigcup_{j \in N} f_j(N, w_{\mathcal{C}}) = \bigcup_{S \subseteq N} w_{\mathcal{C}}(S) = \bigcup_{S \in \mathcal{C}} SMC^v_S.$$

All together, we conclude that

$$f_i(N, w_{\mathcal{C}}) = \bigcup_{S \in \mathcal{C}} SMC^v_S = \bigcup_{\substack{S \in \mathcal{C} \\ S \ni i}} SMC^v_S,$$

where the latter equality is due to the assumption $i \in \bigcap_{S \in \mathcal{C}} S$. So (2.5.28) holds.

Subcase three. Suppose $i \notin \bigcup_{S \in \mathcal{C}} S$, that is $i \notin S$ for all $S \in \mathcal{C}$. We claim $f_i(N, w_{\mathcal{C}}) = \emptyset$, since player i turns out be a destructive player in the set game $(N, w_{\mathcal{C}})$. Indeed, for all $S \subseteq N$ with $S \ni i$, it holds that $S \notin \mathcal{C}$ and so, $w_{\mathcal{C}}(S) = \emptyset$. As in case one recall that the destructive player property follows immediately from semi-marginalistic contribution monotonicity together with global efficiency (applied to the empty set game). This completes the inductive proof of (2.5.28). $\square$

2.6 Some properties of values for set games

In this section, we will discuss some properties, including *standard allocation* for two-person set games and *population monotonic allocation schemes* for set games. Several kinds of values for set games were introduced in the previous sections. Which kind of value is more reasonable for set games? One notices that the values

$$f_i(N, v) = \bigcup_{\substack{T \subseteq N \\ T \ni i}} v(T) \quad \text{and} \quad f_i(v) = v(N),$$

are rather unfair for players, since players will disagree with such allocations. Now let us introduce some other properties of values for set games.

Example 2.6.1. Let $N = \{i, j\}$, then (N, v) is a two-person set game. In the two-person setting, the coalitions consist of $\{i\}$, $\{j\}$ and $\{i, j\}$, and their worths are $v(\{i\})$, $v(\{j\})$ and $v(\{i, j\})$ respectively. How to allocate all the attainable items to the two players? Here the set of all attainable items is given by the union $v(\{i\}) \cup v(\{j\}) \cup v(\{i, j\})$. If any player accept to share items, then player i wants to get at least the items of his individual worth $v(\{i\})$ which he can get by non-cooperation; and the items of $v(\{j\})$ are wanted by player j. In this approach, both players i and j accept to share the items in the intersection of $v(\{i\})$ and $v(\{j\})$, attainable by non-cooperation. Then the remainder is $v(\{i, j\}) - v(\{i\}) - v(\{j\})$. It is reasonable that both players may share the remainder $v(\{i, j\}) - v(\{i\}) - v(\{j\})$ together, because the worth $v(\{i, j\})$ can be obtained only by player i and player j in a cooperative framework. This results in the standard allocation for two-person set games.

$$\begin{aligned}
\omega_i(N, v) &= v(\{i\}) \cup [v(\{i, j\}) - v(\{i\}) - v(\{j\})] \\
&= v(\{i\}) \cup [v(\{i, j\}) - v(\{j\})] \\
\omega_j(N, v) &= v(\{j\}) \cup [v(\{i, j\}) - v(\{i\}) - v(\{j\})] \\
&= v(\{j\}) \cup [v(\{i, j\}) - v(\{i\})].
\end{aligned}$$

Obviously, $f_i(N, v) = v(N)$ (introduced in Section 1.4), $f_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} v(S)$ (introduced in Section 1.4), and, $ICM_i(N, v) = \bigcup_{\substack{S \in N, \\ S \ni i}} \{ \bigcup_{j \in S} [v(S) - v(S \setminus \{j\})]\}$ and $OCM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \left[v(S) - \bigcup_{T \subseteq N \setminus \{i\}} v(T) \right]$ (introduced in Section 2.4) are not standard allocations for two-person set games.

Theorem 2.6.2 For any set game $(N, v) \in G^N(\mathcal{U})$, the following values (cf. Example 2.1.1) are standard allocations for two-person set games,

$$IM_i(N, v) = \bigcup_{\substack{S \subseteq N \\ S \ni i}} \left[v(S) - v(S \setminus \{i\}) \right], \quad \text{for all } i \in N,$$

$$SCM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) - \bigcup_{T\subset S} v(T) \right], \quad \text{for all } i \in N,$$

$$OIM_i(N,v) = \bigcup_{\substack{S\subseteq N\\ S\ni i}} \left[v(S) - \bigcup_{j\in S} v(S\setminus\{j\}) \right], \quad \text{for all } i \in N.$$

It is easy to verify Theorem 2.6.3.

We recall some concepts and results given by Sprumont [68] for TU-games. In most interesting economic applications, the function v is super-additive, so that it is efficient for the players to form the grand coalition N. The question then arises how to divide $v(N)$. Among the most the popular multi-valued concepts, the core proposes a very compelling solution. Formally, the core of the TU-game v is the set

$$\ell(v) := \left\{ X \in \mathbb{R}^n | \sum_{i\in N} X_i = v(N) \text{ and } \sum_{i\in S} X_i \geq v(S) \text{ for all } S \subseteq N \right\}. \tag{2.6.1}$$

If an allocation belongs to the core, no coalition S will unanimously decide to challenge it since there is no way to divide $v(S)$ so as to make every member of S better off.

But the process of coalition formation is a complex one, in which players may not necessarily achieve full efficiency. To deal with the possibility of partial cooperation, we should specify not only how to allocate $v(N)$ but also how to allocate the worth of every coalition S, should the players fail to fully cooperate and S eventually be formed. The need for such a generalized allocation was stressed, among others, by Dutta & Ray [28]. The concept we propose here is in that spirit. Our concern is to guarantee that once a coalition S has decided upon an allocation of $v(S)$, no player will ever be tempted to induce the formation of a coalition smaller than S by using his bargaining skills or by any other means. This amounts to requiring that the payoff to every player increases as the coalition to which he belongs grows larger. Formally, we are searching for what we call a *population monotonic allocation scheme*, a PMAS:

Definition 2.6.3. A vector $x = (x_{iS})_{i \in S,\ S \subseteq N}$ is a *population monotonic allocation scheme* of the TU-game (N, v) if and only if it satisfies the following conditions:

$$\text{For all } S \subseteq N, \quad \sum_{i \in S} x_{iS} = v(S). \tag{2.6.2}$$

$$\text{For all } S, T \subseteq N \text{ and } S \ni i, \quad S \subseteq T \Rightarrow x_{iS} \leq x_{iT}. \tag{2.6.3}$$

To give another justification for our concept, we may reverse the perspective. Instead of regarding N as a given society, which breaks down into smaller groups S, let us reinterpret each S as a society of its own, and N as some maximal conceivable society. In this variable population interpretation of the model, condition (2.6.3) is akin to the population monotonicity axiom introduced by Thomson [73] in the context of bargaining. See also Chun's solidarity axiom [15] for quasi-linear social choice problems and Moulin's variant [48] for the fair division problem.

Condition (2.6.3) has much normative appeal in several contexts. Suppose we want to choose the level of production of the public good and share its cost. Since a newcomer never reduces anyone's consumption and may pay some share of the cost, it seems reasonable to ask that he be in Foley's core [32]. Moulin [48] constructs a PMAS that further satisfies the rather natural condition that no agent enjoys more than the utility from choosing the level of public good and paying 1/n th of the cost. Sprumont [68] has given the conditions for the existence of a PMAS .

Let us discuss the PMAS for set games.

Definition 2.6.4. Let $(N, v) \in G^N(\mathcal{U})$ and $S \subseteq N$. A *subset game* $(S, v_S,)$ of (N, v) is obtained by restricting v to the power set of S, *i.e.*, for any $T \subseteq S$, $v_S(T) = v(T)$.

Definition 2.6.5. Let $(N, v) \in G^N(\mathcal{U})$, (S, v_S) be a subset game and let f be an allocation for set games. $f(N, v)$ is a *population monotonic allocation scheme* of the set game (N, v) if and only if it satisfies the following conditions:

$$\text{for all } S \subseteq N, \quad \bigcup_{i \in S} f_i(S, v_S) = \bigcup_{T \subseteq S} v(T), \tag{2.6.4}$$

$$\text{for all } S, T \subseteq N \text{ and } S \ni i, \quad S \subseteq T \Rightarrow f_i(S, v_S) \subseteq f_i(T, v_T), \tag{2.6.5}$$

where $f(S, v_S)$ are obtained by applying $f(N, v)$ to subset game (S, v_S).

Lemma 2.6.6. A value f for set game (N, v) satisfies

$$f_i(N,v) = \Big[v(N) - v(N \setminus \{i\})\Big] \cup \bigcup_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, v_{N \setminus \{j\}}), \quad \text{for all } i \in N,$$

if and only if

$$f_i(N,v) = \bigcup_{\substack{S \subset N, \\ S \ni i}} \Big[v(S) - v(S \setminus \{i\})\Big] = IM_i(N,v), \quad \text{for all } i \in N,$$

where $f_i(N \setminus \{j\}, v_{N \setminus \{j\}})$ means that $f(N, v)$ is applied to the subset game $(N \setminus \{j\}, v_{N \setminus \{j\}})$. In addition, $T \subset S$ implies that T is a proper subset of S. The value $IM(N, v)$ was introduced in Section 2.2.

Proof. We will proceed by induction on the number $n = |N|$.
(**I**) When $n = 1$, then $N = \{i\}$. It is easy to say that

$$f_i(N,v) = v(\{i\}) - v(\emptyset) = v(\{i\}) = IM_i(N,v).$$

(**II**) Suppose $n = 2$ and $N = \{i, j\}$. By the definition of $IM_i(N, v)$ and the condition of Lemma 2.6.6, we can conclude that

$$\begin{aligned} f_i(N,v) &= \Big[v(N) - v(N \setminus \{i\})\Big] \cup \bigcup_{j \in N \setminus \{i\}} f_i(N \setminus \{j\}, v_{N \setminus \{j\}}) \\ &= \Big[v(\{i,j\}) - v(\{j\})\Big] \cup f_i(i,v) = \Big[v(\{i,j\}) - v(\{j\})\Big] \cup v(\{i\}), \\ IM_i(N,v) &= \bigcup_{\substack{S \in N, \\ S \ni i}} \Big[v(S) - v(S \setminus \{i\})\Big] \\ &= \Big[v(\{i,j\}) - v(\{j\})\Big] \cup v(\{i\}) = f_i(N,v), \end{aligned}$$

and

$$f_j(N,v) = IM_j(N,v).$$

So, if $n = 1, 2$, the lemma holds

(**III**) Suppose $n > 2$. By the hypothesis we obtain that

$$\begin{aligned}
f_i(N,v) &= [v(N) - v(N \setminus \{i\})] \cup \bigcup_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \\
&= [v(N) - v(N \setminus \{i\})] \cup \bigcup_{j \in N\setminus\{i\}} IM_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \\
&= [v(N) - v(N \setminus \{i\})] \cup \bigcup_{j \in N\setminus\{i\}} \bigcup_{\substack{S \subseteq N\setminus\{j\},\\ S \ni i}} \Big[v(S) - v(S \setminus \{i\})\Big] \\
&= [v(N) - v(N \setminus \{i\})] \cup \bigcup_{\substack{S \subset N,\\ S \ni i}} \Big[v(S) - v(S \setminus \{i\})\Big] \\
&= \bigcup_{\substack{S \subseteq N,\\ S \ni i}} \Big[v(S) - v(S \setminus \{i\})\Big] = IM_i(N,v),
\end{aligned}$$

where the second equality follows from the hypothesis. □

Lemma 2.6.7. A value f for set game (N,v) satisfies

$$f_i(N,v) = [v(N) - \nabla_N^v] \cup \bigcup_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}}), \qquad \text{for all } i \in N,$$

if and only if

$$f_i(N,v) = \bigcup_{\substack{S \subseteq N\\ S \ni i}} \Big[v(S) - \nabla_S^v\Big] = SM_i(N,v), \qquad \text{for all } i \in N,$$

where $f_i(N \setminus \{j\}, v_{N\setminus\{j\}})$ means that f is applied to the subset game $(N \setminus \{j\}, v_{N\setminus\{j\}})$. The SM-value is introduced in Section 2.5 and ∇_S^v is independent of players.

Proof. We will proceed by induction on the number $n = |N|$.

(**I**) If $n = 1$ then the lemma holds obviously.

(**II**) If $n = 2$ and $N = \{i, j\}$, by the definition of $SM_i(N,v)$ and the condition of Lemma 2.6.7, we find that

$$\begin{aligned} f_i(N,v) &= [v(N) - \nabla^v_N] \cup \bigcup_{j\in N\setminus\{i\}} f_i(N\setminus\{j\}, v_{N\setminus\{j\}}) \\ &= [v(\{i,j\}) - \nabla^v_{\{i,j\}}] \cup f_i(\{i\}, v) \\ &= [v(\{i,j\}) - \nabla^v_{\{i,j\}}] \cup v(\{i\}), \\ SM_i(N,v) &= \bigcup_{\substack{S\in N\\ S\ni i}} [v(S) - \nabla^v_S] = [v(\{i,j\}) - \nabla^v_{\{i,j\}}] \cup v(\{i\}) = f_i(N,v), \end{aligned}$$

and

$$f_j(N,v) = SM_j(N,v).$$

So, when $n = 1, 2$ the lemma holds.

(**III**) Suppose $n > 2$, by the hypothesis we obtain that

$$\begin{aligned} f_i(N,v) &= [v(N) - \nabla^v_N] \cup \bigcup_{j\in N\setminus\{i\}} f_i(N\setminus\{j\}, v_{N\setminus\{j\}}) \\ &= [v(N) - \nabla^v_N] \cup \bigcup_{j\in N\setminus\{i\}} SM_i(N\setminus\{j\}, v_{N\setminus\{j\}}) \\ &= [v(N) - \nabla^v_N] \cup \bigcup_{j\in N\setminus\{i\}} \{ \bigcup_{\substack{S\subseteq N\setminus\{j\}\\ S\ni i}} [v(S) - \nabla^v_S]\} \\ &= [v(N) - \nabla^v_N] \cup \bigcup_{\substack{S\subset N,\\ S\ni i}} [v(S) - \nabla^v_S] \\ &= \bigcup_{\substack{S\subseteq N\\ S\ni i}} [v(S) - \nabla^v_S] = SM_i(N,v), \end{aligned}$$

where the second equality is obtained by the hypothesis of induction. □

Theorem 2.6.8. For any set game $(N,v) \in G^N(\mathcal{U})$, the IM-value is a population monotonic allocation scheme for set games.

Proof. By the definition of the IM-value, we have

$$\bigcup_{i\in N} IM_i(N,v) = \bigcup_{i\in N}\bigcup_{S\ni i}[v(S) - v(S\setminus\{i\})] = \bigcup_{S\subseteq N} [v(S) - \bigcap_{i\in S} v(S\setminus\{i\})].$$

For any $T \subseteq N$, let $t = |T|$. We will prove the following claim by induction on t. For any $T \subseteq N$,

$$\bigcup_{i \in T} IM_i(T, v_T) = \bigcup_{S \subseteq T} v(S). \tag{2.6.6}$$

If $t = 1$, obviously (2.6.6) holds. Suppose $t \geq 2$. By the hypothesis of induction, we have that, for any $T \subseteq N$,

$$\begin{aligned}
\bigcup_{i \in T} IM_i(T, v_T) &= \bigcup_{S \subseteq T} [v(S) - \bigcap_{i \in S} v(S \setminus \{i\})] \\
&= [v(T) - \bigcap_{i \in T} v(T \setminus \{i\})] \cup [\bigcup_{S \subset T} (v(S) - \bigcap_{i \in S} v(S \setminus \{i\}))] \\
&= [v(T) - \bigcap_{i \in T} v(T \setminus \{i\})] \cup [\bigcup_{i \in T} \bigcup_{S \subseteq T \setminus \{i\}} (v(S) - \bigcap_{j \in S} v(S \setminus \{j\}))] \\
&= [v(T) - \bigcap_{i \in T} v(T \setminus \{i\})] \cup [\bigcup_{i \in T} \bigcup_{S \subseteq T \setminus \{i\}} v(S))] \\
&= [v(T) - \bigcap_{i \in T} v(T \setminus \{i\})] \cup [\bigcup_{S \subset T} v(S)] = \bigcup_{S \subseteq T} v(S),
\end{aligned}$$

where the fourth equality follows from the hypothesis of induction. So, this proves that the IM-value satisfies the condition (2.6.4). Further, by Lemma 2.6.6, we know that for any n, $n > 1$,

$$IM_i(N, v) \supseteq IM_i(N \setminus \{j\}, v_{N \setminus \{j\}}) \supseteq IM_i(N \setminus \{k, j\}, v_{N \setminus \{k,j\}}) \supseteq \cdots \supseteq IM_i(i, v_i),$$

where $i \neq j, k, ...$ Therefore for any $S, T \subseteq N$ with $S \subset T$, we have that, for any $i \in S$,

$$IM_i(S, v_S) \subseteq IM_i(T, v_T).$$

This implies that the IM-value satisfies the condition (2.6.5). □

Theorem 2.6.9. For any set game $(N, v) \in G^N(\mathcal{U})$, the SM-value is a population monotonic allocation scheme for set games. The SM-value is introduced in Section 2.5 and ∇_S^v is independent of players.

Proof. By the definition of the SM-value, we have

$$\bigcup_{i \in N} SM_i(N, v) = \bigcup_{i \in N} \bigcup_{S \ni i} [v(S) - \nabla_S^v] = \bigcup_{S \subseteq N} [v(S) - \nabla_S^v].$$

By induction similar to that used in Lemma 2.5.3, we obtain that, for any $T \subseteq N$,

$$\bigcup_{i \in T} SM_i(T, v_T) = \bigcup_{S \subseteq T} [v(S) - \nabla_S^v] = \bigcup_{S \subseteq T} v(S). \tag{2.6.7}$$

So, (2.6.7) implies that the SM-value satisfies the condition (2.6.4). By Lemma 2.6.7, we have that, for any n, $n > 1$,

$$SM_i(N, v) \supseteq SM_i(N \backslash \{j\}, v_{N \backslash \{j\}}) \supseteq SM_i(N \backslash \{k, j\}, v_{N \backslash \{j,k\}}) \supseteq ... \supseteq SM_i(i, v_i),$$

where $i \neq j, k, ...$ Therefore, for any $S, T \subseteq N$ with $S \subset T$, we have that, for any $i \in S$,

$$SM_i(S, v_S) \subseteq SM_i(T, v_T).$$

This implies that the SM-value satisfies the condition (2.6.5). □

2.7 Core and convexity for set games

The concept solution of core for set games is defined as follows:

Definition 2.7.1. Let $(N, v) \in G^N(\mathcal{U})$ be a set game.

(i) The *set of globally efficient allocations* $X(N, v) \subseteq (2^{\mathcal{U}})^N$ of (N, v) is defined to be

$$X(N, v) := \{x = (x_i)_{i \in N} \in (2^{\mathcal{U}})^N \mid \bigcup_{i \in N} x_i = \bigcup_{T \subseteq N} v(T)\},$$

(ii) The *core* $C(N, v) \subseteq (2^{\mathcal{U}})^N$ of the set game (N, v) is defined to be

$$C(N, v) := \{x \in X(N, v) \mid \bigcup_{i \in S} x_i \supseteq \bigcup_{T \subseteq S} v(T) \quad \text{for all } S \subseteq N\}. \tag{2.7.1}$$

Proposition 2.7.2. Let $(N, v) \in G^N(\mathcal{U})$ be a set game. Then it holds that

$$\begin{aligned} & C(N, v) \\ = \; & \{x \in X(N, v) \mid \bigcup_{i \in S} x_i \supseteq v(S) \quad \text{for all } S \subseteq N\} && (2.7.2) \\ = \; & \{x \in X(N, v) \mid \bigcup_{i \in S} x_i \supseteq v(S) - \bigcup_{T \subset S} v(T) \quad \text{for all } S \subseteq N\}. && (2.7.3) \end{aligned}$$

Obviously, the core concept is invariant under the *monotonic cover*, that is $C(N,w) = C(N,v)$ for the monotonic cover $(N,w) \in G^N(\mathcal{U})$ defined to be $w(S) := \bigcup_{T\subseteq S} v(T)$ for all $S \subseteq N$.

Proof. Let $(N,v) \in G^N(\mathcal{U})$. One type of set inclusions is trivial due to the inclusions $\bigcup_{T\subseteq S} v(T) \supseteq v(S) \supseteq v(S) - \bigcup_{T\subset S} v(T)$ for all $S \subseteq N$. The inverse set of inclusions is explained as follows. Firstly, we claim that $\bigcup_{i\in S} x_i \supseteq v(S)$ for all $S \subseteq N$ implies $\bigcup_{i\in S} x_i \supseteq \bigcup_{T\subseteq S} v(T)$ for all $S \subseteq N$. Indeed, for all $S \subseteq N$, it holds, by assumption, that $v(T) \subseteq \bigcup_{i\in T} x_i \subseteq \bigcup_{i\in S} x_i$ for all $T \subseteq S$ and so, $\bigcup_{T\subseteq S} v(T) \subseteq \bigcup_{i\in S} x_i$. Secondly, we claim that $\bigcup_{i\in S} x_i \supseteq v(S) - \bigcup_{T\subset S} v(T)$ for all $S \subseteq N$ implies $\bigcup_{i\in S} x_i \supseteq v(S)$ for all $S \subseteq N$. This proof proceeds by induction on the number of players in the coalition S. Indeed, for all $S \subseteq N$, it holds, by induction, that $v(T) \subseteq \bigcup_{i\in T} x_i \subseteq \bigcup_{i\in S} x_i$ for all $T \subset S$ and so, $\bigcup_{T\subset S} v(T) \subseteq \bigcup_{i\in S} x_i$. Together with the assumption $v(S) - \bigcup_{T\subset S} v(T) \subseteq \bigcup_{i\in S} x_i$, it follows that $v(S) \subseteq \bigcup_{i\in S} x_i$. □

Definition 2.7.3. We say the set game $(N,v) \in G^N(\mathcal{U})$ is [1]

(i) a *convex set game* if

$$v(S) - v(S\setminus\{i\}) \subseteq v(T) - v(T\setminus\{i\}) \qquad \text{for all } S \subseteq T \subseteq N \text{ and all } i \in S. \tag{2.7.4}$$

(ii) a *semi-convex set game* if

$$v(S) - v(S\cap T) \subseteq v(S\cup T) - v(T) \qquad \text{for all } S,\ T \subseteq N. \tag{2.7.5}$$

(iii) a *pseudo-convex set game* if

$$v(S) \cup v(T) \subseteq v(S\cup T) \cup v(S\cap T) \qquad \text{for all } S,\ T \subseteq N. \tag{2.7.6}$$

[1] Let $\mathcal{CG}$ denote the set of cooperative games. We say the cooperative game $(N,v) \in \mathcal{CG}$ is convex if $v(S)+v(T) \leq v(S\cup T)+v(S\cap T)$ for all $S,\ T \subseteq N$ or equivalently, $v(S)-v(S\setminus\{i\}) \leq v(T) - v(T\setminus\{i\})$ for all $S \subseteq T \subseteq N$ and all $i \in S$ (cf. [66]).

