Cycles in Weighted Graphs and Related Topics

Shenggui Zhang

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Shenggui Zhang
October 2002, Enschede

Preface

This thesis is the result of almost five years of research in the field of graph theory between August 1997 and June 2002. After an introductory chapter the readers will find eight chapters that contain three topics within this research field. These topics have more or less strong connections with each other. Both of the first two topics are on paths and cycles in graphs. One deals with paths and cycles in weighted graphs, and the other deals with paths and cycles in colored graphs. The third topic is on the scattering number, a vulnerability measure of graphs. This parameter is closely related to the existence of Hamilton paths and cycles in graphs.

Papers underlying this thesis

[1] S. Zhang, X. Li and H. J. Broersma, Heavy paths and cycles in weighted graphs, *Discrete Math.* **223** (2000), 327-336. (Chapter 2)

[2] S. Zhang, H. J. Broersma, X. Li and L. Wang, A Fan type condition for heavy cycles in weighted graphs, *Graphs & Combin.* **18** (2002), 193-200. (Chapter 3)

[3] S. Zhang, H. J. Broersma and X. Li, A σ_3 type condition for heavy cycles in weighted graphs, *Discuss. Math. Graph Theory* **21** (2001), 159-166. (Chapter 4)

[4] S. Zhang and X. Li, Long paths and cycles in weighted graphs, *Pure Appl. Math.* **15** (1999), 46-51. (Chapter 5)

[5] H. J. Broersma, X. Li, G. J. Woeginger and S. Zhang, *Paths and cycles in colored graphs,* Working paper. (Chapters 6 and 7)

[6] S. Zhang and Z. Wang, Scattering number in graphs, *Networks* **37** (2001), 102-106. (Chapter 8)

[7] S. Zhang, X. Li and X. Han, Computing the scattering number of graphs, *Int. J. Comput. Math.* **79** (2002), 179-187. (Chapter 9)

Chapter 1

Introduction

This thesis is composed of three parts. Each of the parts is on one topic in graph theory. The first two topics are closely related. One is on paths and cycles in weighted graphs, and the other is on paths and cycles in colored graphs. The third topic of the thesis is on a vulnerability parameter of graphs, the scattering number. This parameter itself also has a strong connection with paths and cycles in graphs.

The first part of the thesis consists of Chapters 2 to 5. In this part, we present several sufficient conditions for a weighted graph to contain paths and cycles with special properties, e.g., with a large sum of edge weights. These results generalize some of the well-known theorems on the existence of long paths and cycles in unweighted graphs.

The second part of the thesis consists of Chapters 6 and 7. In this part, we turn our attention to paths and cycles in (edge-) colored graphs. This part contains some basic results on paths and cycles in colored graphs. Some sufficient conditions for the existence of (long) monochromatic paths and cycles, and long heterochromatic paths and cycles are obtained. We prove that the problem of finding a path with as few different colors as possible between two given vertices in a colored graph is

NP-hard and propose two exact algorithms and an approximation algorithm for finding such a path with the fewest colors. The complexity of the two exact algorithms and the efficiency of the approximation algorithm are analyzed. We also pose a conjecture on the existence of paths and cycles with many different colors.

The third part of the thesis consists of Chapters 8 and 9. In this part, we deal with a vulnerability parameter of graphs, namely the scattering number. The relationships between the scattering number and some other vulnerability parameters of graphs, including the (edge-) connectivity, toughness, integrity and tenacity are discussed. At the same time, the scattering number of grids, hypercubes and the Cartesian products of two complete graphs is determined. It is proved that the scattering number of split graphs can be computed in polynomial time.

We assume that the reader is familiar with the essentials of graph theory. Most of the terminology and notations can be found in Bondy & Murty [14].

In the remainder of this introductory chapter, we will present, together with the relevant terminology and notations, a survey of the main results of the thesis against a background of related results.

1.1 Paths and cycles in weighted graphs

In the first part of this thesis, we mainly deal with paths and cycles in weighted graphs.

Let $G = (V, E)$ be a simple graph. G is called a *weighted graph* if each edge e is assigned a nonnegative number $w(e)$, called the *weight* of e. For a subgraph H of G, we use $V(H)$ and $E(H)$ to denote the sets of vertices and edges of H, respectively. The *weight* of H is defined by

$$w(H) = \sum_{e \in E(H)} w(e).$$

A *heaviest* cycle (path) is one with maximum weight among all cycles (paths) of G. For each vertex $v \in V$, $N_H(v)$ denotes the set, and $d_H(v)$ the number, of vertices of H adjacent to v. We define the *weighted degree* of v in H by

$$d_H^w(v) = \sum_{h \in N_H(v)} w(vh).$$

When no confusion occurs, we will denote $N_G(v)$, $d_G(v)$ and $d_G^w(v)$ by $N(v)$, $d(v)$ and $d^w(v)$, respectively.

If we regard an unweighted graph as a weighted graph in which each edge e is assigned weight $w(e) = 1$, then the weight of a subgraph is simply the number of its edges, the weighted degree of a vertex is the degree of it, and a heaviest cycle (path) is just a longest cycle (path).

In [12] and [13], Bondy & Fan began the study of heavy paths and cycles in weighted graphs by generalizing several classical theorems of Dirac and of Erdös & Gallai on the existence of paths and cycles. A weighted generalization of Ore's theorem was obtained by Bondy, Broersma, van den Heuvel & Veldman [11]. In [8], Bollobás & Scott examined similar questions for weighted digraphs. In [15], Bondy & Tuza gave a weighted generalization of Turán's theorem. Their results were further extended by Füredi & Kündgen in a very recent paper [32].

Bondy & Tuza [15] give several reasons why weighted generalizations of classical extremal theorems in unweighted graphs are of interest. We cite some of the reasons here:

1. The appropriate generalization, if indeed there is one, is not always immediately evident, and a choice must be made between several likely candidates.
2. The standard proofs for unweighted graphs do not usually extend to the weighted context, thus requiring new proof techniques to be developed.
3. The extremal configurations in the weighted case are more varied and have a richer structure than in the unweighted case.
4. Weighted theorems with appropriately chosen weights have the potential to yield new results for unweighted graphs.

In Chapters 2 to 5, some results on paths and cycles in unweighted graphs are generalized to weighted graphs. The proof techniques are modifications of similar techniques in unweighted graphs or extensions of techniques used to prove the earlier results in weighted graphs.

Here we give some further terminology and notations. An (x, z)-*path* is a path connecting the two vertices x and z. If u and v are two vertices on a path P, $P[u, v]$ denotes the segment of P from u to v. For a given vertex y in G, an (x, z)-path is called an (x, y, z)-*path* if it passes through the vertex y. A z-path is a path with z as one of its end-vertices. For a given set $X \subseteq V(G)$, a cycle is called an X-*cycle* if it passes through all the vertices of X. If $X = \{x\}$, an X-cycle is simply denoted by an x-cycle. The distance between two vertices u and v, denoted by $d(u, v)$, is the length of a shortest (u, v)-path. The number of vertices in a maximum independent set of G is denoted by $\alpha(G)$. For a positive integer $k \leq \alpha(G)$ we denote by $\sigma_k(G)$ the minimum value of the degree sum of any k pairwise nonadjacent vertices, and by $\sigma_k^w(G)$ the minimum value of the weighted degree sum of any k pairwise nonadjacent vertices. Instead of $\sigma_1(G)$ and $\sigma_1^w(G)$, we use the notations $\delta(G)$ and $\delta^w(G)$, respectively.

1.1.1 Results on paths in weighted graphs

The following results on the existence of long paths in unweighted graphs are known.

Theorem 1.1.1 (Erdös & Gallai [24])
Let G be a graph on n vertices and m edges. Then G contains a path of length at least $2m/n$.

Theorem 1.1.2 (Dirac [20])
Let G be a graph and d an integer. If $d(v) \geq d$ for every vertex v of G, then G contains (1) *a path of length at least d, and* (2) *if $d \geq 2$, a cycle of length at least $d + 1$.*

Theorem 1.1.3 (Erdös & Gallai [24])
Let G be a 2-connected graph and d an integer. Let x and z be two distinct vertices of G. If $d(v) \geq d$ for all $v \in V(G) \backslash \{x, z\}$, then G contains an (x, z)-path of length at least d.

Theorems 1.1.1 to 1.1.3 are generalized to weighted graphs by the following theorems, respectively.