Proposition 2.7.4. Semi-convexity for set games implies both convexity and pseudo-convexity.

Proof. Let $(N, v) \in \mathcal{G}$ be a semi-convex set game. In order to prove the pseudo-convexity, let $S,\ T \subseteq N$ and suppose $x \in v(S) \cup v(T)$. Without loss of generality, suppose $x \in v(S)$. Clearly, the inclusion (2.7.6) holds if $x \in v(S \cap T)$ and, in case $x \notin v(S \cap T)$, we derive from the semi-convexity (2.7.5) the following chain of inclusions:

$$x \in v(S) - v(S \cap T) \subseteq v(S \cup T) - v(T) \subseteq v(S \cup T) \subseteq v(S \cup T) \cup v(S \cap T).$$

So, semi-convexity implies pseudo-convexity. Moreover, semi-convexity implies convexity since the convexity condition (2.7.4), for any $i \in S \subseteq T \subseteq N$, follows from the semi-convexity condition (2.7.5) applied to the coalitions S and $T \setminus \{i\}$ (satisfying $S \cup T \setminus \{i\} = T$ as well as $S \cap T \setminus \{i\} = S \setminus \{i\}$). □

Definition 2.7.5.

(*i*) v is a *monotonic game*, *i.e.*, for any $S, T \subseteq N$ with $S \subseteq T$,

$$v(S) \subseteq v(T),$$

(*ii*) v is a *super-additive set game*, *i.e.*, for any coalitions $S, T \subseteq N$ with $S \cap T = \emptyset$,

$$v(S) \cup v(T) \subseteq v(S \cup T),$$

(*iii*) v is a *strong super-additive set game*, *i.e.*, for any $S, T \subseteq N$

$$v(S) \cup v(T) \subseteq v(S \cup T).$$

Proposition 2.7.6. For the set game $(N, v) \in G^N(\mathcal{U})$ the following four statements are equivalent.

(i) $v(S) \cup v(T) \subseteq v(S \cup T) \cup v(S \cap T)$ for all $S,\ T \subseteq N$, (Pseudo-convexity)

(ii) $v(S) \cup v(T) \subseteq v(S \cup T)$ for all $S,\ T \subseteq N$ with $S \cap T = \emptyset$, (Super additivity)

(iii) $v(S) \cup v(T) \subseteq v(S \cup T)$ for all $S,\ T \subseteq N$, (Strong super additivity)

(iv) $v(S) \subseteq v(T)$ for all $S \subseteq T \subseteq N$ (Monotonicity).

Proof. Let $(N, v) \in G^N(\mathcal{U})$. We claim that the chain of implications $(i) \Longrightarrow (ii) \Longrightarrow (iv) \Longrightarrow (iii) \Longrightarrow (i)$ holds. Since almost all implications are trivial, it suffices to prove the implication $(ii) \Longrightarrow (iv)$. Let $S \subseteq T \subseteq N$. From part (ii) we derive the chain of inclusions: $v(S) \subseteq v(S) \cup v(T \backslash S) \subseteq v(S \cup (T \backslash S)) = v(T)$. □

Definition 2.7.7. We say the set game $(N, v) \in G^N(\mathcal{U})$ is a *sub-additive set game* if one of the following three equivalent statements holds:

(i) $v(S) \subseteq v(S \setminus \{i\}) \cup v(\{i\})$ for all $S \subseteq N$ and all $i \in S$,

(ii) $v(S) - v(\{i\}) \subseteq v(S \setminus \{i\})$ for all $S \subseteq N$ and all $i \in S$,

(iii) $v(S) - v(S \setminus \{i\}) \subseteq v(\{i\})$ for all $S \subseteq N$ and all $i \in S$.

Note that the subadditivity of the set game (N, v) implies $v(S) \subseteq \bigcup_{j \in S} v(\{j\})$ for all $S \subseteq N$.

Since the core concept is invariant under the monotonic cover, we concentrate on monotonic games in order to formulate an equivalence theorem between the convexity property of the set game and the structure of its core. The main theorem states that for the convexity of the set game it is necessary and sufficient that particularly chosen set allocations belong to the core. We start with the definition of these allocations taking into account some ordering of the player set described by any permutation of the player set. Let Θ^N denote the set of all permutations (one-to-one functions) $\sigma : N \rightarrow N$ of the player set N.

Definition 2.7.8. Let $(N, v) \in G^N(\mathcal{U})$ be a set game. With every permutation $\sigma : N \rightarrow N$, there is associated the *permutation individually marginalistic* PIM-value $PIM^\sigma(N, v) = (PIM_i^\sigma(N, v))_{i \in N} \in (2^{\mathcal{U}})^N$ of the set game (N, v) defined to be [2]

[2] In the framework of cooperative TU-games $(N, v) \in \mathcal{CG}$, the analogous definition of the marginalistic worth vector $x^\sigma(N, v) = (x_i^\sigma(N, v))_{i \in N} \in \mathbb{R}^N$ applies with respect to the subtraction in $\mathbb{R}$ (cf. [66], [39], [60]).

$$PIM_i^\sigma(N,v) := v(P_i^\sigma \cup \{i\}) - v(P_i^\sigma) \qquad \text{for all } i \in N, \tag{2.7.7}$$

where $P_i^\sigma := \{j \in N \mid \sigma(j) < \sigma(i)\}$ represents the coalition consisting of the predecessors of player i in the order σ of N. Clearly, any PIM-value $PIM^\sigma(N,v)$ for monotonic set games satisfies the global efficiency principle since, by construction, $\bigcup_{i\in N} PIM_i^\sigma(N,v) = v(N)$.

We claim that $U_T(R) - U_T(R\setminus\{i\}) \subseteq U_T(S) - U_T(S\setminus\{i\})$ for any $R \subseteq S \subseteq N$ and $i \in R$. If $U_T(R) - U_T(R\setminus\{i\}) = emptyset$, the claim holds. If $U_T(R) - U_T(R\setminus\{i\}) = \mathcal{U}$, then, $U_T(R) = \mathcal{U}$ and $U_T(R\setminus\{i\}) = emptyset$. That implies $T \subseteq R$ and $T \ni i$. So, we have that $U_T(S) - U_T(S\setminus\{i\}) = \mathcal{U}$ for any $R \subseteq S \subseteq N$, since $U_T(S) = \mathcal{U}$ and $U_T(S\setminus\{i\}) = \emptyset$ due to $T \subseteq R \subseteq S$ and $i \in T \nsubseteq S\setminus\{i\}$. Therefore, the unanimity set game U_T is a convex set game. Whereas the element set game E_T has no convexity. By Definition 2.7.8, we obtain that $PIM_i^\sigma(N,U_T) := U_T(P_i^\sigma \cup \{i\}) - U_T(P_i^\sigma) = \mathcal{U}$ if and only if $T \subseteq P_i^\sigma \cup \{i\}$ and $T \nsubseteq P_i^\sigma$. Furthermore, we know that $PIM_i^\sigma(N,U_T) = \mathcal{U}$ if and only if $i \in T$ and $T\setminus\{i\} \subseteq P_i^\sigma$. That means i is the last player of T in the order σ. By Definition 1.3.1 and Proposition 1.3.2 the allocations $X = (x_i)_{i\in N}$ of the core of the unanimity set game U_T satisfy $\bigcup_{j\in S} x_j \supseteq U_T(S)$, for any $S \subseteq N$. The allocation $X \in C(N,U_T)$ is equivalent to $\bigcup_{j\in T} x_j = \mathcal{U}$, due to the fact that $\bigcup_{j\in S} x_j \supseteq U_T(S)$ for any $S \subseteq N$ is equivalent to $\bigcup_{j\in T} x_j = \mathcal{U}$ for $S = T$. The extreme allocations of the core of the unanimity set game U_T are the PIM-value (only one player gets $\mathcal{U}$). The payoff of player i $PIM_i^\sigma(N,U_T) = \mathcal{U}$ if $\sigma(i) > \sigma(j)$ for any $j \in T\setminus\{i\}$ and $i \in T$. Otherwise, $PIM_i^\sigma(N,U_T) = emptyset$. Whereas the allocation X is in the core of the elementary set game E_T if and only if $\bigcup_{j\in T} x_j = \mathcal{U}$.

Theorem 2.7.9. The following four statements for a monotonic set game $(N,v) \in G^M(\mathcal{U})$ are equivalent. [3]

[3] In the framework of cooperative TU-games $(N,v) \in \mathcal{CG}$, a similar equivalence theorem concerning convex games holds as well. The implication $(i) \Longrightarrow (ii)$ is due to Shapley (cf. [66]), while the converse implication $(ii) \Longrightarrow (i)$ is due to Ichiishi (cf. [39]). The remaining

(i) (N, v) is a convex set game, (2.7.4) holds,

(ii) $PIM^\sigma(N, v) \in C(N, v)$ for every permutation $\sigma \in \Theta^N$,

(iii) $PIM^\sigma(N, v) \in C(N, v)$ for every even permutation $\sigma \in \Theta^N$,

(iv) $PIM^\sigma(N, v) \in C(N, v)$ for every odd permutation $\theta \in \Theta^N$.

Proof. We prove the two implications $(i) \Longrightarrow (ii)$ and $(ii) \Longrightarrow (i)$.
Part (i). Suppose (N, v) is a convex set game, (2.7.4) holds. Let $\sigma \in \Theta^N$ be a permutation and let $S \subseteq N$ be a coalition. We verify the associated core constraint $\bigcup_{j \in S} PIM_j^\sigma(N, v) \supseteq v(S)$. For that purpose, we list the members of S in accordance with the order determined by σ, that is write $S := \{i_1, i_2, \ldots, i_s\}$ where s denotes the number of members in S and $\sigma(i_1) < \sigma(i_2) < \ldots < \sigma(i_s)$. For the sake of notation, write $[i_0] := \emptyset$ and $[i_k] := \{i_1, i_2, \ldots, i_k\}$, for all $k \in \{1, 2, \ldots, s\}$. Clearly, the inclusion $[i_{k-1}] \subseteq P_{i_k}^\sigma$ holds for all $k \in \{1, 2, \ldots, s\}$ and so, the convexity property (2.7.4) of (N, v) yields the following inclusion:

$$v(P_{i_k}^\sigma \cup \{i_k\}) - v(P_{i_k}^\sigma) \supseteq v([i_k]) - v([i_{k-1}]), \qquad \text{for all } k \in \{1, 2, \ldots, s\}. \quad (2.7.8)$$

From (2.7.7)–(2.7.8) we deduce the following chain of inclusions:

$$\bigcup_{j \in S} PIM_j^\sigma(N, v) = \bigcup_{k=1}^{s} PIM_{i_k}^\sigma(N, v) \overset{(2.7.7)}{=} \bigcup_{k=1}^{s} \left[v(P_{i_k}^\sigma \cup \{i_k\}) - v(P_{i_k}^\sigma) \right]$$

$$\overset{(2.7.8)}{\supseteq} \bigcup_{k=1}^{s} \left[v([i_k]) - v([i_{k-1}]) \right] = v([i_s]) = v(S),$$

where the last equality but one is due to the monotonicity of the set game (N, v). We conclude that $PIM^\sigma(N, v) \in C(N, v)$ for every permutation $\sigma \in \Theta^n$, provided (N, v) is a convex set game. This proves the implications $(i) \Longrightarrow (ii)$, $(i) \Longrightarrow (iii)$, as well as $(i) \Longrightarrow (iv)$.
Part (ii). Suppose (ii), (iii), or (iv) holds, whatever applies for the moment.

implications $(iii) \Longrightarrow (i)$ and $(iv) \Longrightarrow (i)$ are due to Rafels and Ybern (cf. [60]), who refined Ichiishi's proof by distinguishing between even and odd permutations. Our proof technique, in the framework of set games, resembles Ichiishi's proof technique, applied to cooperative TU-games.

We prove the convexity condition (2.7.4) in the following manner: for all $S \subseteq N$ and all $i \in S$, $j \in S$, $i \neq j$,

$$v(S) - v(S \setminus \{i\}) \supseteq v(S \setminus \{j\}) - v(S \setminus \{i,j\}) \text{ or, equivalently,} \quad (2.7.9)$$

$$v(S) - v(S \setminus \{j\}) \supseteq v(S \setminus \{i\}) - v(S \setminus \{i,j\}). \quad (2.7.10)$$

Let $S \subseteq N$ and $i \in S$, $j \in S$, $i \neq j$. For the sake of notation, write $S := \{i_1, i_2, \ldots, i_{s-1}, i_s\}$ and $N \setminus S := \{i_{s+1}, i_{s+2}, \ldots, i_n\}$, where s denotes the number of members in S ($s \geq 2$) and $i_{s-1} := i$, $i_s := j$. The essential part of the proof is the choice of two appropriate permutations $\sigma \in \Theta^n$, $\theta \in \Theta^n$, defined to be

$$\sigma(i_k) := k \text{ for all } k \in \{1, 2, \ldots, n\},$$

$$\theta(i_k) := k \text{ for all } k \in \{1, 2, \ldots, n\} \setminus \{s-1, s\}, \ \theta(i_{s-1}) := s \text{ and} \theta(i_s) := s-1.$$

Clearly, $\theta = \sigma \circ (i,\ j)$. In words, θ arises from σ by interchanging the two neighbors i and j. Thus, if σ is an even (respectively odd) permutation of N, then θ is an odd (even) permutation of N. By construction, the two PIM-valus $PIM^{\sigma}(N,v)$ and $PIM^{\theta}(N,v)$ satisfy the following equality:

$$\bigcup_{k \in S \setminus \{i,j\}} PIM_k^{\theta}(N,v) = v(S \setminus \{i,j\}) = \bigcup_{k \in S \setminus \{i,j\}} PIM_k^{\sigma}(N,v). \quad (2.7.11)$$

We distinguish two subcases depending on the core membership of the two PIM-values $PIM^{\sigma}(N,v)$ and $PIM^{\theta}(N,v)$.

Subcase one. Suppose $PIM^{\theta}(N,v) \in C(N,v)$. Under these circumstances, the aim is to prove (2.7.9). Since $P_i^{\theta} = S \setminus \{i\}$, we have, by (2.7.7), $PIM_i^{\theta}(N,v) = v(P_i^{\theta} \cup \{i\}) - v(P_i^{\theta}) = v(S) - v(S \setminus \{i\})$. Further, the assumption $PIM^{\theta}(N,v) \in C(N,v)$ implies $v(S \setminus \{j\}) \subseteq \bigcup_{k \in S \setminus \{j\}} PIM_k^{\theta}(N,v)$. In order to prove the inclusion (2.7.9), let $y \in v(S \setminus \{j\}) - v(S \setminus \{i,j\})$. Since $y \in v(S \setminus \{j\})$, it holds that $y \in \bigcup_{k \in S \setminus \{j\}} PIM_k^{\theta}(N,v)$, whereas, by (2.7.11), $y \notin \bigcup_{k \in S \setminus \{i,j\}} PIM_k^{\theta}(N,v)$. Thus, $y \in PIM_i^{\theta}(N,v)$, and consequently, $y \in v(S) - v(S \setminus \{i\})$. So, the convexity condition (2.7.9) holds, provided the core membership $PIM^{\theta}(N,v) \in C(N,v)$ holds.

Subcase two. Suppose $PIM^{\sigma}(N,v) \in C(N,v)$. Under these circumstances, the aim is to prove (2.7.10). Since $P_j^{\sigma} = S \setminus \{j\}$, we have, by (2.7.7), $PIM_j^{\sigma}(N,v) = v(P_j^{\sigma} \cup \{j\}) - v(P_j^{\sigma}) = v(S) - v(S \setminus \{j\})$. Further, the assumption $PIM^{\sigma}(N,v) \in C(N,v)$ implies $v(S \setminus \{i\}) \subseteq \bigcup_{k \in S \setminus \{i\}} PIM_k^{\sigma}(N,v)$.
In order to prove the inclusion (2.7.10), let $z \in v(S \setminus \{i\}) - v(S \setminus \{i,j\})$. Since $z \in v(S \setminus \{i\})$, it holds that $z \in \bigcup_{k \in S \setminus \{i\}} PIM_k^{\sigma}(N,v)$, whereas, by (2.7.11), $z \notin \bigcup_{k \in S \setminus \{i,j\}} PIM_k^{\sigma}(N,v)$. Thus, $z \in PIM_j^{\sigma}(N,v)$ and, consequently, $z \in v(S) - v(S \setminus \{j\})$. So, the convexity condition (2.7.10) holds, provided the core membership $PIM^{\sigma}(N,v) \in C(N,v)$ holds.
Since either σ or θ is an even (respectively odd) permutation of N, this proves the implications $(ii) \Longrightarrow (i)$, $(iii) \Longrightarrow (i)$, as well as $(iv) \Longrightarrow (i)$. $\square$

Chapter 3

A potential approach to solutions for set games

In this chapter, we introduce a new solution by allocating, to any player, the items (taken from a universe) that are attainable for the player, but cannot be blocked (by any coalition not containing the player). The resulting value turns out to be a very important concept for set games to characterize the family of set game solutions that possess a so-called potential representation (similar to the potential approaches applied in both physics and cooperative game theory). An axiomatization of the new value, called overall-coalitionally marginalistic value, is given by three properties, namely a kind of efficiency property, the equal treatment property and a kind of monotonicity property.

3.1 Introduction

In physics the potential is a highly important concept. For instance, a vector field u is said to be conservative if there exists a continuously differentiable function U called potential the gradient of which is identical with the vector field (notation: $\bigtriangledown U = u$). There exist several characterizations of conservative vector fields (e.g., $\bigtriangledown_j u_i = \bigtriangledown_i u_j$, or every contour integral with respect to the vector field u is zero). Surprisingly, the successful treatment of the potential in physics turned out to be reproducible, in the late eighties and the nineties, in the mathematical field of cooperative game theory. Informally, a solution concept f on the universal cooperative game space Γ is said to possess a potential representation if it is the discrete gradient (with reference to the subtraction) of a *real-valued function* P on Γ called potential (notation: $\bigtriangledown P = f$). In other words, if possible, each component of the cooperative game solution (or each player's payoff) may be interpreted as the incremental return, determined by the difference of the potential function evaluation at the given cooperative game and one of its sub-games in which the relevant player is not included. In their innovative paper, Hart & Mas-Colell [36] showed that the well-known cooperative game solution called Shapley value is the unique solution that has a potential representation and satisfies the standard efficiency principle as well. The role of the Shapley value has been strengthened later on by a second fundamental result concerning the family of cooperative game solutions that possess a potential representation. This fundamental equivalence theorem (Calvo & Santos [12]) states that every cooperative game solution with a potential representation is equivalent to the Shapley value in that the solution of the initial cooperative game coincides with the Shapley value of an auxiliary cooperative game.

The main purpose of this chapter is to introduce a new value concept for set games (see Section 3.2) that can be regarded as the counterpart of the Shapley value for cooperative games whenever the potential approach to the solution theory is applied to the space of set games instead of the space of cooperative games. In the yet undeveloped mathematical field called set game theory, a value concept f on the universal set game space $G^N(\mathcal{U})$ is said to possess a potential representation if it is the discrete gradient (with reference to the set difference) of a *set-valued function* P on $G^N(\mathcal{U})$ called potential (notation: $\bigtriangledown P = f$). The fundamental equivalence Theorem 3.3.1. states that every set game solution with a potential representation is equivalent to the so-called potential f–value in that the solution of the initial set game contains the OCM–value of an auxiliary set game (such that, under certain circumstances, the inclusion reduces to an equality). In the introductory part of Section 3.2 the OCM–value is simply introduced by allocating, to any player, the items (taken from a universe) that are attainable for the player, but cannot be blocked by any coalition not containing the player (see Definition 3.2.1).

Based on its explicit descriptions (3.2.1)–(3.2.3), we present, at the end of Section 3.2, a first axiomatization of the OCM–value in terms of its potential representation (with respect to disjoint unions) together with some type of efficiency property (see Theorem 3.2.8). Section 3.4 is devoted to a second axiomatization of the OCM–value, the three axioms of which are presented in terms of the relevant efficiency property, the classical equal treatment property as well as one particular type of a monotonicity property with reference to contributions (see Theorem 3.4.3). Section 3.3 treats, besides the fundamental equivalence Theorem 3.3.1, the relationship between the existence of the potential representation for a value and the law of preservation of (disjoint) unions. In the context of this law, the balanced unions property (3.3.4) for a set game value $f(N,v)$ may be interpreted as the discrete version $\bigtriangledown_j f_i = \bigtriangledown_i f_j$ of the characterization $\bigtriangledown_j u_i = \bigtriangledown_i u_j$ of a conservative vector field u in physics. Roughly speaking, the balanced unions property (3.3.4) is necessary and sufficient for a set game value to possess a potential representation (see Theorem 3.3.6).