Theorem 1.1.4 (Bondy & Fan [13])
Let G be a weighted graph on n vertices. Then G contains a path of weight at least $2w(G)/n$.

Theorem 1.1.5 (Bondy & Fan [12])
Let G be a connected weighted graph on at least two vertices, z a specified vertex of G, and d a real number. Suppose that $d^w(v) \geq d$ for every vertex $v \in V(G) \backslash \{z\}$ and $w(e) > 0$ for every $e \in E(G)$.

(1) *Then G contains a z-path of weight at least d. Moreover, if G contains no z-path of weight more than d, then each block B of G (a) is complete, (b) includes z, and (c) is weighted so that $w(vz) = \alpha$ for all $v \in V(B) \backslash \{z\}$ and $w(uv) = \beta$ for all $u, v \in V(B) \backslash \{z\}$, where $\alpha + \beta(|V(B)| - 2) = d$.*

(2) *If* $d(v) \geq 2$ *for every* $v \in V(G)\backslash\{z\}$, *then* G *contains a cycle of weight more than* d.

Theorem 1.1.6 (Bondy & Fan [12])
Let G *be a 2-connected weighted graph and* d *a real number. Let* x *and* z *be two distinct vertices of* G. *If* $d^w(v) \geq d$ *for all* $v \in V(G)\backslash\{x,z\}$, *then* G *contains an* (x,z)*-path of weight at least* d.

Theorem 1.1.3 was further generalized by Enomoto [22] as follows.

Theorem 1.1.7 (Enomoto [22])
Let G *be a 2-connected graph and* d *an integer. Suppose that* x *and* z *are two distinct vertices of* G *such that* $d(v) \geq d$ *for all* $v \in V(G)\backslash\{x,z\}$. *Then for every given vertex* y, G *contains an* (x,y,z)*-path of length at least* d.

In Chapter 2, we generalize Theorem 1.1.7 to weighted graphs.

Theorem 1.1.8
Let G *be a 2-connected weighted graph and* d *a real number. Let* x *and* z *be two distinct vertices of* G. *Suppose that* $d^w(v) \geq d$ *for all* $v \in V(G)\backslash\{x,z\}$.

(1) *Then for every given vertex* y, G *contains an* (x,y,z)*-path of weight at least* d.
(2) *If* $w(e) > 0$ *for all* $e \in E(G)$ *and for some vertex* $y \in V(G)\backslash\{x,z\}$, G *contains no* (x,y,z)*-path of weight more than* d, *then (a) the component* H_y *of* $G - \{x,z\}$ *that contains* y *is complete; (b)* $V(H_y) \subseteq N(x) \cap N(z)$; *(c)* $w(xv) = \alpha_x, w(zv) = \alpha_z$ *for all* $v \in V(H_y)$ *and* $w(uv) = \beta_y$ *for all* $u,v \in V(H_y)$ *so that* $\alpha_x + \beta_y(|V(H_y)|-1) + \alpha_z = d$. *If* $y \in \{x,z\}$, *then the assertion holds for every component of* $G - \{x,z\}$.

It is clear that Theorem 1.1.8 also generalizes Theorem 1.1.6.

The following result on longest paths in graphs is due to Qiao [54].

Theorem 1.1.9 (QIAO [54])
Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq \frac{c}{2}$ *for every pair of vertices u and v in G with $d(u,v) = 2$. Then either G is hamiltonian or G contains a longest path P such that the degree sum of the two end-vertices of P is at least c.*

In Chapter 5 we give a generalization of the above theorem to weighted graphs as follows.

Theorem 1.1.10
Let G be a 2-connected weighted graph such that $\max\{d^w(u), d^w(v)\} \geq \frac{s}{2}$ *for every pair of vertices u and v in G with $d(u,v) = 2$. Then either G is hamiltonian or G contains a longest path P such that the weighted degree sum of the two end-vertices of P is at least s.*

1.1.2 Results on cycles in weighted graphs

There have been many results on the existence of long cycles in unweighted graphs. The following theorem is well-known.

Theorem 1.1.11 (ERDÖS & GALLAI [24])
Let G be a 2-edge-connected graph on n vertices and m edges. Then G contains a cycle of length at least $\frac{2m}{n-1}$.

For 2-connected graphs, DIRAC [20] proved a stronger result than (2) of Theorem 1.1.2.

Theorem 1.1.12 (DIRAC [20])
let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a Hamilton cycle or a cycle of length at least $2d$.

Theorems 1.1.11 and 1.1.12 are generalized to weighted graphs by the following two theorems, respectively.

Theorem 1.1.13 (Bondy & Fan [13])
Let G be a 2-edge-connected weighted graph on n vertices. Then G contains a cycle of weight at least $\dfrac{2w(G)}{n-1}$.

Theorem 1.1.14 (Bondy & Fan [12])
Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geq d$ for every vertex v in G, then either G contains a cycle of weight at least $2d$ or every heaviest cycle in G is a Hamilton cycle.

Theorem 1.1.12 was further generalized to the following four theorems in different ways.

Theorem 1.1.15 (Pósa [53])
Let G be a 2-connected graph such that $d(u) + d(v) \geq c$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c.

Theorem 1.1.16 (Grötschel [38])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then for every given vertex y, G contains either a Hamilton cycle or a y-cycle of length at least $2d$.

Theorem 1.1.17 (Fan [26])
Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq \dfrac{c}{2}$ for every pair of vertices u and v with $d(u, v) = 2$. Then G contains either a Hamilton cycle or a cycle of length at least c.

Theorem 1.1.18 (Fournier & Fraisse [29])
Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $\dfrac{2m}{k+1}$.

Theorem 1.1.15 was generalized to weighted graphs by Bondy *et al.* in [11].

Theorem 1.1.19 (Bondy *et al.* [11])
Let G be a 2-connected weighted graph such that $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of weight at least s.

In Chapter 5, we provide a further generalization of their result.

Theorem 1.1.20
Let G be a 2-connected weighted graph such that $d(u) + d(v) \geq c$ and $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c and weight at least s.

In Chapter 2 we generalize Theorem 1.1.16 to weighted graphs. The result also extends Theorem 1.1.14.

Theorem 1.1.21
Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geq d$ for every vertex v in G, then for every given vertex y, either G contains a y-cycle of weight at least $2d$ or every heaviest cycle in G is a Hamilton cycle.

As for Theorems 1.1.17 and 1.1.18, we pose the following two problems.

Problem 1.1.22
Let G be a 2-connected weighted graph such that $\max\{d^w(u), d^w(v)\} \geq \frac{s}{2}$ for every pair of vertices u and v with $d(u, v) = 2$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least s?

Problem 1.1.23
Let G be a k-connected weighted graph where $2 \leq k < \alpha(G)$, such that $\sigma^w_{k+1}(G) \geq m$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least $\dfrac{2m}{k+1}$?

In Chapter 3, we provide some counter-examples to Problem 1.1.22, which show that if one wants to generalize Theorem 1.1.17 to weighted graphs, some extra conditions must be added. We prove the following analogue of Theorem 1.1.17 for weighted graphs, which also generalizes Theorem 1.1.17.

Theorem 1.1.24
Let G be a 2-connected weighted graph which satisfies the following conditions:

1. $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq \dfrac{s}{2}$;
2. $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x,y) = 2$;
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least s.

We also show that neither of the last two conditions of Theorem 1.1.24 can be dropped. So, the above theorem is best possible in some sense.

Since all the counter-examples to Problem 1.1.22 we have constructed have connectivity 2, we posed the following research problem in [57].

Problem 1.1.25
If G is a 3-connected weighted graph such that $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq \dfrac{s}{2}$, is it true that G contains either a Hamilton cycle or a cycle of weight at least s?

Enomoto [23] announced that the answer to Problem 1.1.25 is also negative by providing some counter-examples to this problem.

A graph is called a *claw* if it is isomorphic to the complete bipartite graph $K_{1,3}$. The graph $K_{1,3} + e$ (e is an edge) is called a *modified claw.* In [6], Theorem 1.1.17 was generalized as follows.

Theorem 1.1.26 (Bedrossian *et al.* [6])
Let G be a 2-connected graph. If $\max\{d(u), d(v)\} \geq \frac{c}{2}$ for every pair of nonadjacent vertices u and v, which are vertices of an induced claw or an induced modified claw of G, then G contains either a Hamilton cycle or a cycle of length at least c.