3.2 An overall-coalitionally marginalistic value for set games

Definition 3.2.1. The *overall-coalitionally marginalistic* OCM-value on the set game space $G^N(\mathcal{U})$ associates with every set game (N,v) the allocation $OCM(N,v) = (OCM_i(N,v))_{i\in N} \in (2^{\mathcal{U}})^N$, where its allocation to any player is given by

$$\begin{aligned} &OCM_i(N,v) \\ := \; & \Big[\bigcup_{\substack{S\subseteq N\\ S\ni i}} v(S)\Big] - \Big[\bigcup_{T\subseteq N\setminus\{i\}} v(T)\Big] && \text{or, equivalently,} \quad (3.2.1) \\ = \; & \Big[\bigcup_{S\subseteq N} v(S)\Big] - \Big[\bigcup_{T\subseteq N\setminus\{i\}} v(T)\Big] && \text{or, equivalently,} \quad (3.2.2) \\ = \; & \bigcup_{\substack{S\subseteq N\\ S\ni i}} \Big[v(S) - \Big[\bigcup_{T\subseteq N\setminus\{i\}} v(T)\Big]\Big], && \text{for all } i\in N. \quad (3.2.3) \end{aligned}$$

By (3.2.1), the OCM–value of player i in a set game is fully determined by those items that are attainable by player i (through a certain coalition containing i), but cannot be blocked (by any coalition not containing i). In this context we say a coalition $T \subseteq N$ cannot block an item $x \in \mathcal{U}$ whenever the item does not belong to the coalition's worth, that is $x \notin v(T)$.

Remark 3.2.2. The OCM-value is one member out of the family of set games values of the following form:

$$f_i(N, v) = \bigcup_{\substack{S \subseteq N, \\ S \ni i}} \left[v(S) - \triangledown^v_{S,i} \right], \text{ for all } (N, v) \in G^N(\mathcal{U}) \text{ and all } i \in N, \quad (3.2.4)$$

where, for every coalition S and every player i, the expression $\triangledown^v_{S,i}$ depends, to some weak or strong extent, upon the worths of a certain collection of coalitions, somehow determined by S and/or i. By (3.2.3), the *OCM*–value arises from (3.2.4) by choosing $\triangledown^v_{S,i} := \bigcup_{T \subseteq N \setminus \{i\}} v(T)$ for all $i \in N$, with reference to worths of all the coalitions not containing player i. The IM-value as introduced by Aarts, Funaki & Hoede [2], arises from (3.2.4) by choosing $\triangledown^v_{S,i} := v(S \setminus \{i\})$ with reference to the worth of a unique coalition. For the class of *monotonic set games*, it was shown in [2] that the IM-value coincides with the semi-marginalistic value of the form (3.2.4) by choosing $\triangledown^v_{S,i} := \bigcup_{j \in S} v(S \setminus \{j\})$ with reference to the worths of all the sub-coalitions of S with one player less. Or, alternatively, for monotonic set games the IM-value coincides with the value of the form (3.2.4) by choosing $\triangledown^v_{S,i} := \bigcup_{T \subset S} v(T)$ with reference to the worths of all the proper sub-coalitions of S [69]. According to the next lemma, the *OCM*–value differs from these latter values in that another type of efficiency applies.

Definition 3.2.3. Let f be a value on the set game space $G^N(\mathcal{U})$. We say the value f satisfies the axiom of *restricted global efficiency* if

$$\bigcup_{i \in N} f_i(N, v) = \left[\bigcup_{S \subseteq N} v(S) \right] - \left[\bigcap_{i \in N} \bigcup_{T \subseteq N \setminus \{i\}} v(T) \right],$$

$$\text{for all } (N, v) \in G^N(\mathcal{U}). \quad (3.2.5)$$

In words, a restricted globally efficient value allocates those items that are attainable (by some player through a certain coalition containing that player), but cannot be blocked by any coalition not containing a certain player (the existence of which is guaranteed).

Lemma 3.2.4. The OCM–value on $G^N(\mathcal{U})$ satisfies the axiom of restricted global efficiency.

Proof of Lemma 3.2.4.
Let $(N,v) \in G^N(\mathcal{U})$. For the sake of notation, write $\bigtriangledown^v_{N\setminus\{i\}} := \bigcup_{T\subseteq N\setminus\{i\}} v(T)$.
By (3.2.2), the OCM–value of any player i is given by $OCM_i(N,v) = \bigcup_{S\subseteq N}\Big[v(S) - \bigtriangledown^v_{N\setminus\{i\}}\Big]$. From this we derive the following chain of equalities.

$$
\begin{aligned}
&\bigcup_{i\in N} OCM_i(N,v)\\
\overset{(3.2.2)}{=}\ &\bigcup_{i\in N}\bigcup_{S\subseteq N}\Big[v(S) - \bigtriangledown^v_{N\setminus\{i\}}\Big] = \bigcup_{S\subseteq N}\bigcup_{i\in N}\Big[v(S) - \bigtriangledown^v_{N\setminus\{i\}}\Big]\\
=\ &\bigcup_{S\subseteq N}\Big[v(S) - \bigcap_{i\in N}\bigtriangledown^v_{N\setminus\{i\}}\Big] = \Big[\bigcup_{S\subseteq N} v(S)\Big] - \Big[\bigcap_{i\in N}\bigtriangledown^v_{N\setminus\{i\}}\Big].
\end{aligned}
$$

This completes the proof of the restricted global efficiency property for the OCM–value. □

Obviously, the bankruptcy set game (cf. Example 1.2.5) (N,v) is a monotonic game such that $v(S) = v(S\setminus\{i\}) \cup v(\{i\})$ for all $S \subseteq N$ and all $i \in S$. Consequently, the IM-value as introduced by Aarts *et. al.* [2], is determined as follows:

$$
IM_i(N,v) := \bigcup_{\substack{S\subseteq N,\\ S\ni i}}\Big[v(S) - v(S\setminus\{i\})\Big] = v(\{i\}) = D_i \cap E \qquad \text{for all } i \in N.
$$

In words, the marginal value allocates, to any creditor, his claim set in such a way that jointly claimed items have to be shared by various creditors. Due to the monotonicity of the bankruptcy set game, the OCM–value simplifies to

$OCM_i(N,v) = v(N) - v(N \setminus \{i\}) = \left[D_i - \bigcup_{j \in N\setminus\{i\}} D_j\right] \cap E$ for all $i \in N$. In words, the OCM–value allocates, to any creditor, those items that are solely claimed by the creditor (in that jointly claimed items are not shared).

Definition 3.2.5. Let f be a value on the set game space $G^N(\mathcal{U})$.

(i) We say that the value f admits a *potential* [1] if there exists a set-valued function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ satisfying $P_f(\emptyset, v_\emptyset) := \emptyset$ and

$$P_f(N,v) = f_i(N,v) \cup P_f(N \setminus \{i\}, v_{N\setminus\{i\}}),$$

$$\text{for all } (N,v) \in G^N(\mathcal{U}) \text{ and all } i \in N. \tag{3.2.6}$$

We say that the value f admits a *potential* P_f *with disjoint unions* if (3.2.6) refers to a disjoint union, i.e., $f_i(N,v) \cap P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) = \emptyset$, for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$.

(ii) The mapping $F_f : G^N(\mathcal{U}) \to G^N(\mathcal{U})$ associates with every set game $(N,v) \in G^N(\mathcal{U})$ its *solution set game* $(N, F_f^v) \in G^N(\mathcal{U})$, defined to be $F_f^v(\emptyset) := \emptyset$ and

$$F_f^v(S) := \bigcup_{j \in S} f_j(S, v_S), \qquad \text{for all } S \subseteq N,\ S \neq \emptyset, \tag{3.2.7}$$

where (S, v_S) is a subset game, $v_S(T) = v(T)$ for any $T \subseteq S$.

By (3.2.6), the set-valued potential function P_f is supposed to be monotonic (with respect to inclusion of sets, *i.e.*, player sets of sub-games). Assuming the monotonicity of P_f, (3.2.6) reduces to the following equality:

$$P_f(N,v) - P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) = f_i(N,v) - P_f(N \setminus \{i\}, v_{N\setminus\{i\}}),$$

for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$. In words, for any set game, the complementary part of two subsequent set-valued potential evaluations agrees with

[1] A single-valued cooperative game solution f is said to admit a potential if there exists a real-valued function $P_f : \mathcal{CG} \to \mathbb{R}$ satisfying $P_f(\emptyset, v_\emptyset) := 0$ and $f_i(N,v) = P_f(N,v) - P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$ for all $(N,v) \in \mathcal{CG}$ and all $i \in N$ (Hart & Mas-Colell [36],Calvo & Santos [12], Ortmann [58]).

the part of the solution which is not yet covered by the smallest potential evaluation. Next, we claim that the OCM-value admits a very natural potential, composed of the union of the worths of all the coalitions in a set game. In fact, we establish that the OCM–value is fully determined by its potential representation (with disjoint unions), together with the restricted global efficiency property.

Proposition 3.2.6. The OCM–value admits a set-valued potential function $P_{OCM} : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$, with disjoint unions, given by

$$P_{OCM}(N,v) = \bigcup_{S \subseteq N} v(S), \qquad \text{for all } (N,v) \in G^N(\mathcal{U}). \tag{3.2.8}$$

Proof of Proposition 3.2.6.
Let $(N,v) \in G^N(\mathcal{U})$ and $i \in N$. With the help of (3.2.8), we verify the potential representation (3.2.6) as follows:

$$\begin{aligned} & OCM_i(N,v) \cup P_{OCM}(N \setminus \{i\}, v_{N\setminus\{i\}}) \\ \overset{(3.2.1)}{=} & \left[\left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} v(S)\right] - \left[\bigcup_{T\subseteq N\setminus\{i\}} v(T)\right]\right] \cup \left[\bigcup_{S\subseteq N\setminus\{i\}} v(S)\right] \\ = & \left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} v(S)\right] \cup \left[\bigcup_{S\subseteq N\setminus\{i\}} v(S)\right] = \bigcup_{S\subseteq N} v(S) = P_{OCM}(N,v). \end{aligned}$$

So, (3.2.6) holds. Particularly, in the context of the OCM–value, (3.2.6) refers to a disjoint union. The above proof clarifies that the very same potential function (3.2.8) is applicable for a potential representation (but not with disjoint unions) of any value f of the form

$$f_i(N,v) = \left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} v(S)\right] - \beta_i^v, \quad \text{where } \beta_i^v \subseteq \bigcup_{T\subseteq N\setminus\{i\}} v(T) \text{ for all } i \in N.$$

□

Theorem 3.2.7. (Characterization of the OCM–value)
The OCM–value on $G^N(\mathcal{U})$ is the unique solution f on $G^N(\mathcal{U})$ that satisfies the axiom of restricted global efficiency and admits a potential with disjoint unions.

Proof of the uniqueness part of Theorem 3.2.7.
Let f be a value on $G^N(\mathcal{U})$ satisfying the restricted global efficiency and admitting a set-valued potential function $P_f : \mathcal{G}^N(\mathcal{U}) \to 2^{\mathcal{U}}$ with disjoint unions. Let $(N, v) \in G^N(\mathcal{U})$. By the potential representation (3.2.6) with disjoint unions, it holds that

$$f_i(N,v) \overset{(3.2.6)}{=} P_f(N,v) - P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) \quad \text{for all } i \in N, \text{ and so}$$

$$F_f^v(N) \overset{(3.2.7)}{=} \bigcup_{i\in N} f_i(N,v) = P_f(N,v) - \bigcap_{i\in N} P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$$

or, equivalently,

$$P_f(N,v) = F_f^v(N) \cup \bigcap_{i\in N} P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$$

on the understanding that

$$F_f^v(N) \overset{(3.2.5)}{=} \left[\bigcup_{S\subseteq N} v(S)\right] - \left[\bigcap_{i\in N} \bigcup_{T\subseteq N\setminus\{i\}} v(T)\right]$$

(restricted global efficiency).

In words, the potential function P_f is uniquely determined in a recursive manner (as a matter of fact, P_f is given by (3.2.8)) and since (3.2.6) refers to a disjoint union, the value f is uniquely determined too (and equals the OCM–value). □

3.3 Characterizations of values that admit a potential

We now treat an equivalence theorem concerning set game values that possess a potential representation, the main result of which is referring to the OCM-value, as given by (3.2.1). Until further notice, no efficiency constraints are imposed upon a value.

Theorem 3.3.1. (Equivalence Theorem) [2] Consider the setting of Definitions 3.2.1 and 3.2.6.

[2] A single-valued cooperative game solution f admits a real-valued potential function $P_f : \mathcal{CG} \to \mathbb{R}$ if and only if the value of any cooperative game equals the Shapley–value of the associated solution cooperative game, that is $f(N,v) = Sh(N, F_f^v)$ for all $(N,v) \in \mathcal{CG}$, where the solution game $(N, F_f^v) \in \mathcal{CG}$ is defined to be $F_f^v(\emptyset) := 0$ and $F_f^v(S) := \sum_{j\in S} f_j(S, v_S)$, for all $S \subseteq N$, $S \neq \emptyset$ (Calvo & Santos [12]).

(i) If a value f on $G^N(\mathcal{U})$ admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$, then the following holds:

$$P_f(N,v) = \bigcup_{S\subseteq N} F_f^v(S) \text{ and } OCM(N,F_f^v) \subseteq f(N,v) \text{ for all } (N,v) \in G^N(\mathcal{U}). \tag{3.3.1}$$

In words, the value of any set game contains the OCM–value of the associated solution set game.

(ii) If $OCM(N,F_f^v) \subseteq f(N,v)$ for all $(N,v) \in G^N(\mathcal{U})$, then the value f admits a potential, the set-valued function P_f of which is given by (3.3.1).

Proof of Theorem 3.3.1.

Let f be a value on the set game space $G^N(\mathcal{U})$.

Suppose the value f admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$. We prove, by induction on the number n, of players that the unique potential function P_f is given by $P_f(N,v) = \bigcup_{S\subseteq N} F_f^v(S)$ for all $(N,v) \in G^N(\mathcal{U})$.

In case $n = 1$, say we consider the game $(\{i\},v)$, then, by (3.2.6), it holds that $P_f(\{i\},v) = f_i(\{i\},v) \cup P_f(\emptyset,v_\emptyset) = f_i(\{i\},v) = F_f^v(\{i\})$. From now on, let $(N,v) \in G^N(\mathcal{U})$ satisfy $n \geq 2$. From (3.2.6), that is $P_f(N,v) = f_i(N,v) \cup P_f(N\setminus\{i\}, v_{N\setminus\{i\}})$ for all $i \in N$, we derive, by taking the union over all $i \in N$, the following:

$$\begin{aligned}
P_f(N,v) &= \left[\bigcup_{i\in N} f_i(N,v)\right] \cup \left[\bigcup_{i\in N} P_f(N\setminus\{i\}, v_{N\setminus\{i\}})\right] \\
&\overset{(3.2.7)}{=} F_f^v(N) \cup \left[\bigcup_{i\in N}\left[\bigcup_{S\subseteq N\setminus\{i\}} F_f^v(S)\right]\right] \\
&\qquad \text{(due to the induction hypothesis)} \\
&= F_f^v(N) \cup \left[\bigcup_{S\subset N} F_f^v(S)\right] = \bigcup_{S\subseteq N} F_f^v(S).
\end{aligned}$$

From the determination of the potential function P_f, we deduce that, for all $i \in N$, it holds that

$$P_f(N,v) - P_f(N\setminus\{i\}, v_{N\setminus\{i\}}) = \left[\bigcup_{S\subseteq N} F_f^v(S)\right] - \left[\bigcup_{T\subseteq N\setminus\{i\}} F_f^v(T)\right] \overset{(3.2.2)}{=} OCM_i(N,F_f^v),$$

$$(3.3.2)$$

where the latter equality is due to the alternative description (3.2.2) of the OCM–value. So far, we conclude that

$$OCM_i(N, F_f^v) = P_f(N, v) - P_f(N\setminus\{i\}, v_{N\setminus\{i\}}) \subseteq f_i(N, v), \qquad \text{for all } i \in N,$$

where the latter inclusion arises from the assumption $P_f(N, v) = f_i(N, v) \cup P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$. This completes the proof of the statement in part (i). To prove the statement in part (ii), suppose that the inclusion $OCM(N, F_f^v) \subseteq f(N, v)$ holds. Define the potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$, as given by (3.3.1). Let $(N, v) \in G^N(\mathcal{U})$. Since the very same reasoning as in (3.3.2) applies, we arrive at

$$P_f(N, v) - P_f(N\setminus\{i\}, v_{N\setminus\{i\}}) \overset{(3.3.2)}{=} OCM_i(N, F_f^v) \subseteq f_i(N, v) \qquad \text{for all } i \in N.$$

This implies that , for all $i \in N$, the inclusion $P_f(N, v) \subseteq f_i(N, v) \cup P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$ holds, whereas the inverse inclusion holds due to the inclusions $P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) \subseteq P_f(N, v)$ and

$$f_i(N, v) \subseteq \bigcup_{j\in N} f_j(N, v) \overset{(3.2.7)}{=} F_f^v(N) \subseteq \bigcup_{S\subseteq N} F_f^v(S) \overset{(3.3.1)}{=} P_f(N, v).$$

So, (3.2.6) holds and thus the value f admits a potential. □

Remark 3.3.2. Two very slight changes within the proof of Theorem 3.3.1 establish the following equivalence: a value f on $G^N(\mathcal{U})$ admits a potential function P_f with *disjoint unions* if and only if the *equality* $OCM(N, F_f^v) = f(N, v)$ holds for all $(N, v) \in G^N(\mathcal{U})$. Particularly, in the setting of the OCM–value, by Proposition 3.2.7, the equality $OCM(N, F_{OCM}^v) = OCM(N, v)$ holds for all $(N, v) \in G^N(\mathcal{U})$. Without going into details, we state that a direct and computational proof of the latter equality can be based on the following two relationships (to be proved by induction): firstly, $\bigcup_{\substack{S\subseteq N,\\ S\ni i}} F_{OCM}^v(S) = \bigcup_{S\subseteq N} v(S)$, for all $(N, v) \in G^N(\mathcal{U})$ and all $i \in N$, and secondly $\bigcup_{T\subseteq M} F_{OCM}^w(T) = \bigcup_{T\subseteq R} w(T)$, for all $(R, w) \in G^R(\mathcal{U})$.

Corollary 3.3.3. For every globally efficient value f (see (**??**)) it holds that $OCM(N, F_f^v) = OCM(N, v)$, for all $(N, v) \in G^N(\mathcal{U})$. Consequently, the following two statements for a globally efficient value f are equivalent:

(i) the value f admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$,

(ii) $OCM(N, v) \subseteq f(N, v), \qquad$ for all $(N, v) \in G^N(\mathcal{U})$.

In particular, the set-valued potential function, if it exists, is given by

$$P_f(N, v) = \bigcup_{S \subseteq N} v(S), \qquad \text{for all } (N, v) \in G^N(\mathcal{U}). \tag{3.3.3}$$

In words, a globally efficient value admits a set-valued potential function if and only if the value contains the OCM–value (which is not globally efficient by Lemma 3.2.4). Moreover, the associated set-valued potential function, if it exists, does not depend upon the particular choice of any globally efficient value. For instance, the IM-value, as mentioned in Remark 3.2.2, admits a potential by Corollary 3.3.3(ii) since the trivial inclusion $OCM(N, v) \subseteq IM(N, v)$ holds for all $(N, v) \in G^N(\mathcal{U})$.

Proof of Corollary 3.3.3.
Let f be a globally efficient value on the set game space $G^N(\mathcal{U})$ and $(N, v) \in G^N(\mathcal{U})$. By its global efficiency, it holds that $\bigcup_{j \in S} f_j(S, v_S) = \bigcup_{R \subseteq S} v_S(R)$ or, equivalently, $F_f^v(S) = \bigcup_{R \subseteq S} v(R)$, for all $S \subseteq N$, $S \neq \emptyset$. From the alternative description (3.2.2) of the OCM–value, we derive, for all $i \in N$, the following:

$$\begin{aligned} OCM_i(N, F_f^v) &\overset{(3.2.2)}{=} \left[\bigcup_{S \subseteq N} F_f^v(S)\right] - \left[\bigcup_{T \subseteq N \setminus \{i\}} F_f^v(T)\right] \\ &= \left[\bigcup_{S \subseteq N} \bigcup_{R \subseteq S} v(R)\right] - \left[\bigcup_{T \subseteq N \setminus \{i\}} \bigcup_{R \subseteq T} v(R)\right] \\ &= \left[\bigcup_{S \subseteq N} v(S)\right] - \left[\bigcup_{T \subseteq N \setminus \{i\}} v(T)\right] \end{aligned}$$

$$\overset{(3.2.2)}{=} \quad OCM_i(N,v) \qquad\qquad \text{whereas}$$

$$P_f(N,v) \overset{(3.3.1)}{=} \bigcup_{S\subseteq N} F_f^v(S) = \bigcup_{S\subseteq N}\bigcup_{R\subseteq S} v(R) = \bigcup_{S\subseteq N} v(S).$$

This proves the statement $OCM(N,F_f^v) = OCM(N,v)$, for all $(N,v) \in G^N(\mathcal{U})$ as well as (3.3.3). □

The remainder of this section is devoted to a property of a value, which turns out to be sufficient to guarantee the existence of a potential representation of the value.

Definition 3.3.4. We say a value f on the set game space $G^N(\mathcal{U})$ satisfies the *balanced union property* [3] if, for any pair of players, the union of their allocated items is independent of their order to form the grand coalition in the final stage, i.e., for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$, $j \in N$, $i \neq j$, it holds that

$$f_i(N,v) \cup f_j(N\setminus\{i\}, v_{N\setminus\{i\}}) = f_j(N,v) \cup f_i(N\setminus\{j\}, v_{N\setminus\{j\}}). \tag{3.3.4}$$

In other words, we say f *preserves unions* (in physics notation: $\bigtriangledown_j f_i = \bigtriangledown_i f_j$ for all $i,\ j \in N$, $i \neq j$). Moreover, we say that the value f satisfies the *balanced union property with disjoint unions* (or equivalently, preserves disjoint unions) if (3.3.4) refers to a disjoint union, i.e., $f_i(N,v) \cap f_j(N\setminus\{i\}, v_{N\setminus\{i\}}) = \emptyset$, for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$, $j \in N$, $i \neq j$.

Lemma 3.3.5. The OCM–value on $G^N(\mathcal{U})$ preserves disjoint unions, i.e., satisfies the balanced unions property (3.3.4) with disjoint unions.

Proof of Lemma 3.3.5.
Let $(N,v) \in G^N(\mathcal{U})$ and $i \in N$, $j \in N$, $i \neq j$. In order to verify (3.3.4), we

[3] A single-valued cooperative game solution f on $\mathcal{CG}$ is said to satisfy the balanced contributions property (or to preserve discrete differences) if it holds that $f_i(N,v) - f_i(N\setminus\{j\}, v_{N\setminus\{j\}}) = f_j(N,v) - f_j(N\setminus\{i\}, v_{N\setminus\{i\}})$, for all $(N,v) \in \mathcal{CG}$ and all $i \in N$, $j \in N$, $i \neq j$. ([51])

derive from (3.2.1) the following chain of equalities:

$$
\begin{aligned}
& OCM_i(N,v) \cup OCM_j(N \setminus \{i\}, v_{N\setminus\{i\}}) \\
\overset{(3.2.1)}{=} & \left[\left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} v(S) \right] - \left[\bigcup_{T\subseteq N\setminus\{i\}} v(T) \right] \right] \\
& \cup \left[\left[\bigcup_{\substack{T\subseteq N\setminus\{i\},\\ T\ni j}} v(T) \right] - \left[\bigcup_{R\subseteq N\setminus\{i,j\}} v(R) \right] \right] \\
= & \left[\left[\bigcup_{\substack{S\subseteq N,\\ S\ni i,\ S\ni j}} v(S) \right] \cup \left[\bigcup_{\substack{S\subseteq N\setminus\{j\},\\ S\ni i}} v(S) \right] \right. \\
& \left. \cup \left[\bigcup_{\substack{T\subseteq N\setminus\{i\},\\ T\ni j}} v(T) \right] \right] - \left[\bigcup_{R\subseteq N\setminus\{i,j\}} v(R) \right].
\end{aligned}
$$

In words, the latter expression is symmetric in the two players i and j. So, (3.3.4) holds and thus, the OCM–value preserves unions. Notice that (3.3.4) refers to a disjoint union since $\bigcup_{\substack{T\subseteq N\setminus\{i\},\\ T\ni j}} v(T) \subseteq \bigcup_{T\subseteq N\setminus\{i\}} v(T)$. □

Theorem 3.3.6. Let f be a value on the set game space $G^N(\mathcal{U})$. [4]

(i) If f satisfies the balanced unions property (3.3.4), then f admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$.

(ii) If f admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ with disjoint unions, then f satisfies the balanced unions property (3.3.4) with disjoint unions.

Proof of Theorem 3.3.6.

Suppose the value f has the balanced unions property (3.3.4), that is f preserves unions. Define the potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ recursively by

$P_f(N,v) := f_i(N,v) \cup P_f(N\setminus\{i\}, v_{N\setminus\{i\}})$, for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$.

[4] A single-valued cooperative game solution f on $\mathcal{CG}$ admits a real-valued potential function $P_f : \mathcal{CG} \to \mathbb{R}$ if and only if f preserves discrete differences, that is f satisfies the balanced contributions property (Myerson [51], & Calvo & Santos [12]).

We show, by induction on the number n of players, that the potential function P_f is well-defined. In case $n = 1$, say $(\{i\}, v)$, then, by definition, $P_f(\{i\}, v) = f_i(\{i\}, v) \cup P_f(\emptyset, v_\emptyset) = f_i(\{i\}, v)$. From now on, let $(N, v) \in G^N(\mathcal{U})$ satisfy $n \geq 2$ and $i \in N$, $j \in N$, $i \neq j$. By applying the induction hypothesis twice, to $P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$ as well as to $P_f(N \setminus \{j\}, v_{N\setminus\{j\}})$, together with the assumption (3.3.4), we obtain the following chain of equalities:

$$\begin{aligned}
& f_i(N, v) \cup P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) \\
\overset{(IH)}{=} \quad & f_i(N, v) \cup \Big[f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) \cup P_f(N \setminus \{i, j\}, v_{N\setminus\{i,j\}}) \Big] \\
\overset{(3.3.4)}{=} \quad & f_j(N, v) \cup \Big[f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \cup P_f(N \setminus \{i, j\}, v_{N\setminus\{i,j\}}) \Big] \\
\overset{(IH)}{=} \quad & f_j(N, v) \cup P_f(N \setminus \{j\}, v_{N\setminus\{j\}}).
\end{aligned}$$

So the potential function P_f is well-defined, provided (3.3.4) holds. This proves the implication mentioned in part (i). In order to prove the implication mentioned in part (ii), suppose f admits a set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ with disjoint unions. Let $(N, v) \in G^N(\mathcal{U})$ and $i \in N$, $j \in N$, $i \neq j$. By the potential representation (3.2.6), it holds that $f_i(N, v) = P_f(N, v) - P_f(N \setminus \{i\}, v_{N\setminus\{i\}})$ as well as $f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) = P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) - P_f(N \setminus \{i, j\}, v_{N\setminus\{i,j\}})$, where $P_f(N \setminus \{i, j\}, v_{N\setminus\{i,j\}}) \subseteq P_f(N \setminus \{i\}, v_{N\setminus\{i\}}) \subseteq P_f(N, v)$. Now it follows immediately that $f_i(N, v) \cup f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) = P_f(N, v) - P_f(N \setminus \{i, j\}, v_{N\setminus\{i,j\}})$. We conclude that the expression $f_i(N, v) \cup f_j(N \setminus \{i\}, v_{N\setminus\{i\}})$ is symmetric in the two players i and j, so that it concerns a disjoint union. This proves the implication mentioned in part (ii). □

Example 3.3.7. Let the value f on the set game space $G^N(\mathcal{U})$ be given by

$$f_i(N, v) := \bigcup_{\substack{S \subseteq N, \\ S \ni i}} C_S^v, \qquad \text{for all } (N, v) \in G^N(\mathcal{U}) \text{ and all } i \in N, \tag{3.3.5}$$

on the understanding that, for every $S \subseteq N$, $S \neq \emptyset$, the so-called *contribution* C_S^v only depends upon the coalition S and not upon any player $i \in S$ (e.g., $C_S^v := v(S) - \bigcup_{T \subset S} v(T)$). Write $C_\emptyset^v := \emptyset$. A value f of the form (3.3.5) has the following properties:

(i) the value f preserves balanced unions, that is (3.3.4) holds,

(ii) $F_f^v(S) = \bigcup_{R \subseteq S} C_R^v$ for all $S \subseteq N$ and $OCM(N, F_f^v) \subseteq f(N,v)$ for all $(N,v) \in G^N(\mathcal{U})$,

(iii) the potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ satisfies $P_f(N,v) = \bigcup_{S \subseteq N} C_S^v$ for all $(N,v) \in G^N(\mathcal{U})$.

Proof of Example 3.3.7.
Let $(N,v) \in G^N(\mathcal{U})$ and $i \in N$, $j \in N$, $i \neq j$. Then it holds that

$$f_i(N,v) \cup f_j(N\backslash\{i\}, v_{N\backslash\{i\}}) = \Bigg[\bigcup_{\substack{S \subseteq N, \\ S \ni i,\, S \ni j}} C_S^v \Bigg] \cup \Bigg[\bigcup_{\substack{S \subseteq N\backslash\{j\}, \\ S \ni i}} C_S^v \Bigg] \cup \Bigg[\bigcup_{\substack{S \subseteq N\backslash\{i\}, \\ S \ni j}} C_S^v \Bigg].$$

Since the latter expression at the right hand side is symmetric with respect to the two players i and j, the value f preserves balanced unions. Furthermore, for all $S \subseteq N$, $S \neq \emptyset$, it holds that

$$F_f^v(S) \overset{(3.2.7)}{=} \bigcup_{j \in S} f_j(S, v_S) = \bigcup_{j \in S} \bigcup_{\substack{R \subseteq S, \\ R \ni j}} C_R^v = \bigcup_{R \subseteq S} C_R^v \qquad \text{and thus,}$$

$$\begin{aligned}
OCM_i(N, F_f^v) &\overset{(3.2.2)}{=} \Bigg[\bigcup_{S \subseteq N} F_f^v(S) \Bigg] - \Bigg[\bigcup_{T \subseteq N\backslash\{i\}} F_f^v(T) \Bigg] \\
&= \Bigg[\bigcup_{S \subseteq N} \bigcup_{R \subseteq S} C_R^v \Bigg] - \Bigg[\bigcup_{T \subseteq N\backslash\{i\}} \bigcup_{R \subseteq T} C_R^v \Bigg] \\
&= \Bigg[\bigcup_{S \subseteq N} C_S^v \Bigg] - \Bigg[\bigcup_{T \subseteq N\backslash\{i\}} C_T^v \Bigg] \\
&\subseteq \bigcup_{\substack{S \subseteq N, \\ S \ni i}} C_S^v \overset{(3.3.5)}{=} f_i(N,v).
\end{aligned}$$

By Theorem 3.3(ii), the value f admits a potential, the set-valued potential function $P_f : G^N(\mathcal{U}) \to 2^{\mathcal{U}}$ of which is given by $P_f(N,v) \overset{(3.3.1)}{=} \bigcup_{S \subseteq N} F_f^v(S) = \bigcup_{S \subseteq N} \bigcup_{R \subseteq S} C_R^v = \bigcup_{S \subseteq N} C_S^v$. □

Theorem 3.3.8. Let f be a value on the set game space $G^N(\mathcal{U})$. [5]

(i) If f has the balanced union property (3.3.4), then f satisfies the next recursive formula: for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$, it holds that

$$f_i(N,v) \cup F_f^v(N \setminus \{i\}) = F_f^v(N) \cup \left[\bigcup_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \right] \quad (3.3.6)$$

(ii) If f has the balanced union property (3.3.4) with disjoint unions, then f satisfies the recursive formula (3.3.6) with disjoint unions, that is $f_i(N,v) \cap F_f^v(N \setminus \{i\}) = \emptyset$, for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$.

Proof of Theorem 3.3.8.

First suppose (3.3.4) holds, that is the value f preserves unions. Fix the set game $(N,v) \in G^N(\mathcal{U})$ as well as $i \in N$. By (3.3.4), we derive, by taking the union over all $j \in N \setminus \{i\}$, the following:

$$\begin{aligned} & f_i(N,v) \cup \left[\bigcup_{j \in N\setminus\{i\}} f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) \right] \\ = & \left[\bigcup_{j \in N\setminus\{i\}} f_j(N,v) \right] \cup \left[\bigcup_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \right]. \end{aligned}$$

By adding $f_i(N,v)$ to both sides of the latter equality and recalling (3.2.7) concerning the solution game, we arrive at the following equality:

$$f_i(N,v) \cup F_f^v(N \setminus \{i\}) = F_f^v(N) \cup \left[\bigcup_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}}) \right].$$

So the recursive formula (3.3.6) holds. This proves the implication mentioned in part (i). The implication mentioned in part (ii) follows immediately by the observation that, for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$, it holds that $f_i(N,v) \cap F_f^v(N \setminus \{i\}) = f_i(N,v) \cap \left[\bigcup_{j \in N\setminus\{i\}} f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) \right] = \emptyset$, whenever $f_i(N,v) \cap f_j(N \setminus \{i\}, v_{N\setminus\{i\}}) = \emptyset$ for all $j \in N \setminus \{i\}$. □

[5] A single-valued cooperative game solution f on $\mathcal{CG}$ admits a real-valued potential function $P_f : \mathcal{CG} \to \mathbb{R}$ if and only if f satisfies the next recursive formula: $|N| \cdot f_i(N,v) = F_f^v(N) - F_f^v(N \setminus \{i\}) + \sum_{j \in N\setminus\{i\}} f_i(N \setminus \{j\}, v_{N\setminus\{j\}})$ for all $(N,v) \in \mathcal{CG}$ and all $i \in N$ (Calvo & Santos [12]).

3.4 An axiomatization of the overall-coalitionally marginalistic value for set games

The purpose of this section is to present an axiomatic characterization of the OCM–value. To be exact, we show that the OCM–value is fully determined by restricted global efficiency, as treated in Section 3.2, and the so-called equal treatment property together with a type of monotonicity property. The proof technique is based on the decomposition of any set game into a union of simple set games. Recall that for simple set games the worth of any coalition equals either the empty set or a singleton consisting of one arbitrary, but fixed item.

Definition 3.4.1. Let f be a value on the set game space $G^N(\mathcal{U})$. We say the value f possesses the *overall coalitionally marginalistic contribution monotonicity property* [6] if

$$f_i(N,v) \subseteq f_i(N,w) \qquad \text{for all } (N,v),(N,w) \in G^N(\mathcal{U}), \text{ and all } i \in N, \quad (3.4.1)$$

satisfying $C^v_{S,i} \subseteq C^w_{S,i}$ for all $S \subseteq N$ with $i \in S$, where the *overall coalitionally marginalistic contribution* is given by $C^v_{S,i} := v(S) - \bigcup_{T \subseteq N \setminus \{i\}} v(T)$. In words, with respect to two different set games, the larger the player's contributions in the game, the more items allocated to the player.

Lemma 3.4.2. The OCM–value on $G^N(\mathcal{U})$ has the equal treatment property, the overall coalitionally marginalistic contributions monotonicity property, the null player and the destructive player properties.

Proof of Lemma 3.4.2.
Let $(N,v) \in G^N(\mathcal{U})$. In order to prove the equal treatment property for the OCM–value, let the pair $i \in N$ $j \in N$, $i \neq j$, be substitutes in (N,v). By

[6] A single-valued cooperative game solution f on $\mathcal{CG}$ is said to satisfy the *strong monotonicity property* if it holds that $f_i(N,v) \leq f_i(N,w)$, for all $(N,v) \in \mathcal{CG}$, $(N,w) \in \mathcal{CG}$, and all $i \in N$, satisfying $v(S) - v(S \setminus \{i\}) \leq w(S) - w(S \setminus \{i\})$, for all $S \subseteq N$ with $i \in S$. The Shapley value Sh, as given by [64], possesses this property and in fact, is fully characterized by strong monotonicity, together with the efficiency property (Young [75]).

(3.2.2), the OCM–value of player i is given by

$$
\begin{aligned}
&OCM_i(N,v) \\
\overset{(3.2.2)}{=}\;& \Big[\bigcup_{S\subseteq N} v(S)\Big] - \Big[\Big[\bigcup_{\substack{T\subseteq N\setminus\{i\},\\ T\not\ni j}} v(T)\Big] \cup \Big[\bigcup_{\substack{T\subseteq N\setminus\{i\},\\ T\ni j}} v(T)\Big]\Big] \\
=\;& \Big[\bigcup_{S\subseteq N} v(S)\Big] - \Big[\Big[\bigcup_{T\subseteq N\setminus\{i,j\}} v(T)\Big] \cup \Big[\bigcup_{S\subseteq N\setminus\{i,j\}} v(S\cup\{j\})\Big]\Big].
\end{aligned}
$$

Since, by assumption, $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \setminus \{i,j\}$, it follows that the players i and j are interchangeable on the right-hand side of the latter equality and thus $OCM_i(N,v) = OCM_j(N,v)$ for any pair i, j of substitutes in the set game (N,v). This proves the equal treatment property for the OCM–value. The null player property for the OCM–value follows immediately from the inclusion $OCM_i(N,v) \subseteq \bigcup_{T\subseteq N\setminus\{i\}} [v(T\cup\{i\})) - v(T)]$, for all $i \in N$. Clearly, by (3.2.1), $OCM_i(N,v) = \emptyset$ for every destructive player i in the set game (N,v). Finally, by (3.2.3), the OCM–value possesses the overall coalitionally marginalistic contribution monotonicity property. This completes the proof of all four properties for the OCM–value. □

Theorem 3.4.3. (Axiomatization) Consider the setting of Definitions 3.2, 3.2(ii) and 3.4.
The OCM-value on the set game space $G^N(\mathcal{U})$ (with reference to a fixed player set N) is the unique solution f on $G^N(\mathcal{U})$ having the restricted global efficiency, equal treatment and overall coalitionally margianlistic contribution monotonicity properties.

The proof of Theorem 3.4 proceeds in three steps. The first preliminary result provides another interpretation of the value in that the OCM–value represents the *maximal* solution having the restricted global efficiency and overall coalitionally marginalistic contribution monotonicity properties.

Proposition 3.4.4. If a value f on $G^N(\mathcal{U})$ has the restricted global efficiency and contribution monotonicity properties, then the inclusion $f_i(N,v) \subseteq OCM_i(N,v)$ holds for all $(N,v) \in G^N(\mathcal{U})$ and all $i \in N$.

Proof of Proposition 3.4.4.

Suppose a value f on $G^N(\mathcal{U})$ has the restricted global efficiency and overall coalitionally marginalistic contribution monotonicity properties. Let (N, v) be a set game and $i \in N$. In order to show the inclusion $f_i(N, v) \subseteq OCM_i(N, v)$, let $x \in f_i(N, v)$, but assume, on the contrary, $x \notin OCM_i(N, v)$. Define a new set game (N, w) as follows:

$$w(S) := \begin{cases} v(S) - \{x\} & \text{for all } S \subseteq N \text{ with } x \in v(S); \\ v(S) & \text{for all } S \subseteq N \text{ with } x \in \mathcal{U} - v(S). \end{cases}$$

Notice that $x \notin w(S)$ for all $S \subseteq N$. From this observation, together with the restricted global efficiency (3.2.5) of f applied to the set game (N, w), we derive the following chain of inclusions:

$$\begin{aligned} f_i(N, w) &\subseteq \bigcup_{j \in N} f_j(N, w) \overset{(3.2.5)}{=} \Big[\bigcup_{S \subseteq N} w(S)\Big] - \Big[\bigcap_{k \in N} \bigcup_{T \subseteq N \setminus \{k\}} w(T)\Big] \\ &\subseteq \bigcup_{S \subseteq N} w(S) \subseteq \mathcal{U} - \{x\}, \qquad \text{particularly, } x \notin f_i(N, w). \end{aligned}$$

Next we claim that $C^w_{S,i} = C^v_{S,i}$ for all $S \subseteq N$ with $i \in S$ (where $C^v_{S,i} := v(S) - \bigcup_{T \subseteq N \setminus \{i\}} v(T)$). Consequently, $f_i(N, w) = f_i(N, v)$ by the overall coalitionally marginalistic contribution monotonicity (3.4.1) of f, but this equality contradicts the facts that $x \in f_i(N, v)$ and $x \notin f_i(N, w)$. This contradiction completes the proof, provided we establish the claim above-mentioned. For the sake of notation, write $\triangledown^v_{N \setminus \{i\}} := \bigcup_{T \subseteq N \setminus \{i\}} v(T)$.

Let $S \subseteq N$ with $i \in S$. We distinguish two cases. If $x \notin v(S)$, then $w(S) = v(S)$ and it holds that

$$C^w_{S,i} = w(S) - \triangledown^w_{N \setminus \{i\}} = v(S) - \triangledown^w_{N \setminus \{i\}} = v(S) - \triangledown^v_{N \setminus \{i\}} = C^v_{S,i}.$$

If $x \in v(S)$, then $w(S) = v(S) - \{x\}$ as well as $x \in \triangledown^v_{N \setminus \{i\}}$ (because of the assumption $x \notin OCM_i(N, v)$) and thus it holds that

$$C^w_{S,i} = w(S) - \triangledown^w_{N \setminus \{i\}} = \Big[v(S) - \{x\}\Big] - \triangledown^w_{N \setminus \{i\}} = v(S) - \triangledown^v_{N \setminus \{i\}} = C^v_{S,i}.$$

This completes the proof of the remaining claim. Further, this proof indicates that the restricted global efficiency may be replaced by any weak form of (global) efficiency, that is $\bigcup_{j\in N} f_j(N,w) \subseteq \bigcup_{S\subseteq N} w(S)$ for every set game (N,w). In addition, the definition of the expression $\nabla^w_{N\backslash\{i\}}$ does not matter so much. □

The final part of the preliminary results (for the sake of the proof of Theorem 3.4) deals with simple set games, which will be treated as the components of a decomposition for any arbitrary set game.

Proposition 3.4.5. (Decomposition results for set games and the *OCM*–value)
Let (N,v) be a set game, $x \in \mathcal{U}$, and $S \subseteq N$. Recall $C^v_{S,i} := v(S) - \bigcup_{T\subseteq N\backslash\{i\}} v(T)$ for all $i \in N$.
(i) Let $i \in N$. The following equivalence holds:

$$C^{v_x}_{S,i} = \{x\} \Longleftrightarrow x \in C^v_{S,i}, \tag{3.4.2}$$

$$(ii)\quad v = \cup_{y\in\mathcal{U}} v_y \qquad \text{that is,} \qquad v(T) = \cup_{y\in\mathcal{U}} v_y(T), \qquad \text{for all } T \subseteq N, \tag{3.4.3}$$

$$(iii)\quad OCM_i(N,v) = \bigcup_{y\in\mathcal{U}} OCM_i(N,v_y), \qquad \text{for all } i \in N. \tag{3.4.4}$$

(iv) If a value f on $G^N(\mathcal{U})$ possesses the overall coaltionally marginalistic contribution monotonicity property, then it holds that $f_i(N,v_x) \subseteq f_i(N,v)$ for all $i \in N$ and all $x \in \mathcal{U}$.