This theorem was generalized to weighted graphs by Fujisawa [31].

Theorem 1.1.27 (Fujisawa [31])
Let G be a 2-connected weighted graph which satisfies the following conditions:

1. *For each induced claw and each induced modified claw of G, all its nonadjacent pairs of vertices x and y satisfy $\max\{d^w(x), d^w(y)\} \geq \frac{s}{2}$;*
2. *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least s.

It is not difficult to see that this theorem generalizes Theorem 1.1.24.

If the answer to the question of Problem 1.1.23 is positive, then the result would be best possible and it would generalize Theorem 1.1.18 and Theorem 1.1.19.

It seems very difficult to settle this problem, even for the case $k = 2$. In Chapter 4, we prove that the answer to the case $k = 2$ of Problem 1.1.23 is positive if we add some extra conditions. These extra conditions were motivated by Theorem 1.1.24. Our result is an analogue and also a generalization of Theorem 1.1.18 to weighted graphs in the case $k = 2$.

Theorem 1.1.28
Let G be a 2-connected weighted graph which satisfies the following conditions:

1. *The weighted degree sum of every three independent vertices is at least m;*
2. *$w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$;*
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $\frac{2m}{3}$.

There are many other results on the existence of long cycles in graphs. One may wonder whether these results have weighted generalizations. Here we list some of them.

Theorem 1.1.29 (Locke [49])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then for every two vertices x and z, G contains either a Hamilton cycle or an $\{x, z\}$-cycle of length at least $2d$.

Theorem 1.1.30 (Egawa, Glas & Locke [21])
Let G be a k-connected graph where $k \geq 2$, and d an integer, and let X be a set of k vertices of G. If $d(v) \geq d$ for every vertex v in G, then G contains either a Hamilton cycle or an X-cycle of length at least $2d$.

Theorem 1.1.31 (Enomoto [22])
Let G be a 2-connected graph such that $\sigma_2(G) \geq c$. Then for every vertex y, G contains either a Hamilton cycle or a y-cycle of length at least c.

Theorem 1.1.32 (Zhu [58])
Let G be a 2-connected graph. Then G contains either a Hamilton cycle or a cycle of length at least $2\sigma_2(G) - 2\delta(G)$.

Theorems 1.1.29 to 1.1.32 suggest the following problems, respectively.

Problem 1.1.33
Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geq d$ for every vertex v in G, is it true that for every two vertices x and z, G contains either a Hamilton cycle or an $\{x, z\}$-cycle of weight at least $2d$?

Problem 1.1.34
Let G be a k-connected weighted graph where $k \geq 2$, and d a real number, and let X be a set of k vertices of G. If $d^w(v) \geq d$ for every vertex v in G, is it true that G contains either a Hamilton cycle or an X-cycle of weight at least $2d$?

Problem 1.1.35
Let G be a 2-connected weighted graph such that $\sigma_2^w(G) \geq s$. Is it true that for every vertex y, G contains either a Hamilton cycle or a y-cycle of weight at least s?

Problem 1.1.36
Let G be a 2-connected weighted graph. Is it true that G contains either a Hamilton cycle or a cycle of weight at least $2\sigma_2^w(G) - 2\delta^w(G)$?

1.2 Paths and cycles in colored graphs

In the second part of this thesis, we are concerned paths and cycles in colored graphs.

Let $G = (V, E)$ be a graph. By an *edge-coloring* of G we will mean a function $C : E \to \mathbb{N}$. If G is assigned such a coloring, then we say that G is a *colored graph*, denote the colored graph by (G, C), and call $C(e)$ the *color* of the edge $e \in E$. All edges with the same color form a *color class* of the graph. We note that C is not necessarily a proper edge-coloring, i.e., two adjacent edges may have the same color. For a subgraph H of G, we let $C(H) = \cup_{e \in E(H)}\{C(e)\}$ and $c(H) = |C(H)|$. For a vertex v of G, the *color neighborhood* $CN(v)$ of v is defined as the set $\{C(e) : e$ is incident with $v\}$ and the *color degree* $d^c(v) = |CN(v)|$. A path (cycle) is called *monochromatic* if all the edges of it have the same color; and it is called *heterochromatic* if all the edges of it have different colors.