Proof of Proposition 3.4.6.
The decomposition statement (3.4.3) of the set game (N,v) is trivial since $\mathcal{U} = v(T) \cup \Big[\mathcal{U} - v(T)\Big]$ for all $T \subseteq N$. The decomposition statement (3.4.4) of the *OCM*–value of the set game (N,v) is a direct consequence of the equivalence (3.4.2) because, for all $i \in N$, it holds that

$$\bigcup_{y\in\mathcal{U}} OCM_i(N,v_y) \overset{(3.2.3)}{=} \bigcup_{y\in\mathcal{U}} \bigcup_{\substack{S\subseteq N,\\ S\ni i}} C^{v_y}_{S,i} = \bigcup_{\substack{S\subseteq N,\\ S\ni i}} \bigcup_{y\in\mathcal{U}} C^{v_y}_{S,i} \overset{(3.4.2)}{=} \bigcup_{\substack{S\subseteq N,\\ S\ni i}} C^v_{S,i} \overset{(3.2.3)}{=} OCM_i(N,v).$$

The statement in part (iv) is a direct consequence of the equivalence (3.4.2) too due to the inclusion $C^{v_x}_{S,i} \subseteq C^{v}_{S,i}$ for all $S \subseteq N$ with $i \in S$, and all $x \in \mathcal{U}$. It remains to prove, for every $i \in N$, the equivalence (3.4.2) as follows.

$$\begin{aligned}
& C^{v_x}_{S,i} = \{x\} \\
\iff\ & v_x(S) - \bigcup_{T \subseteq N \setminus \{i\}} v_x(T) = \{x\} \\
\iff\ & v_x(S) = \{x\} \quad \text{and} \quad \bigcup_{T \subseteq N \setminus \{i\}} v_x(T) = \emptyset \\
\iff\ & v_x(S) = \{x\} \quad \text{and} \quad v_x(T) = \emptyset, \quad \text{for all } T \subseteq N \setminus \{i\} \\
\iff\ & x \in v(S) \quad \text{and} \quad x \notin v(T), \quad \text{for all } T \subseteq N \setminus \{i\} \\
\iff\ & x \in v(S) - \bigcup_{T \subseteq N \setminus \{i\}} v(T) \\
\iff\ & x \in C^{v}_{S,i}.
\end{aligned}$$

□

Proof of the uniqueness part of Theorem 3.4.3.
Suppose a value f on $G^N(\mathcal{U})$ has the restricted global efficiency, equal treatment, and overall coalitionally marginalistic contribution monotonicity properties. Let (N, v) be a set game and $i \in N$. We show that $f_i(N, v) = OCM_i(N, v)$. By Propositions 3.4.4 and 3.4.5 (iii)-(iv), we obtain the following relationships:

$$OCM_i(N, v) = \bigcup_{y \in \mathcal{U}} OCM_i(N, v_y) \text{ as well as } \bigcup_{y \in \mathcal{U}} f_i(N, v_y) \subseteq f_i(N, v) \subseteq OCM_i(N, v)$$

Fixing the set game (N, v), player i and item $x \in \mathcal{U}$, it suffices to show that

$$OCM_i(N, v_x) = f_i(N, v_x), \qquad \text{for every simple set game } (N, v_x). \tag{3.4.5}$$

The proof of (3.4.5) proceeds by induction on the number of winning coalitions in the set game $(N, C^{v_x}_i)$, defined to be $C^{v_x}_i(S) := C^{v_x}_{S,i}$ for all $S \subseteq N$. Coalition S is said to be winning in the set game $(N, C^{v_x}_i)$ if it holds that $C^{v_x}_i(S) = \{x\}$

or, equivalently, $x \in C^v_{S,i}$ (see (3.4.2)). We distinguish two cases, whether or not there exists a unique winning coalition.

Case one. Suppose there exists a unique winning coalition S_1 in the set game $(N, C_i^{v_x})$, that is $C_i^{v_x}(S_1) = \{x\}$ and $C_i^{v_x}(S) = \emptyset$, for all $S \neq S_1$.
By the equivalence (3.4.2), $x \in C^v_{S_1,i} := v(S_1) - \bigcup_{T\subseteq N\setminus\{i\}} v(T)$. Particularly, $x \in v(S_1)$ as well as $x \notin v(T)$ for all $T \subseteq N \setminus \{i\}$ or, equivalently, by (**??**), $v_x(S_1) = \{x\}$ and $v_x(T) = \emptyset$ for all $T \subseteq N \setminus \{i\}$. Our first claim is the following:

$$f_j(N, v_x) = OCM_j(N, v_x) = \emptyset \qquad \text{for all } j \in N \setminus S_1. \tag{3.4.6}$$

Indeed, for all $j \in N \setminus S_1$, it holds that $C^{v_x}_{S,j} := v_x(S) - \bigcup_{T\subseteq N\setminus\{j\}} v_x(T) = \emptyset$ for all $S \subseteq N$ (due to $S_1 \subseteq N \setminus \{j\}$ and $v_x(S_1) = \{x\}$). From this, together with Proposition 3.4 applied to the simple set game (N, v_x), we deduce the following chain of inclusions:

$$f_j(N, v_x) \subseteq OCM_j(N, v_x) \overset{(3.2.3)}{=} \bigcup_{\substack{S\subseteq N,\\ S\ni j}} C^{v_x}_{S,j} = \emptyset, \qquad \text{for all } j \in N \setminus S_1,$$

so (3.4.6) holds. Our second claim is the following: $C^{(C_i^{v_x})}_{S,i} = C^{v_x}_{S,i}$ for all $S \subseteq N$, and thus

$$f_i(N, C_i^{v_x}) = f_i(N, v_x) \qquad \text{and} \qquad OCM_i(N, C_i^{v_x}) = OCM_i(N, v_x). \tag{3.4.7}$$

Indeed, since $v_x(T) = \emptyset$ for all $T \subseteq N \setminus \{i\}$, we get $C_i^{v_x}(T) = C^{v_x}_{T,i} = v_x(T) - \bigcup_{R\subseteq N\setminus\{i\}} v_x(R) = \emptyset$ for all $T \subseteq N \setminus \{i\}$ and thus, $C^{(C_i^{v_x})}_{S,i} = C_i^{v_x}(S) - \bigcup_{T\subseteq N\setminus\{i\}} C_i^{v_x}(T) = C_i^{v_x}(S) = C^{v_x}_{S,i}$ for all $S \subseteq N$. From this, together with the overall coalitionally marginalistic contribution monotonicity property for both f and the OCM-value, we derive $f_i(N, C_i^{v_x}) = f_i(N, v_x)$ as well as $OCM_i(N, C_i^{v_x}) = OCM_i(N, v_x)$. So (3.4.7) holds.

In case $i \in N \setminus S_1$, then (3.4.6) yields $f_i(N, v_x) = OCM_i(N, v_x) = \emptyset$, so (3.4.5) holds. It remains to consider the case $i \in S_1$. In view of (3.4.7), we aim to prove, instead of (3.4.5), the equivalent equality $f_i(N, C_i^{v_x}) = OCM_i(N, C_i^{v_x})$.

Firstly, note that, for all $j \in N \setminus S_1$, we have $C_i^{v_x}(S) = \emptyset$, for all $S \subseteq N$ with $j \in S$, and thus, by (3.2.3), $OCM_j(N, C_i^{v_x}) = \emptyset$ for all $j \in N \setminus S_1$. So far, from this, together with Proposition 3.4, we conclude that $f_j(N, C_i^{v_x}) = OCM_j(N, C_i^{v_x}) = \emptyset$ for all $j \in N \setminus S_1$.

Secondly, the restricted global efficiency property (3.2.5) for both f and the OCM-value, applied to the set game $(N, C_i^{v_x})$, yields

$$\bigcup_{k \in N} OCM_k(N, C_i^{v_x}) = \bigcup_{k \in N} f_k(N, C_i^{v_x}), \quad \text{which equals } \{x\} \text{ or, equivalently,}$$
$$\bigcup_{k \in S_1} OCM_k(N, C_i^{v_x}) = \bigcup_{k \in S_1} f_k(N, C_i^{v_x}), \qquad \text{which equals } \{x\},$$

since $f_j(N, C_i^{v_x}) = OCM_j(N, C_i^{v_x}) = \emptyset$, for all $j \in N \setminus S_1$. Note that any pair of players in S_1 are substitutes in the contributions set game $(N, C_i^{v_x})$ (since S_1 is the unique winning coalition). From the equal treatment property for both f and the OCM-value, applied to the game $(N, C_i^{v_x})$, we derive $f_k(N, C_i^{v_x}) = f_i(N, C_i^{v_x})$ as well as $OCM_k(N, C_i^{v_x}) = OCM_i(N, C_i^{v_x})$ for all $k \in S_1$, given $i \in S_1$. In summary, the latter efficiency equality simplifies to $f_i(N, C_i^{v_x}) = OCM_i(N, C_i^{v_x}) = \{x\}$. From this and (3.4.7), we conclude that $f_i(N, v_x) = f_i(N, C_i^{v_x}) = OCM_i(N, C_i^{v_x}) = OCM_i(N, v_x)$. This completes the proof of (3.4.5) if there exists one winning coalition in the game $(N, C_i^{v_x})$.

Case two. Suppose there are at least two winning coalitions in the set game $(N, C_i^{v_x})$, say, among others, coalition S_1. Particularly, it holds that $C_i^{v_x}(S_1) = \{x\}$ or, equivalently, $x \in C^v_{S_1,i}$.
Define two new set games $(N, v_{1,i})$ and $(N, v_{2,i})$, arising from the contributions game (N, C_i^v) such that $v_{1,i}$ is almost the contributions set game C_i^v and $v_{2,i}$ almost the empty set game. To be exact,

$$v_{1,i}(S) := \begin{cases} C^v_{S,i} & \text{for all } S \neq S_1; \\ \emptyset & \text{for } S = S_1; \end{cases} \tag{3.4.8}$$

$$v_{2,i}(S) := \begin{cases} \emptyset & \text{for all } S \neq S_1; \\ C^v_{S,i} & \text{for } S = S_1. \end{cases} \tag{3.4.9}$$

From the descriptions (3.4.8)–(3.4.9) of both set games, together with the equivalence (3.4.2), we obtain that their associated simple set games

$(N,(v_{1,i})_x)$ and $(N,(v_{2,i})_x)$ are given by

$$(v_{1,i})_x(S) := \begin{cases} C^{v_x}_{S,i} & \text{for all } S \neq S_1; \\ \emptyset & \text{for } S = S_1; \end{cases} \tag{3.4.10}$$

$$(v_{2,i})_x(S) := \begin{cases} \emptyset & \text{for all } S \neq S_1; \\ C^{v_x}_{S,i} & \text{for } S = S_1. \end{cases} \tag{3.4.11}$$

Note that, for all $S \subseteq N$, the inclusions $(v_{1,i})_x(S) \subseteq v_x(S)$ and $(v_{2,i})_x(S) \subseteq v_x(S)$ hold. Concerning the contributions in both simple set games, as given by (3.4.10)–(3.4.11), we claim the following:

$$C^{(v_{1,i})_x}_{S_1,i} = \emptyset \qquad \text{and} \qquad C^{(v_{1,i})_x}_{S,i} = C^{v_x}_{S,i} \qquad \text{for all } S \neq S_1; \tag{3.4.12}$$

$$C^{(v_{2,i})_x}_{S_1,i} = C^{v_x}_{S_1,i} \qquad \text{and} \qquad C^{(v_{2,i})_x}_{S,i} = \emptyset \qquad \text{for all } S \neq S_1. \tag{3.4.13}$$

In order to verify (3.4.12), for all $S \neq S_1$, the following chain of equalities holds:

$$\begin{aligned} & C^{(v_{1,i})_x}_{S,i} \\ = \; & (v_{1,i})_x(S) - \bigcup_{T \subseteq N \setminus \{i\}} (v_{1,i})_x(T) \\ \stackrel{(3.4.10)}{=} \; & C^{v_x}_{S,i} - \bigcup_{T \subseteq N \setminus \{i\}} (v_{1,i})_x(T) \\ = \; & \left[v_x(S) - \bigcup_{T \subseteq N \setminus \{i\}} v_x(T) \right] - \left[\bigcup_{T \subseteq N \setminus \{i\}} (v_{1,i})_x(T) \right] \\ = \; & \left[v_x(S) - \bigcup_{T \subseteq N \setminus \{i\}} v_x(T) \right] \qquad \text{(since } (v_{1,i})_x(R) \subseteq v_x(R) \text{ for all } R \subseteq N) \\ = \; & C^{v_x}_{S,i}. \end{aligned}$$

So (3.4.12) holds and, similarly, (3.4.13) holds. Clearly, it concerns a disjoint union in that $C^{v_x}_{S,i} = C^{(v_{1,i})_x}_{S,i} \cup C^{(v_{2,i})_x}_{S,i}$, for all $S \subseteq N$. From this we deduce the following chain of equalities:

$$\begin{aligned} OCM_i(N, v_x) &\overset{(3.2.3)}{=} \bigcup_{\substack{S\subseteq N,\\ S\ni i}} C_{S,i}^{v_x} = \bigcup_{\substack{S\subseteq N,\\ S\ni i}} \left[C_{S,i}^{(v_{1,i})_x} \cup C_{S,i}^{(v_{2,i})_x}\right] \\ &= \left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} C_{S,i}^{(v_{1,i})_x}\right] \cup \left[\bigcup_{\substack{S\subseteq N,\\ S\ni i}} C_{S,i}^{(v_{2,i})_x}\right] \\ &\overset{(3.2.3)}{=} OCM_i(N, (v_{1,i})_x) \cup OCM_i(N, (v_{2,i})_x). \end{aligned}$$

By (3.4.13), the contributions set game $(N, C_i^{(v_{2,i})_x})$ has a unique winning coalition S_1, whereas, by (3.4.12), the collection of winning coalitions in the contributions set game $(N, C_i^{(v_{1,i})_x})$ is identical to the one in the initial contributions set game $(N, C_i^{v_x})$, except for coalition S_1. The induction hypothesis (3.4.5) applied to both set games $(N, (v_{1,i})_x)$ and $(N, (v_{2,i})_x)$ yields

$$f_i(N, (v_{1,i})_x) = OCM_i(N, (v_{1,i})_x) \text{ as well as } f_i(N, (v_{2,i})_x) = OCM_i(N, (v_{2,i})_x).$$

Further, from the inclusion $C_{S,i}^{(v_{1,i})_x} \subseteq C_{S,i}^{v_x}$, for all $S \subseteq N$, together with the overall coaltionally marginalistic contribution monotonicity property for f, we derive the inclusion $f_i(N, (v_{1,i})_x) \subseteq f_i(N, v_x)$ and similarly, $f_i(N, (v_{2,i})_x) \subseteq f_i(N, v_x)$. Finally, we conclude that the following chain of inclusions holds:

$$\begin{aligned} & OCM_i(N, v_x) \\ = \;& OCM_i(N, (v_{1,i})_x) \cup OCM_i(N, (v_{2,i})_x) \\ = \;& f_i(N, (v_{1,i})_x) \cup f_i(N, (v_{2,i})_x) \qquad \text{(by the induction hypothesis)} \\ \subseteq \;& f_i(N, v_x) \qquad \text{(by the overall coaltionally marginalistic} \\ & \text{contribution monotonicity property of } f) \\ \subseteq \;& OCM_i(N, v_x) \qquad \text{(by Proposition 3.4.4).} \end{aligned}$$

We arrive at the equality $f_i(N, v_x) = OCM_i(N, v_x)$. This completes both the inductive proof of (3.4.5) and the full proof of Theorem 3.4.3. □

Chapter 4

Restricted set games

In this chapter we consider restrictions on set games. For cooperative games with worths in the reals restrictions have been considered on the coalitions. For a set of $|N|$ players only a subset of the $2^{|N|}$ coalitions is considered to the feasible. The restriction made can be by considering partition systems, like in Section 4.1, or by considering a matroid structure, like in Section 4.2. Results for such restricted games are now considered for restricted set games.

A restriction on the possible values of the worths of coalitions can also be considered. One way *e.g.* is to prescribe that certain coalitions have worth zero in the reals. However, set games also allow the choice of specific restricted subsets of the universe of elements as worths of the coalitions. Of course, in a general setting one might restrict the set of feasible coalitions, subsets of $|N|$, and also specify the set of allowed worths, subsets of $\mathcal{U}$. The choice of the restriction will highly depend on the application that one has in mind.

In Section 4.3 power indexes are considered as values where the worths are restricted to $\{0, 1\}$.

4.1 Set games restricted by partition systems

In this section a general cooperation structure is considered, which is an extension of the graph-communication structure (Myerson [50]). Recall that a set system on a finite ground set N is a pair $(N, \mathcal{F})$, with $\mathcal{F} \subseteq 2^N$. The sets belonging to $\mathcal{F}$ are called *feasible coalitions*. For any $S \subseteq N$, maximal feasible subsets of S are called *components* of S. Some basic knowledge refer to Bilbao [8].

Definition 4.1.1. A *partition system* is a set system $(N, \mathcal{F})$ satisfying:

(P1) $\emptyset \in \mathcal{F}$, and $\{i\} \in \mathcal{F}$, for every $i \in N$, and

(P2) for all $S \subseteq N$, the components of S, denoted by $\amalg(S) = \{T_1, \cdots, T_p\}$, form a partition of S.

Proposition 4.1.2. A set system $(N, \mathcal{F})$ which has property (P1) is a partition system if and only if $F \in \mathcal{F}$, $G \in \mathcal{F}$ and $G \cap F \neq \emptyset$ imply $F \cup G \in \mathcal{F}$.

Proof. Suppose $F \cup G \notin \mathcal{F}$ for some pair $\{F, G\} \subset \mathcal{F}$ with $G \cap F \neq \emptyset$. Then there exist two components $\{T_1, T_2\}$ of $F \cup G$, such that $T_1 \supseteq F$ and $T_2 \supseteq G$. Hence, $T_1 \cap T_2 \supseteq F \cap G \neq \emptyset$, which contradicts property (P2).

Conversely, if $(N, \mathcal{F})$ has property (P1), then $S = \bigcup_{i \in S} \{i\}$ for every $S \subseteq N$. Let $\amalg(S) = \{T_1, \cdots, T_p\}$ be the family of components of S. If $\amalg(S)$ is not a partition of S, then $T_i \cap T_j \neq \emptyset$, and hence $T_i \cup T_j \in \mathcal{F}$, which contradicts the maximality of T_i and T_j. □

Example 4.1.3. The following collections of subsets of N, given by $\mathcal{F} = 2^N$ and $\mathcal{F} = \{\emptyset, \{1\}, \cdots, \{n\}\}$ are the maximal and minimal partition systems.

Example 4.1.4. In a sequencing situation there is a queue, consisting of n customers waiting to be served at a counter. Curiel, Pederzoil and Tijs [17] introduced sequencing games (N, v), defined by $v(S) := \sum\{v(T) : T \in \amalg(S)\}$, where $v(T)$ is equal to the maximal cost savings the coalition can obtain by rearranging their positions in the queue. The components of $\amalg(S)$ are the maximal intervals of S in a total order on N. If (N, v) is a sequencing game then the collection $\mathcal{F} = \{T \subseteq N : T$ is an interval of $N\}$, is a partition system.

Example 4.1.5. In a communication structure given by a graph $G = (N, E)$, the set system $(N, \mathcal{F})$ with $\mathcal{F} = \{S \subseteq N : (S, E(S))$ is a connected subgraph of $G\}$, is a partition system.

Example 4.1.6. A hypergraph communication structure is a triple $(N, \mathcal{H}, v)$, where (N, v) is a game and $\mathcal{H} \subseteq 2^N$. The idea of modelling communication by means of conferences $H \in \mathcal{H}$ is due to Myerson [51].

Definition 4.1.7. Let (N, v) be a set game and let $(N, \mathcal{F})$ be a partition system. The $\mathcal{F}$*−restricted set game* $(N, v^{\mathcal{F}})$, is defined by

$$v^{\mathcal{F}}(S) = \bigcup_{T \in \amalg(S)} v(T), \qquad \text{for all } S \subseteq N,$$

where $\amalg(S)$ is the collection of the components of S.

If $S \in \mathcal{F}$ then $v^{\mathcal{F}}(S) = v(S)$. Note that if the partition system $(N, \mathcal{F})$ is defined by a communication structure, the corresponding restricted game is called a *graph-restricted* set game.

Proposition 4.1.8. Let $(N, \mathcal{F})$ be a partition system. For any $T \subseteq N$ and any $j \in T$, let $\amalg(T) = \{T_1, \cdots, T_m\}$ and $\amalg(T \setminus \{j\}) = \{R_1, \cdots, R_k\}$. Then, for any $R_i \in \amalg(T \setminus \{j\})$ there exists one $T_l \in \amalg(T)$ satisfying $T_l \supseteq R_i$.

Proof. Suppose that there is one component R_i of $T \setminus \{j\}$ with $R_i \nsubseteq T_l$ for all $T_l \in \amalg(T)$. $R_i \subseteq T \setminus \{j\} \subseteq T$ implies that there exists one $T_p \in \amalg(T)$ satisfying $T_p \cap R_i \neq \emptyset$. By Proposition 4.1.2, we know that $T_p \cap R_i \neq \emptyset$ implies $T_p \cup R_i \in \amalg(T)$. This gives a contradiction with the property of maximal feasible coalition of T_p. □

We also obtain the following result.

Proposition 4.1.9. Let $(N, \mathcal{F})$ be a partition system and $S \subseteq T \subseteq N$. Then for any $S_i \in \amalg(S)$ there exists one $T_j \in \amalg(T)$ satisfying $T_j \supseteq S_i$.

Let us consider the unanimity set game U_T and the elementary set game E_T for any $T \subseteq N$ in the context of partition systems:

Proposition 4.1.10. Let $(N, \mathcal{F})$ be a partition system. We have that

$$U_T(S) = \bigcap_{T_i \in \amalg(T)} U_{T_i}(S) = \bigcap_{T_i \in \amalg(T)} \bigcup_{S_j \in \amalg(S)} U_{T_i}(S_j), \tag{4.1.1}$$

$$E_T(S) = \bigcap_{T_i \in \amalg(T)} E_{T_i}(S) = \bigcap_{T_i \in \amalg(T)} \bigcup_{S_j \in \amalg(S)} E_{T_i}(S_j). \tag{4.1.2}$$

Proof. We will prove that (4.1.1) holds in two cases.

$$(i) \qquad U_T(S) = \emptyset \;\Leftrightarrow\; S \not\supseteq T$$
$$\Leftrightarrow \text{ there exist some components } T_i \text{ of } T$$
$$\text{such that } S_j \not\supseteq T_i \text{ for all } S_j \in \Pi(S),$$
$$\Leftrightarrow \bigcup_{S_j \in \Pi(S)} U_{T_i}(S_j) = \emptyset$$
$$\Leftrightarrow \bigcap_{T_i \in \Pi(T)} U_{T_i}(S) = \bigcap_{T_i \in \Pi(T)} \bigcup_{S_j \in \Pi(S)} U_{T_i}(S_j) = \emptyset.$$

$$(ii) \qquad U_T(S) = \mathcal{U} \;\Leftrightarrow\; S \supseteq T$$
$$\Leftrightarrow \text{ there exists one component } S_j \text{ of } S$$
$$\text{such that } S_j \supseteq T_i \text{ for any } T_i \in \Pi(T),$$
$$\Leftrightarrow \bigcup_{S_j \in \Pi(S)} U_{T_i}(S_j) = \mathcal{U}$$
$$\Leftrightarrow \bigcap_{T_i \in \Pi(T)} U_{T_i}(S) = \bigcap_{T_i \in \Pi(T)} \bigcup_{S_j \in \Pi(S)} U_{T_i}(S_j) = \mathcal{U}.$$

Also, we can obtain that, for any $S \subseteq N$,

$$E_T(S) = \bigcap_{T_i \in \Pi(T)} E_{T_i}(S) = \bigcap_{T_i \in \Pi(T)} \bigcup_{S_j \in \Pi(S)} E_{T_i}(S_j). \qquad \square$$

Proposition 4.1.11. Let (N, v) be a set game and let $(N, \mathcal{F})$ be a partition system. If (N, v) is a monotonic set game, then so is the $\mathcal{F}-$restricted set game $(N, v^{\mathcal{F}})$.

Proof. Let $S \subseteq T \subseteq N$. For any $\mathcal{F}-$restricted set game $(N, v^{\mathcal{F}})$, $v^{\mathcal{F}}(S) = \bigcup_{S_i \in \Pi(S)} v(S_i)$ and $v^{\mathcal{F}}(T) = \bigcup_{T_j \in \Pi(T)} v(T_j)$. Given $S_i \in \Pi(S)$, by Proposition 4.1.9 there exists a unique $T_j \in \Pi(T)$, with $T_j \supseteq S_i$, such that $v(S_i) \subseteq v(T_j)$, which is due to the monotonicity of the set game (N, v). So, we can conclude that, for any $S \subseteq T \subseteq N$,

$$v^{\mathcal{F}}(S) = \bigcup_{S_i \in \Pi(S)} v(S_i) \subseteq \bigcup_{T_j \in \Pi(T)} v(T_j) = v^{\mathcal{F}}(T). \qquad \square$$

In Chapter 2 and Chapter 3, some values for set games are given. These values are related to some kinds of marginalistic contribution of players. We now discuss these values for the $\mathcal{F}-$restricted set game $(N, v^{\mathcal{F}})$ with a partition system $(N, \mathcal{F})$. Let (N, v) be a set game and let $(N, \mathcal{F})$ be a partition system.

(i) Individually marginalistic contribution of player i: for any $T \ni i$ and $T \subseteq N$,

$$\begin{aligned} v^{\mathcal{F}}(T) - v^{\mathcal{F}}(T \setminus \{i\}) &= \Big[\bigcup_{R\in \amalg(T)} v(R)\Big] - \Big[\bigcup_{S\in \amalg(T\setminus\{i\})} v(S)\Big] \\ &= \bigcup_{R\in \amalg(T)} \Big[v(R) - \bigcup_{S\in \amalg(T\setminus\{i\})} v(S)\Big] \\ &= v(R) - \bigcup_{S\in \amalg(T\setminus\{i\})} v(S), \end{aligned} \tag{4.1.3}$$

where $R \ni i$ and $R \in \amalg(T)$. The third equality follows from Proposition 4.1.8.

(ii) An individually marginalistic value for the $\mathcal{F}$-restricted set game $(N, v^{\mathcal{F}})$ is defined by

$$\begin{aligned} IM_i(N, v^{\mathcal{F}}) &= \bigcup_{\substack{T\subseteq N\\ T\ni i}} [v^{\mathcal{F}}(T) - v^{\mathcal{F}}(T \setminus i)] \\ &= \bigcup_{\substack{T\in \mathcal{F}\\ T\ni i}} \Big[v(T) - \bigcup_{S\in \amalg(T\setminus\{i\})} v(S)\Big], \end{aligned} \tag{4.1.4}$$

where the third equality follows from (4.1.3). If $\{T|T \in \mathcal{F} \text{ and } T \ni i\} = \{i\}$, then $IM_i(N, v^{\mathcal{F}}) = v(\{i\})$.

(iii) An overall individually marginalistic value for the $\mathcal{F}$-restricted monotonic set game $(N, v^{\mathcal{F}})$ is defined by

$$OIM_i(N, v^{\mathcal{F}}) = \bigcup_{\substack{T\subseteq N\\ T\ni i}} [v^{\mathcal{F}}(T) - \bigcup_{j\in T} v^{\mathcal{F}}(T \setminus \{j\})]$$

$$= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \Big[\bigcup_{R\in \amalg(T)} v(R) - \bigcup_{j\in T} \bigcup_{S\in \amalg(T\setminus\{j\})} v(S)\Big]$$

$$= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \Big[\bigcup_{R\in \amalg(T)} v(R) - \bigcup_{\substack{S\in \mathcal{F}\\ S\subset T}} v(S)\Big]$$

and

$$\bigcup_{R\in\Pi(T)} v(R) - \bigcup_{\substack{S\subset T\\ S\in\mathcal{F}}} v(S) = \begin{cases} v(T) - \bigcup_{\substack{S\subset T\\ S\in\mathcal{F}}} v(S) & \text{if } T\in\mathcal{F}, \\ \emptyset & \text{if } T\notin\mathcal{F}. \end{cases} \tag{4.1.5}$$

If $T \notin \mathcal{F}$ then $\Pi(T) \subseteq \{S|S\subset T \text{ and } S\in\mathcal{F}\}$. Therefore we have that, for any $i \in N$,

$$OIM_i(N, v^{\mathcal{F}}) = \bigcup_{\substack{T\ni i\\ T\in\mathcal{F}}} \left[v(T) - \bigcup_{\substack{R\subset T\\ R\in\mathcal{F}}} v(R) \right].$$

Proposition 4.1.12. Let (N, v) be a monotonic set game and let $(N, \mathcal{F})$ be a partition system. Then, the following two values are equivalent. For any $i \in N$,

$$IM_i(N, v^{\mathcal{F}}) = OIM_i(N, v^{\mathcal{F}}).$$

Proof. By Proposition 4.1.11, we have that the $\mathcal{F}$-restricted set game $(N, v^{\mathcal{F}})$ is also monotonic set game. Then, the conclusion of Proposition 4.1.12 follows from Theorem 2.2.1. □

Proposition 4.1.13. Let (N, v) be a set game and let $(N, \mathcal{F})$ be a partition system.
(i) A sub-coalitionally marginalistic value for the $\mathcal{F}$–restricted set game $(N, v^{\mathcal{F}})$ is

$$SCM_i(N, v^{\mathcal{F}}) = \bigcup_{\substack{T\ni i\\ T\in\mathcal{F}}} [v(T) - \bigcup_{\substack{S\subset T\\ S\in\mathcal{F}}} v(S)], \quad \text{for any } T\ni i \text{ and } T\subseteq N.$$

(ii) An individually co-marginalistic value for the set game $(N, v^{\mathcal{F}})$ restricted by partition system is

$$ICM_i(N, v^{\mathcal{F}}) = \bigcup_{\substack{S\in\Pi(T)\\ T\ni i\\ T\subseteq N}} \left[v(S) - \bigcap_{j\in T} \bigcup_{R\in\Pi(T\setminus\{j\})} v(R) \right], \text{ for any } T\ni i \text{ and } T\subseteq N.$$

Proof. By the definitions of values in Example 2.1.1 and the definition of

$\mathcal{F}$–restricted set game, we can obtain the following fact.

$$
\begin{aligned}
(i) \quad SCM_i(N, v^{\mathcal{F}}) &= \bigcup_{\substack{T\subseteq N\\ T\ni i}} [v^{\mathcal{F}}(T) - \bigcup_{R\subset T} v^{\mathcal{F}}(R)] \\
&= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \left[\bigcup_{S\in \amalg(T)} v(S) - \bigcup_{R\subset T} \bigcup_{Q\in \amalg(R)} v(Q) \right] \\
&= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \left[\bigcup_{S\in \amalg(T)} v(S) - \bigcup_{\substack{R\subset T\\ R\in\mathcal{F}}} v(R) \right].
\end{aligned}
$$

Let us simplify the above expression of the SCM-value.

$$
\bigcup_{S\in\amalg(T)} v(S) - \bigcup_{\substack{R\subset T\\ R\in\mathcal{F}}} v(R) = \begin{cases} v(T) - \bigcup_{\substack{R\subset T\\ R\in\mathcal{F}}} v(R) & \text{if } T\in\mathcal{F}, \\ \emptyset & \text{if } T\notin\mathcal{F}. \end{cases}
$$

From the fact that $T \notin \mathcal{F}$ and $S \in \amalg(T)$ implies $S \subset T$ and $S \in \mathcal{F}$, we have $\bigcup_{S\in\amalg(T)} v(S) - \bigcup_{\substack{R\subset T\\ R\in\mathcal{F}}} v(R) = \emptyset$. Then the SCM-value for the $\mathcal{F}$-restricted set game $(N, v^{\mathcal{F}})$ is described as follows. For any $i \in N$,

$$
SCM_i(N, v^{\mathcal{F}}) = \bigcup_{\substack{T\ni i\\ T\in\mathcal{F}}} [v(T) - \bigcup_{\substack{S\subset T\\ S\in\mathcal{F}}} v(S)].
$$

$$
\begin{aligned}
(ii) \quad ICM_i(N, v^{\mathcal{F}}) &= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \left[v^{\mathcal{F}}(T) - \bigcap_{j\in T} v^{\mathcal{F}}(T\setminus\{j\}) \right] \\
&= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \left[(\bigcup_{S\in\amalg(T)} v(S)) - \bigcap_{j\in T} \bigcup_{R\in\amalg(T\setminus\{j\})} v(R) \right] \\
&= \bigcup_{\substack{T\subseteq N\\ T\ni i}} \bigcup_{S\in\amalg(T)} \left[v(S) - \bigcap_{j\in T} \bigcup_{R\in\amalg(T\setminus\{j\})} v(R) \right] \\
&= \bigcup_{\substack{S\in\amalg(T)\\ T\ni i\\ T\subseteq N}} \left[v(S) - \bigcap_{j\in T} \bigcup_{R\in\amalg(T\setminus\{j\})} v(R) \right].
\end{aligned}
$$

□

We conclude that values for games restricted by partition systems can be extended to set games in a straightforward way.

4.2 Values for set games restricted by matroids

An alternative restriction of the coalitions by a partition system, next to simply indicating an arbitrary set of feasible coalitions, is restriction by matroids. This has been discussed in the literature *a.o.* by Bilbao, Driessen, Jimenez Losada, and Lebron, [9], [10]. It is remarkable that their results do not heavily depend on the fact that the feasible coalitions are restricted by the matroid structure. The focus in their paper is on the effects on the values of the fact that there is a restriction on the coalition of some kind. We will mainly focus on the effect of having sets instead of real numbers for the worth of a coalition. In this section we will consider feasible coalitions of players by using a class of combinatorial geometries called matroids.

Let us assume that there are two rules of cooperation between players. Firstly, if a coalition can form, then any sub-coalition is also feasible. In general, the players that take part in the formation of a coalition have common interests. Therefore, any subset of these players has at least the same common interests. Secondly, if there are two coalitions where the cardinality differs in one element, a player of the largest can join with the smallest making a feasible coalition. For this reason, we will define the feasible coalitions by using combinatorial geometries called matroids. Bilbao *et al.*[8],[9],[10] have given a detailed treatment of the Shapley value for TU-games on matroids. In the following we will discuss the values for set games on matroids. Let us recall some basic knowledge about matroid.

Definition 4.2.1 A *matroid* is a pair $(N, \mathcal{M})$ consisting of a finite set N and a collection $\mathcal{M}$ of subsets of N having the following properties:
(i). $\emptyset \in \mathcal{M}$.
(ii). If $S \in \mathcal{M}$ and $T \subseteq S$, then $T \in \mathcal{M}$.
(iii). If $T, S \in \mathcal{M}$ with $|S| = |T| + 1$, then there exists $i \in S \setminus T$ such that $T \cup \{i\} \in \mathcal{M}$.

The sets belonging to $\mathcal{M}$ are called *independent sets*. A subset of N not belonging to $\mathcal{M}$ is called *dependent*. A set system $(N, \mathcal{I})$ satisfying (i) and (ii) is called an *independence system*.

Example 4.2.2. Let F be a finite subset of a vector space E and let $\mathcal{M}$ be the set of linearly independent subsets of vectors of F. Then $(F, \mathcal{M})$ is a matriod.

Example 4.2.3. Let E be the set of edges of a graph G and let $\mathcal{M} = \mathcal{M}(G)$ be a family that consists of those subsets of E that do not contain a cycle of G. The matroid $(E, \mathcal{M})$ is called a graphic matroid or forest matroid of G.

Example 4.2.4. For $k \in N$, $0 < k \leq n$, the family $\{S \subseteq N : |S| \leq k\}$ of all subsets of N of size at most k is the *uniform matroid* denoted by U_n^k. In particular, 2^N is called a *free matroid*.

Example 4.2.5. A pair of distinct element i, j of N are *parallel* in the matroid $(N, \mathcal{M})$ if $\{i, j\}$ is a minimal dependent set and we denote this by $i \parallel j$. If $i, j \in N$, and $i \neq j$, the family

$$\mathcal{M}_n(i \parallel j) = \{S \subseteq N : \{i, j\} \not\subseteq S\}$$

is a matroid.

The *rank function* $r : 2^N \to \mathbb{Z}_+$ of a matroid $\mathcal{M}$ on N is defined by

$$r(T) := max\{|S| : S \subseteq T, S \in \mathcal{M}\} \quad \text{for all } T \subseteq N.$$

Notice that $T \in \mathcal{M}$ if and only if $r(T) = |T|$. The following two theorems (cf. [8]) axiomatize matroids in terms of their rank functions.

Theorem 4.2.6. A function $r : 2^N \to \mathbb{Z}_+$ is the rank function of a matroid on N if and only if, for all $S, T \subseteq N$, the following holds:
(i) $0 \leq r(T) \leq |T|$.
(ii) $r(S) \leq r(T)$ whenever $S \subseteq T$.
(iii) $r(S \cup T) + r(S \cap T) \leq r(T) + r(S)$ for all $S, T \subseteq N$.

Theorem 4.2.7. A function $r : 2^N \to \mathbb{Z}_+$ is the rank function of a matroid on N if and only if, for all $T \subseteq N$ and all $i, j \in N \setminus T$, the following holds:
(i) $r(\emptyset) = 0$.
(ii) $r(T) \leq r(T \cup \{i\}) \leq r(T) + 1$.
(iii) if $r(T \cup \{i\}) = r(T \cup \{j\}) = r(T)$ then $r(T \cup \{i, j\}) = r(T)$.

In the setting of the above theorems, the function r determines uniquely the corresponding matroid through $\mathcal{M} = \{S \subseteq N : r(S) = |S|\}$. Elements of a given matroid are called *independent* sets and furthermore, for a given set system $(N, \mathcal{M})$ and $S \subseteq N$, a subset $B \subseteq S$, $B \in \mathcal{M}$ is called a *basis* of S if $B \cup \{i\} \notin \mathcal{M}$ for all $i \in S \setminus B$. That is, B is a basis of S when B is a maximal feasible subset of S. The property (ii) in Definition 4.2.1 implies that the subsets of any basis are independent sets and thus $2^B \subseteq \mathcal{M}$ for every basis $B \in B(\mathcal{M})$, where $B(\mathcal{M}) = \{B \in \mathcal{M} : |B| = r(N)\}$ is the family of bases of $\mathcal{M}$. For every $S \subseteq N$, all bases of S have the same cardinality, as has been shown in Theorem 1.5.1 (Chapter 8) [8]. We call the bases of N the *bases* of the matroid $(N, \mathcal{M})$. The rank of the matroid $(N, \mathcal{M})$ is the cardinality of any basis. Then the family of the bases is $B(\mathcal{M}) = \{B \in \mathcal{M} : |B| = r\}$, where r is the rank of $(N, \mathcal{M})$. We will suppose that $\bigcup\limits_{S \in \mathcal{M}} \{i : i \in S\} = N$.

Definition 4.2.8. A set game on a matroid $\mathcal{M}$ is a quadruple $(N, v, \mathcal{M}, \mathcal{U})$, where N is a finite set of players, v is a mapping $v : \mathcal{M} \to 2^{\mathcal{U}}$ with $v(\emptyset) = \emptyset$, $\mathcal{M}$ is a matroid, and $\mathcal{U}$, named universe, is an abstract finite set. For any basis $B \in B(\mathcal{M})$, a set game (B, v_B) is obtained by restricting v to the power set of B, *i.e.*, for any $S \subseteq B$, $v_B(S) = v(S)$.

If no confusion arises, we call $(N, v, \mathcal{M})$ itself a set game on matroid $\mathcal{M}$. Let $G^{\mathcal{M}}(\mathcal{U})$ be the collection of all set games on the matroid $\mathcal{M} \neq \emptyset$ and let $G^B(\mathcal{U})$ be the collection of all set games with players set B. In the following, we will discuss some values for set games on the space $G^{\mathcal{M}}(\mathcal{U})$. A value f on $G^{\mathcal{M}}(\mathcal{U})$ is a mapping $f : G^{\mathcal{M}}(\mathcal{U}) \to (2^{\mathcal{U}})^n$, which associates with any set game $(\mathcal{M}, v) \in G^{\mathcal{M}}(\mathcal{U})$ a set-valued vector $f(N, v, \mathcal{M}) = (f_i(N, v, \mathcal{M}))_{i \in N} \in (2^{\mathcal{U}})^n$, or shortly $f(N, v, \mathcal{M})$. Now we recall four different values mentioned in Chapter 2 and Chapter 3, and discuss these values for set games restricted by matroids.

Example 4.2.9. For any set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$,

(i) The individually marginalistic IM-value for set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$

is defined by

$$\begin{aligned} IM_i(N,v,\mathcal{M}) &:= \bigcup_{\substack{S\in\mathcal{M}\\ S\ni i}} \Big[v(S) - v(S\setminus\{i\})\Big] \qquad (4.2.1)\\ &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} \bigcup_{\substack{S\subseteq B\\ S\ni i}} \Big[v(S) - v(S\setminus\{i\})\Big] \quad \text{or, equivalently}\\ IM_i(N,v,\mathcal{M}) &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} IM_i(B,v_B)\\ &= \bigcup_{B\in B(\mathcal{M})} IM_i(B,v_B), \quad \text{for all } i\in N. \end{aligned}$$

(ii) The overall-individually marginalistic OIM-value for set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ is defined by

$$\begin{aligned} OIM_i(N,v,\mathcal{M}) &:= \bigcup_{\substack{S\in\mathcal{M}\\ S\ni i}} \Big[v(S) - \bigcup_{j\in S} v(S\setminus\{j\})\Big] \qquad (4.2.2)\\ &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} \bigcup_{\substack{S\subseteq B\\ S\ni i}} \Big[v(S) - \bigcup_{j\in S} v(S\setminus\{j\})\Big] \quad \text{or, equivalently}\\ OIM_i(N,v,\mathcal{M}) &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} OIM_i(B,v_B)\\ &= \bigcup_{B\in B(\mathcal{M})} OIM_i(B,v_B), \quad \text{for all } i\in N. \end{aligned}$$

(iii) The sub-coalitionally marginalistic SCM-value for set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ is defined by

$$\begin{aligned} SCM_i(N,v,\mathcal{M}) &:= \bigcup_{\substack{S\in\mathcal{M}\\ S\ni i}} \Big[v(S) - \bigcup_{T\subset S} v(T)\Big] \qquad (4.2.3)\\ &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} \bigcup_{\substack{S\subseteq B\\ S\ni i}} \Big[v(S) - \bigcup_{T\subset S} v(T)\Big] \quad \text{or, equivalently}\\ SCM_i(N,v,\mathcal{M}) &= \bigcup_{\substack{B\in B(\mathcal{M})\\ B\ni i}} SCM_i(B,v_B)\\ &= \bigcup_{B\in B(\mathcal{M})} SCM_i(B,v_B), \quad \text{for all } i\in N. \end{aligned}$$

(iv) The individually co-marginalistic ICM-value for set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ is described as

$$\begin{aligned} ICM_i(N, v, \mathcal{M}) &:= \bigcup_{\substack{S \in \mathcal{M} \\ S \ni i}} \left[v(S) - \bigcap_{j \in S} v(S \setminus \{j\}) \right] \qquad (4.2.4) \\ &= \bigcup_{\substack{B \in B(\mathcal{M}) \\ B \ni i}} \bigcup_{\substack{S \subseteq B \\ S \ni i}} \left[v(S) - \bigcap_{j \in S} v(S \setminus \{j\}) \right] \quad \text{or, equivalently} \\ ICM_i(N, v, \mathcal{M}) &= \bigcup_{\substack{B \in B(\mathcal{M}) \\ B \ni i}} ICM_i(B, v_B) \\ &= \bigcup_{B \in B(\mathcal{M})} SCM_i(B, v_B), \quad \text{for all } i \in N. \end{aligned}$$

For any set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$, the individually marginalistic value, the overall-individually marginalistic value, the sub-coalitionally marginalistic value and the individually co-marginalistic value are described by the corresponding values for set games (B, v_B) $(B \in B(\mathcal{M}))$, respectively. In the context of set games on matroids we have

Definition 4.2.10. Let f be a value on the set game space $G^{\mathcal{M}}(\mathcal{U})$. We say the value f has

(i) the *global efficiency* property, if the solution allocates all the attainable items to the players, that is

$$\bigcup_{i \in N} f_i(N, v, \mathcal{M}) = \bigcup_{S \in \mathcal{M}} v(S), \qquad \text{for all } (N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U}). \qquad (4.2.5)$$

(ii) the *equal treatment property*, if $f_i(N, v, \mathcal{M}) = f_j(N, v, \mathcal{M})$ for any pair $i \in N$, $j \in N$, $i \neq j$, of *substitutes* in the set game $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ (i.e., $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \in \mathcal{M}$ with $S \subseteq N \setminus \{i, j\}$). In words, two substitutes in a set game $(N, v, \mathcal{M})$ are allocated the same items.