If we regard an uncolored graph as a colored graph in which all edges have different colors, then the number of colors of a subgraph is simply the number of its edges, and the color degree of a vertex is the degree of it.

We are not aware of any papers dealing with the existence of monochromatic or heterochromatic paths and cycles in general colored graphs. All existing results we could find deal with colored complete graphs. In [36] GIRAUD studied the existence of monochromatic triangles and heterochromatic triangles in colored complete graph. A problem on the conditions for a colored complete graph to contain heterochromatic Hamilton cycles was mentioned in [25] by ERDÖS, NEŠETŘIL & RÖDL. This problem was studied by HAHN & THOMASSEN [40], RÖDL & WINKLER (see [30]), FRIEZE & REED [30], and ALBERT, FRIEZE & REED [1]. Most of the results in these papers are proved by using probabilistic methods.

In Chapters 6 and 7, we deal with paths and cycles in general colored graphs.

The first problem we consider in this part is under what conditions a general colored graph contains a monochromatic path or a monochromatic cycle. It is clear that every colored graph contains at least one monochromatic path. But not every colored graph contains monochromatic cycles.

The *arboricity* $a(G)$ of a graph G is defined as the minimum number of edge-disjoint forests into which G can be decomposed. Clearly, it is also the minimum number of colors necessary to color the edges of G so that no cycle of G is monochromatic. The first result is immediate from the above observation.

Proposition 1.2.1
Let G be a colored graph. If $c(G) < a(G)$, then G contains at least one monochromatic cycle.

The arboricity $a(G)$ of a graph G can be determined in polynomial time [48, 52]. So there are polynomial-time algorithms to determine whether a colored graph contains monochromatic cycles.

The following result on the existence of monochromatic paths and cycles with prescribed lengths is obvious.

Proposition 1.2.2
Let G be a colored graph with color classes $E_1, E_2, \ldots, E_c$. Then G has a monochromatic path (cycle) of length at least l if and only if for some i with $1 \leq i \leq c$, the induced subgraph $G[E_i]$ has a path (cycle) of length at least l.

It is not difficult to see that the problem of deciding whether there is a monochromatic path (cycle) of length at least l in a colored graph is **NP**-complete.

There are many results on the existence of long paths and cycles in (uncolored) graphs. The following two results are well-known, and Theorem 1.2.3 was already mentioned in Section 1.1.1.

Theorem 1.2.3 (Erdös & Gallai [24])
Let G be a graph of order n and size m. Then G contains a path of length at least $\frac{2m}{n}$.

Theorem 1.2.4 (Erdös & Gallai [24])
Let G be a graph of order n and size m such that $m \geq n$. Then G contains a cycle of length at least $\dfrac{2m}{n-1}$.

These two theorems can be easily generalized to colored graphs by the following two results, respectively.

Proposition 1.2.5
Let G be a colored graph of order n and size m. Then G contains a monochromatic path of length at least $\dfrac{2m}{c(G)n}$.

Proposition 1.2.6
Let G be a colored graph of order n and size m such that $m \geq c(G)n$. Then G contains a monochromatic cycle of length at least $\dfrac{2m}{c(G)(n-1)}$.

Although these propositions are easy to prove, they cannot be improved, even in the case $c(G) \neq 1$, as shown by the examples presented in Chapter 6.

The problem of deciding whether there is a heterochromatic path (cycle) of length at least l in a colored graph is also **NP**-complete. We give the following two results on the existence of long heterochromatic paths and cycles in colored graphs.

Proposition 1.2.7
Let G be a colored graph of order n. Then G contains a heterochromatic path of length at least $\dfrac{2c(G)}{n}$.

Proposition 1.2.8
Let G be a colored graph of order n such that $c(G) \geq n$. Then G contains a heterochromatic cycle of length at least $\dfrac{2c(G)}{n-1}$.

Propositions 1.2.7 and 1.2.8 generalize Theorem 1.2.3 and 1.2.4, respectively.

Furthermore, we obtained the following two results on the existence of long heterochromatic paths.

Proposition 1.2.9
Let G be a colored graph and k an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then for every vertex z of G there exists a heterochromatic z-path of length at least $\lceil \frac{k+1}{2} \rceil$.