(iii) the *individually marginalistic monotonicity* property, if

$$f_i(N, v, \mathcal{M}) \subseteq f_i(N, w, \mathcal{M}), \qquad \text{for all } (N, v, \mathcal{M}), (N, w, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U}) \text{ and all } i \in N,$$

(4.2.6)

satisfying $C^v_{S,i} \subseteq C^w_{S,i}$ for all $S \in \mathcal{M}$ with $i \in S$, where the individually marginalistic contribution $C^v_{S,i} := v(S) - v(S \setminus \{i\})$.

In Section 2.5 we gave a proof of Proposition 2.5.9 and a proof of Theorem 3.4.3 in Section 3.4 Rather similar proof techniques for the following Proposition 4.2.11 respectively Lemma 4.2.12. We just state the results of Proposition 4.2.11 and Lemma 4.2.12.

Proposition 4.2.11. If a value f on $G^{\mathcal{M}}(\mathcal{U})$ has the global efficiency and individually marginalistic monotonicity property, then $f_i(N, v, \mathcal{M}) \subseteq IM_i(N, v, \mathcal{M})$ holds, for all $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ and all $i \in N$.

Lemma 4.2.12. There exists a unique value f on the space $G^N(\mathcal{U})$ having the global efficiency property, the equal treatment property and the individually marginalistic monotonicity property. The value f is described as follows: for any $i \in B$,

$$f_i(N, v) = \bigcup_{\substack{S \subseteq N, \\ S \ni i}} [v(S) - v(S \setminus \{i\})] = IM_i(N, v).$$

Theorem 4.2.13. There exists a unique value f on the space $G^{\mathcal{M}}(\mathcal{U})$ having the global efficiency property, the equal treatment property and the individually marginalistic monotonicity property. The value f is described as follows: for any $i \in N$ and for all $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$,

$$f_i(N, v, \mathcal{M}) = \bigcup_{\substack{S \in \mathcal{M}, \\ S \ni i}} [v(S) - v(S \setminus \{i\})] = IM_i(N, v, \mathcal{M}).$$

Proof. It is easily proved that the individually marginalistic value $IM(N, v, \mathcal{M})$ has the global efficiency property, the equal treatment property and the individually marginalistic monotonicity property. In order to show the uniqueness of Theorem 4.2.13, we only prove the following chain: for any $i \in N$,

$$\begin{aligned} \bigcup_{B \in B(\mathcal{M})} IM_i(B, v_B) &= IM_i(N, v, \mathcal{M}) \supseteq f_i(N, v, \mathcal{M}) \\ &\supseteq \bigcup_{B \in B(\mathcal{M})} f_i(B, v_B) = \bigcup_{B \in B(\mathcal{M})} IM_i(B, v_B). \end{aligned} \quad (4.2.7)$$

By Proposition 4.2.11 and Lemma 4.2.12, the remaining part of the proof is that, for any $i \in N$, the conclusion $f_i(N, v, \mathcal{M}) \supseteq \bigcup_{B \in B(\mathcal{M})} f_i(B, v_B)$ holds. It is sufficient to verify that for any $i \in B$ and for any basis $B \in B(\mathcal{M})$ of the matroid $\mathcal{M}$, $f_i(N, v, \mathcal{M}) \supseteq f_i(B, v_B)$. Let us proceed in three steps.

For any $S \in \mathcal{M}$ with $S \ni i$, $(\bigcup_{T \in \mathcal{M}} (v(T) \cap E_T))(S) - (\bigcup_{T \in \mathcal{M}} (v(T) \cap E_T))(S \setminus \{i\}) \supseteq (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S) - (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S \setminus \{i\})$ since, if $S \subseteq B$ with $i \in S$, $(\bigcup_{T \in \mathcal{M}} (v(T) \cap E_T))(S) - (\bigcup_{T \in \mathcal{M}} (v(T) \cap E_T))(S \setminus \{i\}) = v(S) - v(S \setminus \{i\}) = (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S) - (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S \setminus \{i\})$ and, otherwise, $(\bigcup_{T \subseteq B} (v(T) \cap E_T))(S) - (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S \setminus \{i\}) = \emptyset$. By the individually marginalistic monotonicity of the value f, we have that

$$f_i(N, v, \mathcal{M}) = f_i(N, \bigcup_{T \in \mathcal{M}} (v(T) \cap E_T), \mathcal{M}) \supseteq f_i(N, \bigcup_{T \subseteq B} (v(T) \cap E_T), \mathcal{M}). \quad (4.2.8)$$

Next we claim that, for any $i \in N$ and for any $i \in B$,

$$f_i(N, \bigcup_{T \subseteq B} (v(T) \cap E_T), \mathcal{M}) = f_i(N, \bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}). \quad (4.2.9)$$

The reason is that, for any $S \in \mathcal{M}$ with $S \ni i$, the conclusion $(\bigcup_{T \subseteq B} (v(T) \cap E_T))(S) - (\bigcup_{T \subseteq B} (v(T) \cap E_T))(S \setminus \{i\}) = (\bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T])(S) - (\bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T])(S \setminus \{i\})$ holds.

We also have that, for any $T \subseteq B$, $f_i(N, \bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}) \supseteq f_i(N, [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M})$, due to $(\bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T])(S) - (\bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T])(S \setminus \{i\}) \supseteq [(v(T) - v(T \setminus \{i\})) \cap E_T])(S) - [(v(T) - v(T \setminus \{i\})) \cap E_T])(S \setminus \{i\})$, for all $S \in \mathcal{M}$ with $S \ni i$. Then, we obtain that $f_i(N, \bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}) \supseteq \bigcup_{T \subseteq B} f_i(N, [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M})$. Because the value f has the global efficiency property, the equal treatment property and the individually marginalistic monotonicity property, we have that $f_i(N, [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}) = v(T) - v(T \setminus \{i\})$. Furthermore, we have $\bigcup_{T \subseteq B} f_i(N, [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}) = \bigcup_{T \subseteq B} (v(T) -$

$v(T \setminus \{i\})) = f_i(B, v_B)$, the last equality following from Lemma 4.2.12. Thus, we know that

$$f_i(N, \bigcup_{T \subseteq B} [(v(T) - v(T \setminus \{i\})) \cap E_T], \mathcal{M}) \supseteq f_i(B, v_B). \tag{4.2.10}$$

Combining (4.2.8), (4.2.9) and (4.2.10), we read that, for any $i \in B$ and any $B \in B(\mathcal{M})$,

$$f_i(N, v, \mathcal{M}) \supseteq f_i(B, v_B), \tag{4.2.11}$$

and

$$f_i(N, v, \mathcal{M}) \supseteq \bigcup_{B \in B(\mathcal{M})} f_i(B, v_B). \tag{4.2.12}$$

By (4.2.7) and (4.2.12), that completes the proof. □

Definition 4.2.14. Let $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ be a set game and $B \in B(\mathcal{M})$. The cores of the set game $(N, v, \mathcal{M})$ and the set game (B, v_B) are defined respectively by

$$C(N, v, \mathcal{M}) := \{X = (x_i)_{i \in N} \mid \bigcup_{i \in N} x_i = \bigcup_{S \in \mathcal{M}} v(S) \text{ and } \bigcup_{i \in S} x_i \supseteq v(S), \text{ for all } S \in \mathcal{M}\},$$

$$C(B, v_B) := \{X = (x_i)_{i \in B} \mid \bigcup_{i \in B} x_i = \bigcup_{S \subseteq B} v(S) \text{ and } \bigcup_{i \in S} x_i \supseteq v(S), \text{ for all } S \subseteq B\}.$$

Theorem 4.2.15. Let $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ a set game and $B \in B(\mathcal{M})$. If $X^{\mathcal{M}} = (x_i^{\mathcal{M}})_{i \in N} \in C(N, v, \mathcal{M})$, then, $X^B = (x_i^B)_{i \in B} \in C(B, v)$, where $x_i^B = x_i^{\mathcal{M}} - \left[\bigcup_{\substack{S \in \mathcal{M} \\ S \not\subseteq 2^B}} v(S) - \bigcup_{S \subseteq B} v(S) \right]$.

Proof. By the description of x_i^B, we have that, for all $T \subseteq B$, $B \in B(\mathcal{M})$

fixed,

$$\begin{aligned}
\bigcup_{i\in T} x_i^B &= \bigcup_{i\in T}\left\{x_i^{\mathcal{M}} - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big]\right\}\\
&= \bigcup_{i\in T} x_i^{\mathcal{M}} - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big]\\
&\supseteq v(T) - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big] = v(T), \quad \text{for any } T\subseteq B.
\end{aligned}$$

We obtain that

$$\bigcup_{i\in B} x_i^B \supseteq \bigcup_{S\subseteq B} v(S). \tag{4.2.13}$$

On the other hand

$$\begin{aligned}
&\bigcup_{i\in B} x_i^B\\
=\ & \bigcup_{i\in B} x_i^{\mathcal{M}} - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big] \subseteq \bigcup_{i\in N} x_i^{\mathcal{M}} - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big]\\
=\ & \bigcup_{S\in\mathcal{M}} v(S) - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big]\\
=\ & \Big[\bigcup_{S\subseteq B} v(S) \cup [\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)]\Big] - \Big[\bigcup_{\substack{S\in\mathcal{M}\\ S\not\subseteq 2^B}} v(S) - \bigcup_{S\subseteq B} v(S)\Big]\\
=\ & \bigcup_{S\subseteq B} v(S)\\
i.e.\ & \bigcup_{i\in B} x_i^B \subseteq \bigcup_{S\subseteq B} v(S).
\end{aligned} \tag{4.2.14}$$

By (4.2.13) and (4.2.14), we have that $\bigcup_{i\in B} x_i^B = \bigcup_{S\subseteq B} v(S)$. Thus, we conclude that $X^B = (x_i^B)_{i\in B} \in C(B, v_B)$. □

Definition 4.2.16. For any $B_1 \in B(\mathcal{M})$ and $B_2 \in B(\mathcal{M})$, let $X^{B_1} = (x_i^{B_1})_{i\in B_1} \in (2^{\mathcal{U}})^{B_1}$ and $X^{B_2} = (x_i^{B_2})_{i\in B_2} \in (2^{\mathcal{U}})^{B_2}$. The allocation X is defined by the following operator: $X = X^{B_1} \bullet X^{B_2} = (x_i)_{i \in B_1 \cup B_2}$, if $i \in B_1$ and $i \in B_2$, then $x_i = x_i^{B_1} \cup x_i^{B_2}$; if $i \in B_1$ but $i \notin B_2$, then $x_i = x_i^{B_1}$; and if $i \in B_2$ but $i \notin B_1$, then $x_i = x_i^{B_2}$.

Theorem 4.2.17. Let $(N, v, \mathcal{M}) \in G^{\mathcal{M}}(\mathcal{U})$ a set game and $B \in B(\mathcal{M})$. If $X^B = (x_i^B)_{i \in B} \in C(B, v)$, for any $B \in B(\mathcal{M})$, then, $X^{\mathcal{M}} = (x_i^{\mathcal{M}})_{i \in \mathcal{M}} \in C(N, v, \mathcal{M})$, where $X^{\mathcal{M}} = X^{B_1} \bullet X^{B_2} \cdots \bullet X^{B_i} \bullet \cdots \bullet X^{B_k}$ and $B(\mathcal{M}) = \{B_1, B_2, \cdots, B_i\ , \cdots, B_k\}$.

Proof. For any $S \in \mathcal{M}$, there exists a basis $B \in B(\mathcal{M})$ such that $B \supseteq S$. Then, by the Definition 4.2.16, we know that, for any $S \in \mathcal{M}$,

$$\bigcup_{i \in S \subseteq B} x_i^{\mathcal{M}} \supseteq \bigcup_{i \in S \subseteq B} x_i^B \supseteq v(S).$$

By Definition 4.2.16, we also have that, for any $S \subseteq B \in B(\mathcal{M})$,

$$\bigcup_{i \in N} x_i^{\mathcal{M}} \supseteq \bigcup_{i \in B} x_i^{\mathcal{M}} \supseteq \bigcup_{i \in B} x_i^B = \bigcup_{S \subseteq B} v(S).$$

Furthermore, we have that

$$\bigcup_{i \in N} x_i^{\mathcal{M}} \supseteq \bigcup_{B \in B(\mathcal{M})} \bigcup_{S \subseteq B} v(S) = \bigcup_{S \in \mathcal{M}} v(S).$$

Thus, we have the conclusion that

$$\bigcup_{S \in \mathcal{M}} v(S) = \bigcup_{B \in B(\mathcal{M})} \bigcup_{S \subseteq B} v(S) = \bigcup_{B \in B(\mathcal{M})} \bigcup_{i \in B} x_i^B \supseteq \bigcup_{i \in N} x_i^{\mathcal{M}} \supseteq \bigcup_{S \in \mathcal{M}} v(S).$$

□

By Theorem 4.2.15 and Theorem 4.2.17, we know that there exists a one to one mapping between the allocations of the core of the set game $(N, v, \mathcal{M})$ and the allocations of the core elements of the set games (B, v_B) $(B \in B(\mathcal{M}))$. That means that the corresponding allocations of the core element for the set game $(N, v, \mathcal{M})$ can be deduced from allocations of the core elements for set games (B, v_B) $(B \in B(\mathcal{M}))$ and, conversely.

We conclude that the translation of some of the results of Bilbao *et al.* [9] to set games is straightforward, although the proofs are considerably different. It seems important to remark that the restriction of feasible coalitions by the matroid only plays a role where the bases are considered.

4.3 Power indices

In the two aforegoing sections the restriction was imposed on the coalition. The same effect is obtained when the worth of an unfeasible coalition is the number 0 or the empty set for normal cooperative games or set games respectively.

A rather important restriction is met when voting procedures are considered. Winning coalitions can be given worth 1 and losing coalitions worth 0. A well-known power index is the Shapley-Shubik index. It is defined by

$$f_i(v) := \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{(n-|T|)!(|T|-1)!}{n!}[v(T) - v(T \backslash \{i\})] \quad \text{for all } i \in N,$$

giving the power of player i for a given decision rule, that is expressed by stating which coalitions are winning. The only contributions come from coalitions T for which $v(T) = 1$ and $v(T \backslash \{i\}) = 0$, so that player i is a *swinger*. Another well-known power index of this kind is the Banzhaf index,

$$f_i(v) := \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{1}{2^{n-1}}[v(T) - v(T \backslash \{i\})] \quad \text{for all } i \in N.$$

The idea behind this index was generalized by Hoede and Bakker [37] in the following way.

The n players are supposed to have inclinations i to vote in favor of or against some proposal, which can be described by a $(0,1)$-vector $\vec{i}$. Before the voting they can influence each other, which leads to some decision vector $\vec{d}$, a $(0,1)$- vector describing the actual vote cast. In

$$\vec{d} = O\vec{i},$$

the operator O describes the influencing process. With $gd(\vec{d})$ for the group decision on basis of the decision vector d, the Hoede-Bakker index reads

$$f_i(v) := \sum_{i, i_i = 1} \frac{1}{2^{n-1}} gd(O\vec{i}), \quad \text{for all } i \in N.$$

Note that player i is assumed to be in favor, $i_i = 1$, that there are 2^{n-1} inclination vectors $\vec{i}$, that $\vec{i}$ determines a coalition S by its components 1 including player i, that $gd(\vec{Oi}) = v(S)$ and, finally, note that in this index contributions come from all coalitions that contain player i and are winning. The index might be written as

$$f_i(N, v) = \sum_{\substack{T \subseteq N \\ T \ni i}} \frac{1}{2^{n-1}} v(T), \quad \text{for all } i \in N.$$

But then the description of the influencing process has disappeared.

The transition to the set game setting is quite easy. The universe $\mathcal{U}$ is chosen to consist of one element $\mathcal{U}$, that stands for "victory" or "win". $v(S) = \mathcal{U}$ and $v(S) = \emptyset$ are the only two worths that come into play, for winning and losing coalitions respectively. The index can be maintained with this change. However, an interpretation of the allocation to a player has to be given. If player i gets allocation $\frac{3}{16}\mathcal{U}$, when using the Banzhaf index for example, the set $\{\mathcal{U}\}$ is allocated with a numerical coefficient. Hence the interpretation is quite obvious. Player i is a swinger for victory "in 3 out of 16 cases". Using the Hoede-Bakker index the interpretation is that player i obtains victory in 3 out of 16 cases. Values for set games that are pure sets will be called *qualitative* values. Values that contain numerical coefficients will be called *quantitative* values for set games.

Chapter 5

A quantitative approach to solutions for set games

In Chapter 5 a quantitative method to set games is introduced. The worth of a coalition for a set game is described by a vector, its components being 1 if the worth of coalition for the set game includes the corresponding item, otherwise 0. Two types of quantitative marginalistic values for the corresponding quantitative set games, which are similar to power indexes for simple TU-games, are characterized by some standard properties. The Shapley value and the Banzhaf value for TU-games are extended to k-dimensional Euclidean space.

5.1 Introduction

In [2], a marginal value for set games was defined by $\psi_i(v) = \bigcup_{\substack{T\subseteq N \\ T\ni i}} (v(T) - v(T \setminus \{i\}))$ for all $i \in N$. Let us apply the solution ψ to Example 1.2.1 introduced in Section 1.2. We have $\psi_1(v)=$ {kite, a jig-saw puzzle, cards, a reading book}, $\psi_2(v)=$ {a set of fishing gear, a jig-saw puzzle, cards, a reading book}, $\psi_3(v)=${a set of fishing gear, a jig-saw puzzle, cards, a reading book}. It is clear that one item of $\mathcal{U}$ can be shared by different players from the allocation of Example 1.2.1. One item of the universe allocated by a solution is shared by some players, which means the item may be used equally by these players. Then the solution is just a qualitative allocation and cannot reflect that different players have different contributions to one item.

In order to have a *quantitative allocation*, let us quantify the set game $(N, v, \mathcal{U})$ first. If the items of the universe are ordered, then, we can denote $\mathcal{U}$ by a $k-$dimensional vector $\mathcal{U}_k = (1, 1, ..., 1)$, where k is the number of items of the universe, and the ith component of the vector $\mathcal{U}_k$ denotes the ith ordered item. The characteristic function v is a mapping from 2^N to R^k with $v(\emptyset) = (0, 0, \cdots, 0) :=\mathbf{0}$. $v(S)$, called the worth of a coalition $S \subseteq N$, is then a $k-$dimensional vector of which each component is 0 or 1. If the ith component of $v(S)$ is 1, then the coalition S can obtain the ith item. Otherwise, it cannot. Let us replace $\mathcal{U}$ by $\mathcal{U}_k$ in the set game$(N, v, \mathcal{U})$, then the triple $(N, v, \mathcal{U}_k)$ denotes the set game $(N, v, \mathcal{U})$ that is quantified as above. Let Γ^k denote the collection of set games $(N, v, \mathcal{U}_k)$. A value f for set game $(N, v, \mathcal{U}_k)$ is defined as a mapping from Γ^k into $(R^k)^n$, which is associated with a $n-$dimensional vector $(f_i(v))_{i\in N}$. For any $i \in N$, $f_i(v)$ is a $k-$dimensional vector where the jth component denotes the portion of the jth ordered item shared by player i if allocated according to value f. In TU-games, the Shapley value is a classical value. If the Shapley value is extended to $k-$dimensional Euclidean space, then we can define a *quantitative marginal value* for set game $(N, v, \mathcal{U}_k)$ as follows:

$$\Psi_i(N, v) = \sum_{\substack{S\subseteq N \\ S\ni i}} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})), \quad \text{for all } i \in N. \quad (5.1.1)$$

The player i cannot share the corresponding item if one component of vector $\Psi_i(v)$ is negative. Now, let us discuss the allocation of the example mentioned above according to the value Ψ.

Example 5.1.1. Let $N = \{1, 2, 3\}$, the items of $\mathcal{U}$ are ordered as $\mathcal{U}$=$\{$ a set of fishing gear, a kite, a jig-saw puzzle, cards, a reading book $\}$. Then, we have $\mathcal{U}_5$=$\{$ 1, 1, 1, 1, 1 $\}$, $v(1)$ =(0, 1, 1, 0, 0), $v(2)$ =(0, 0, 0, 0, 1), $v(3)$ =(1, 0, 0, 0, 0), $v(1, 2)$ =(0, 1, 0, 1, 0), $v(2, 3)$ =(1, 0, 1, 0, 0), $v(1, 3)$ =(0, 1, 0, 1, 0), $v(1, 2, 3)$ =(1, 1, 1, 1, 1). Applying the value Ψ to this example, we have that $\Psi_1(v) = (\frac{-1}{6}, 1, \frac{1}{3}, \frac{2}{3}, \frac{1}{6})$, $\Psi_2(v) = (\frac{1}{3}, 0, \frac{1}{3}, \frac{1}{6}, \frac{2}{3})$, $\Psi_3(v) = (\frac{5}{6}, 0, \frac{1}{3}, \frac{1}{6}, \frac{1}{6})$, where the first component of $\Psi_1(v)$ is $-\frac{1}{6}$. It means that player 1 cannot share the set of fishing gear. If comparing the result of the value Ψ with that of the ψ value for Example 1.2.1, we can conclude that the quantitative marginal value Ψ not only absorbs the advantage of the value ψ that a item of universe can be shared by more than one player, but also quantitatively allocates the universe to all players.

5.2 Quantitative values for set games

In the sequel, we often identify the set game $(N, v, \mathcal{U}_k)$ with its characteristic function v. In $k-$dimensional Euclidean space, let $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)$ denote the ith unit vector and let $\mathbf{0}$=$(0, 0, \cdots, 0)$. For any $a, b \in R^k$, we denote $a \vee b$=$\{\max\{a_1, b_1\}, \max\{a_2, b_2\}, \cdots, \max\{a_k, b_k\}\}$, and $a \wedge b$=$\{\min\{a_1, b_1\}, \min\{a_2, b_2\}, \cdots, \min\{a_k, b_k\}\}$. For set games $v, w \in \Gamma^k$, $v \vee w$ and $v \wedge w$ denote the games defined by

$$(v \vee w)(S) := v(S) \vee w(S), \qquad \text{for all } S \subseteq N, \tag{5.2.1}$$

$$(v \wedge w)(S) := v(S) \wedge w(S), \qquad \text{for all } S \subseteq N. \tag{5.2.2}$$

Let us introduce some properties for a value f for set games $(N, v, \mathcal{U}_k)$, which are similar to those for TU-games.

Efficiency:	$\sum_{i \in N} f_i(v) = v(N)$.
Anonymity:	for all $v \in \Gamma_k$ and for all permutations σ of N,
	$f_{\sigma(i)}(v) = f_i(\sigma v)$, for all $i \in N$, where the game
	$\sigma v(S)$ is defined by $\sigma v(S) = v(\sigma(S))$, for all $S \subseteq N$.
Null Player Property:	$f_i(v) = \mathbf{0}$, if player i is a null player,
	$v(S) = v(S \setminus \{i\})$ for any $S \ni i$.
Transfer Property:	for all $v, w \in \Gamma_k$ and $v \vee w$, $v \wedge w \in \Gamma_k$
	$f(v \vee w) + f(v \wedge w) = f(v) + f(w)$
Banzhaf Sum Property:	$\sum_{i \in N} f_i(v) = \sum_{i \in N} \sum_{S \ni i} (1/2^{n-1})[v(S) - v(S \setminus \{i\})]$.
Additivity:	for all $v, w \in \Gamma$, $f(v + w) = f(v) + f(w)$.

Two kinds of games are defined as follows:

$$U_T^i(S) = \begin{cases} e_i & S \supseteq T \\ \mathbf{0} & S \not\supseteq T \end{cases} \quad E_T^i(S) = \begin{cases} e_i & S = T \\ \mathbf{0} & S \neq T. \end{cases}$$

Theorem 5.2.1. A solution f on the set games space Γ^k has the anonymity property, null player property, transfer property, and Banzhaf sum property if and only if, for any $i \in N$,

$$f_i(N,v) = \Phi_i(N,v) = \sum_{S \ni i} (1/2^{n-1})[v(S) - v(S \setminus \{i\})].$$

Proof. It is clear that the solution Φ has the four properties mentioned in the theorem. Suppose that a solution f has these four properties as well. We prove that f coincides with the value Φ. It is easily seen that $f_i(N, U_T^k) = \mathbf{0}$, for any $i \in N \setminus T$ since player $i \in N \setminus T$ is a null player, and $f_i(N, U_T^k) = f_j(N, U_T^k)$ for any pair $i, j \in T$ with $i \neq j$ by the anonymity property. Finally, by the Banzhaf sum property we have $f(N, U_T^i) = \Phi(N, U_T^i)$. Furthermore, we know that for

$$\begin{aligned} v &= (\vee_{T \in w_1(v)} E_T^1, \vee_{T \in w_2(v)} E_T^2, \cdots, \vee_{T \in w_k(v))} E_T^k) \\ &= (\vee_{T \in w_1(v)} E_T^1, 0, \cdots, 0) \vee (0, \vee_{T \in w_2(v)} E_T^2, 0, \cdots, 0) \\ &\quad \vee \cdots \vee (0, 0, \cdots, 0, \vee_{T \in w_k(v))} E_T^k \\ &:= E_1 \vee E_2 \vee \cdots \vee E_k, \end{aligned} \tag{5.2.5}$$

where $w_i(v)$ is a collection of some coalitions, namely, $w_i(v) = \{T | v(T) \geq e_i\}$. Then, we have

$$f(N,v) = f(N, E_1 \vee E_2 \vee \cdots \vee E_k) = f(N, E_1) + f(N, E_2) + \cdots + f(N, E_k).$$

Next, we use induction on $m = |w_i(v)|$ to prove $f(E_i) = \Phi(E_i)$.

(1) $m = 1$, then $w_i(v) = \{T\}$. Using the transfer property and the fact that $(U_T^i - E_T^i) \vee E_T^i = U_T^i$ and $(U_T^i - E_T^i) \wedge E_T^i = \mathbf{0}$, we have

$$\begin{aligned} f(N, U_T^i) &= f(N, (U_T^i - E_T^i) \vee E_T^i) + f(N, (U_T^i - E_T^i) \wedge E_T^i) \\ &= f(N, U_T^i - E_T^i) + f(N, E_T^i) \end{aligned}$$

and

$$f(N, E_T^i) = f(N, U_T^i) - f(N, U_T^i - E_T^i) = \Phi(N, U_T^i) - \Phi(N, U_T^i - E_T^i) = \Phi(N, E_T^i).$$

(2) If $m \geq 2$, the hypothesis is that, for $|w_i(v)| < m$, $f(N, E_i) = \Phi(N, E_i)$. Given a game v with $|w_i(v)| = m$, for one coalition $T \in w_i(v)$, we have that $E_i = (E_i - E_T^i) \vee E_T^i$ and $(E_i - E_T^i) \wedge E_T^i = \mathbf{0}$. By the transfer property and the hypothesis, we obtain that $f(N, (E_i - E_T^i) \vee E_T^i) + f(N, (E_i - E_T^i) \wedge E_T^i) = f(N, E_i - E_T^i) + f(N, E_T^i)$ and $f(N, E_i) = f(N, E_i - E_T^i) + f(N, E_T^i) = \Phi(N, E_i - E_T^i) + \Phi(N, E_T^i) = \Phi(N, E_i)$. We can conclude that $f(N, v) = f(N, E_1) + f(N, E_2) + \cdots + f(N, E_k) = \Phi(N, E_1) + \Phi(N, E_2) + \cdots + \Phi(N, E_k) = \Phi(N, v)$. □

In the same way one can prove the following theorem.

Theorem 5.2.2. A solution f on set games Γ^k have the efficiency property, anonymity property, null player property and transfer property if and only if, for any $i \in N$,

$$f_i(N, v) = \Phi_i(N, v) = \sum_{S \ni i} \frac{(n-s)!(s-1)!}{n!}[v(S) - v(S \setminus \{i\})].$$

where s is the cardinality of the set S.

Remark 5.2.3. Theorem 2.1 and Theorem 2.2 are extensions of the Banzhaf power index and the Shapley-Shubik index respectively to $k-$dimensional Euclidean space.

5.3 The extension of the Shapley value and the Banzhaf value

Let us extend games $(N, v, \mathbb{R})$ to a new type of cooperative games $(N, v, \mathbb{R}^k)$, where N is a set of finite players $(n = |N| \geq 2)$, v is a mapping from 2^N to $\mathbb{R}^k$ with $v(\emptyset) = \mathbf{0}$ and $\mathbb{R}^k$ is a $k-$dimensional Euclidean space. For any subset S of N, $v(S) \in \mathbb{R}^k$ we consider the profit of coalition S. A value f on games $(N, v, \mathbb{R}^k)$ is described as a mapping from the collection of all games $(N, v, \mathbb{R}^k)$ into $(\mathbb{R}^k)^n$. The class of games $(N, v, \mathbb{R}^k)$ will be denoted by Γ and a game $(N, v, \mathbb{R}^k)$ is identified with its characteristic function v.

It is well known that the unanimity games of the coalitions S are a basis for the collection of games $(N, v, \mathbb{R})$. Let $v, w \in \Gamma$ and $\alpha \in R$. We define $(v + w)(S) = v(S) + w(S)$ and $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$. With respect to this addition and multiplication, the unanimity games of the coalitions S for game $(N, v, \mathbb{R}^k)$, defined by U^i_T, are a basis of Γ as well.

Lemma 5.3.1. For any game $(N, v, \mathbb{R}^k)$ and all the unanimity games U^i_T, we have

$$v = \sum_{i=1}^{k} \sum_{T \subseteq N} \sum_{R \subseteq T} (-1)^{t-r}[e_i(v(R)U^i_T)],$$

where the games $e_i(v(R)U^i_T)$ are defined by

$$(e_i(v(R)U^i_T))(S) = \begin{cases} (e_i v(R))e_i & \text{if } S \supseteq T \\ \mathbf{0} & \text{if } S \not\supseteq T, \end{cases}$$

for all $i \in N$ and $T \subseteq N$ $(T \neq \emptyset)$. $t = |T|$ is the cardinality of set T.

Proof. We have to verify that, for all $S \in N$,

$$v(S) = \sum_{i=1}^{k} \sum_{T \subseteq N} \sum_{R \subseteq T} (-1)^{t-r}[e_i(v(R)U^i_T)](S). \tag{5.3.1}$$

For $S \in N$, we have $v(S) = (v(S)e_1, v(S)e_2, \cdots, v(S)e_k)$. In order to prove (5.3.1), we only check that, for all $i = 1, 2, \cdots, k$ and for any $S \subseteq N$, we have $v(S)e_i = \sum\limits_{T \subseteq N} \sum\limits_{R \subseteq T} (-1)^{t-r}(v(R)U^i_T)(S)$. By the definition of the unanimity game, we obtain that

$$\begin{aligned}
\sum_{T \subseteq N} \sum_{R \subseteq T} (-1)^{t-r} v(R) U^i_T(S) &= \sum_{T \subseteq S} \sum_{R \subseteq T} (-1)^{t-r} v(R) e_i \\
&= e_i(\sum_{T \subseteq S} \sum_{R \subseteq T} (-1)^{t-r} v(R)) \\
&= e_i\Big[\sum_{R \subseteq S} \sum_{t=r}^{s} (-1)^{t-r} \binom{s-r}{t-r}\Big] v(R) \\
&= v(R)e_i.
\end{aligned}$$

In the expression $\sum\limits_{R \subseteq S} [\sum\limits_{t=r}^{s} (-1)^{t-r} \binom{s-r}{t-r}] v(R)$, if $s \neq r$, the expression between brackets equals zero. $\square$

Lemma 5.3.2. $dim(\Gamma) = k2^{n-1}$.

Proof. By Lemma 5.3.1, we only prove that all the unanimity games U_T^i are linearly independent. Suppose there exist real numbers α_T^i, with $T \subseteq N$ and $i = 1, 2, \cdots, k$, such that $\sum_{i=1}^{k}\sum_{T\subseteq N} \alpha_T^i U_T^i(S) = 0$ for all $S \subseteq N$. We prove by induction on t that $\alpha_T^i = 0$, for all $T \subseteq N$, $T \neq \emptyset$ and $i = 1, 2, \cdots, k$.

(1). If $t = 1$, say $T = \{j\}$, then the requirement for $S = T$ yields $0 = \sum_{i=1}^{k}\sum_{R\subseteq N} \alpha_R^i U_R^i(T) = \sum_{i=1}^{k} \alpha_T^i e_i$. So,$0 = (\alpha_T^1, \alpha_T^2, \cdots, \alpha_T^i)$.

(2) Let $2 \leq t \leq n$ and suppose that $\alpha_R^i = 0$, for all $R \subseteq N$ with $1 \leq r = |R| < t$ and $i = 1, 2, \cdots, k$. We have that $0 = \sum_{i=1}^{k}\sum_{R\subseteq N} \alpha_R^i U_R^i(T) = \sum_{i=1}^{k}\sum_{\substack{R\subseteq N\\ r\geq t}} \alpha_R^i U_R^i(T) = \sum_{i=1}^{k} \alpha_T^i U_T^i(T) = (\alpha_T^1, \alpha_T^2, \cdots, \alpha_T^k)$. So, the set of all unanimity games is a basis for the class of cooperative games $(N, v, \mathbb{R}^k)$. □

The solutions Ψ and Φ mentioned in Section 2 for set Games $(N, v, \mathcal{U}^k)$ are similar to power indexes for simple games. In the following, we will discuss the values Ψ and Φ for games $(N, v, \mathbb{R}^k)$.

Theorem 5.3.3. A solution f on Γ has the properties of efficiency, anonymity, null player and additivity if and only if, for all $i \in N$,

$$f_i(N, v) = \Psi_i(N, v) = \sum_{\substack{S\subseteq N\\ S\ni i}} \frac{(s-1)!(n-s)!}{n!}(v(S) - v(S \setminus \{i\})).$$

Proof. It is easy to check that the solution Ψ on Γ satisfies the axioms of efficiency, null player, anonymity and additivity. Suppose that a value f has the four mentioned properties. By Lemma 5.3.1 and the additivity property, we have

$$\begin{aligned} f_j(N, v) &= f_j(\sum_{i=1}^{k}\sum_{T\subseteq N}\sum_{R\subseteq T}(-1)^{t-r}[e_i(v(R)U_T^i)]) \\ &= \sum_{i=1}^{k}\sum_{T\subseteq N}\sum_{R\subseteq T}(-1)^{t-r} f_j(e_i(v(R)U_T^i)). \end{aligned}$$

By the null player property, $f_j(e_i(v(R)U_T^i)) =\mathbf{0}$, whenever $j \in N \setminus T$. But

$f_j(e_i(v(R)U_T^i)) = f_h(e_i(v(R)U_T^i))$ for any $j, h \in T$. So we have

$$
\begin{aligned}
f_j(N,v) &= \sum_{i=1}^{k} \sum_{\substack{T\subseteq N\\ T\ni j}} \sum_{R\subseteq T} (-1)^{t-r} \frac{v(R)e_i}{t} e_i \\
&= \sum_{i=1}^{k} \{ \sum_{\substack{T\subseteq N\\ T\ni j}} [\sum_{\substack{S\cup\{j\}\subseteq T\\ S\not\ni j}} (-1)^{t-s-1} \frac{v(S\cup\{j\})e_i}{t} \\
&\quad + \sum_{\substack{S\subseteq T\\ S\not\ni j}} (-1)^{t-s} \frac{v(S)e_i}{t}]\} e_i \\
&= \sum_{i=1}^{k} \{ \sum_{\substack{T\subseteq N\\ T\ni j}} [\sum_{\substack{S\cup\{j\}\subseteq T\\ S\not\ni j}} (-1)^{t-s-1} \frac{1}{t} (v(S\cup\{j\}) - v(S))e_i]\} e_i \\
&= \sum_{i=1}^{k} \{ \sum_{S\subseteq N-j} [\sum_{t=s+1}^{n} \binom{n-s-1}{t-s-1} (-1)^{t-s-1} \frac{1}{t}][(v(S\cup\{j\}) - v(S))e_i]\} e_i \\
&= \sum_{i=1}^{k} \{ \sum_{S\subseteq N-j} \frac{(n-s-1)!s!}{n!} [(v(S\cup\{j\}) - v(S))e_i]\} e_i \\
&= \sum_{i=1}^{k} \{ \sum_{R\ni j} \frac{(n-r)!(r-1)!}{n!} [(v(R) - v(R\setminus\{j\}))e_i]\} e_i \\
&= \sum_{R\ni j} \frac{(n-r)!(r-1)!}{n!} (v(R) - v(R\setminus\{j\})) \\
&= \Psi_j(N,v),
\end{aligned}
$$

□

In an analogous way , we gain the characterization of the solution Φ on Γ.

Theorem 5.3.4: A solution f on Γ has the property of anonymity, the null player property, the additivity property and the Banzhaf sum property if and only if, for all $i \in N$,

$$f_i(N,v) = \Phi_i(N,v) = \sum_{\substack{S\subseteq N\\ S\ni i}} \frac{1}{2^{n-1}} (v(S) - v(S\setminus\{i\})).$$

By representing sets by characteristic $(0,1)-$vectors the two values considered were easily extended to $k-$dimensional Euclidean space. However, like in Section 4.3, where we consider power indices with $k = 1$, there is an important problem concerning the interpretation of the numerical coefficients in these quantitative values. That interpretation clearly depends on the application under consideration.

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Notations

$\mathbb{R}$	the set of real numbers
$\emptyset$	the empty set
$\|T\|$	the cardinality of the set T
N	the set of players
n	the cardinality of the set N
$\mathcal{U}$	the universe
$G^N(\mathcal{U})$	the space of set games
$G^M(\mathcal{U})$	the space of monotonic set games
$G^{\mathcal{M}}(\mathcal{U})$	the space of set games on matroids
$2^N = \{S \mid S \subseteq N\}$	the collection of subsets of N
$2^{\mathcal{U}} = \{R \mid R \subseteq \mathcal{U}\}$	the collection of subsets of $\mathcal{U}$
$A \subset B$	the proper subset A of B
$A \subseteq B$	the subset A of B
$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$	the intersection of sets A and B
$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$	the union of sets A and B
$a = (a_1, \cdots, a_i, \cdots, a_n)$	the n-dimensional vector a
$ab = \sum_{i=1}^n a_i b_i$	the inner product of vectors a and b

Index

Summary

This monograph deals with a new type of cooperative games, named set games, in characteristic function form and solution concepts for these games. The contributions to this work, which consists of five chapters, are as follows.

In Chapter 1 a new type of cooperative games in characteristic function form (called set games) is introduced. It was firstly described by Hoede in 1992. Of particular interest is that the worth of a coalition for set game is expressed by a set instead of by a real number as for TU-games. For this class of games we define the notion of value, being a solution concept and also several axioms, which are similar to those for TU-games (Transferable Utility games). The Chapter gives a survey of results on set games derived by Aarts *et al.*

In Chapter 2 we propose several values, which have some relationship with the marginalistic contribution with respect to any coalition containing the player, for set games. An individually marginalistic value for monotonic set games, which is similar to the Shapley value for TU-games, was characterized by Aarts, *et al.* The axiomatization of a sub-coalitionally marginalistic value, which coincides with the marginalistic value for monotonic set games, is shown using unanimity set games. A value analogous to the solidarity value (introduced by Nowak & Radzik) for set games, named individually co-marginalistic value, is proved to uniquely satisfy some axiom properties using elementary set games. Furthermore, we give a class of values, called semi- marginalistic values. These values satisfy axioms like global efficiency, equal treatment and monotonicity. In order to characterize this class of values, we provide a rather different method, which is based on simple set games. In the last section, some

properties of values for set games are presented. The concept of core, being another type of solution, is described too.

In Chapter 3 we introduce a new value by allocating, to any player, the items (taken from a universe) that are attainable for the player, but cannot be blocked (by any coalition not containing the player). The resulting value turns out to be an important concept for set games to characterize the family of set game solutions that possess a so-called potential representation (similar to the potential approaches applied in both physics and cooperative game theory). An axiomatization of the new value, called overall coalitionally marginalistic value, is given by three properties, namely a kind of efficiency property, the substitution property and a kind of monotonicity property.

Two kinds of restricted set games are discussed in Chapter 4, one restriction on set games by partition systems, another by matroids. Several values for restricted set games are discussed. We characterize the individually marginalistic value for set games on matroids by several standard axioms and prove that an element of the core for normal set games is extendable to an element of the core for set games on matroids.

In Chapter 5 a quantitative method to set games is introduced, in which the worth of a coalition of a set game is described by a vector, its components being 1 if the worth of the coalition includes the corresponding item, otherwise being 0. Two types of quantitative marginalistic values for the corresponding quantitative set games, which are similar to power indexes for simple TU-games, are characterized by some standard properties. The Shapley value and the Banzhaf value for TU-games are extended to k-dimensional Euclidean space.

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Curriculum Vitae

Hao Sun was born on November 6, 1967, in Chengcheng County of Shaanxi Province, P.R. China. From 1976 until 1983 he attended primary and middle school in his hometown. From September 1983, he has received primary education in Shaanxi Dali Teachers School for three years. After this, he has worked at a middle school in his hometown. During his working periods, he has studied pure mathematics at Shaanxi Education Institute twice, a first time from September 1988 to July 1990 and a second time from September 1991 to July 1993. After receiving his Bachelor's degree in July 1993, he became a graduate student at Northwestern Polytechnical University. He specialized in probability and statistics, and completed his Master's degree thesis, entitled "A Large Sample Theory for Partial Linear Regression Models", under the supervision of Professor Xuanmin Zhao. In April 1996, he graduated with his Master's degree and began his job as a teacher in Northwestern Polytechnical University. He has been teaching mathematics and doing research on statistics theory and game theory at the university in these years .

In March 1999 he started as a Ph.D. student under the supervision of Prof. Dr. Cornelis Hoede and Dr. Theo Driessen of the University of Twente, and Prof. Dr. Xueliang Li of Northwestern Polytechnical University in Xi'an. One of the main results of the study during the past five years is the thesis that is now in the hand of the reader.