Proposition 1.2.10
Let G be a colored graph and s an integer. Suppose that $|CN(u) \cup CN(v)| \geq s > 1$ for every pair of distinct vertices u and v of G. Then G contains a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$.

We also give some sufficient conditions for the existence of heterochromatic triangles or quadrilaterals.

Proposition 1.2.11
Let G be a colored graph of order $n \geq 4$, such that $|CN(u) \cup CN(v)| \geq n-1$ for every pair of distinct vertices u and v of G. Then G contains at least one heterochromatic triangle or one heterochromatic quadrilateral.

Although the proofs of the results in Propositions 1.2.9 to 1.2.11 are easy, it can be shown that these results are best possible in the sense that there exist some graphs showing that they cannot be improved. However, we think that perhaps much stronger results are possible to obtain if one excludes some small counter-examples or simple classes of counter-examples. The proof techniques we applied here do not seem to be strong enough for obtaining such improvements. Maybe an approach using probabilistic proof techniques could yield such improvements, but we did not consider them in the context of this thesis.

In Chapter 7 we consider the complexity aspect of finding a path between two given vertices, with as few different colors as possible in a colored graph.

Problem 1.2.12

INSTANCE: *Graph* $G = (V, E)$ *with a coloring* $C : E \to \mathbb{N}$ *and two given vertices* s_0 *and* t_0*, positive integer* $K \leq c(G)$.

QUESTION: *Is there a path* P *from* s_0 *to* t_0 *such that* $c(P) \leq K$*?*

In this thesis, by using the fact that the **3-SATISFIABILITY** problem is **NP**-complete, we show that Problem 1.2.12 is **NP**-complete too.

Theorem 1.2.13

Problem 1.2.12 is ***NP****-complete.*

Corollary 1.2.14

Finding a path with as few colors as possible in a colored graph is ***NP****-hard.*

We propose two exact algorithms for finding a path with the fewest colors. The complexity of the two exact algorithms is analyzed.

It is also of interest to consider approximation algorithms for finding the paths with the fewest colors between two given vertices. If we use a shortest path between two vertices as an approximation solution for a path with the fewest colors, the approximation ratio is $c(G)$. We design another approximate algorithm which is similar to Dijkstra's Algorithm for finding the shortest paths.

Algorithm 1.2.15

Step 1. *Set* $C(s_0) = \emptyset$, $C(S_0) = 0$, $c(v) = \infty$ *for* $v \neq s_0$, $S_0 = \{s_0\}$ *and* $i = 0$.

Step 2. *For each* $v \in V \backslash S_i$*, replace* $C(v)$ *by* $C(u_i) \cup \{C(u_i v)\}$ *if* $c(v) > |C(u_i) \cup \{C(u_i v)\}|$ *and set* $c(v) = |C(v)|$*. Compute* $\min_{v \in V \backslash S_i} \{c(v)\}$ *and let* u_{i+1} *denote a vertex for which this minimum is attained. Set* $S_{i+1} = S_i \cup \{u_{i+1}\}$*.*

Step 3. *If* $u_{i+1} = t_0$*, stop. Otherwise, replace* i *by* $i + 1$ *and go to Step 2.*

We show that the approximation factor of this algorithm can become arbitrarily large.

It is easy to see that the problem of finding paths and cycles with as many colors as possible in a colored graph is also **NP**-hard.

We pose the following conjecture which would be best possible.

Conjecture 1.2.16
Let G *be a colored graph and* d *a positive integer. If* $d^c(v) \geq d$ *for every vertex* v *of* G*, then* G *contains* (1) *a path with at least* $d - 1$ *colors, and* (2) *a cycle with at least* d *colors.*

Our attempt to prove Conjecture 1.2.16 in a similar way as in the proof of Theorem 1.1.5 on weighted graphs failed.

An example shows that imposing a higher connectivity on the graphs in Conjecture 1.2.16 can not guarantee cycles with more colors, contrary to analogous results for long cycles in graphs with large minimum degree.

These facts and our experiences indicate that the problems on paths and cycles in colored graphs are even harder than similar problems on paths and cycles in weighted graphs.

1.3 The scattering number of a graph

The third part of this thesis is devoted to a vulnerability parameter of graphs, the scattering number of graphs.

Let $G = (V, E)$ be a graph. By $\kappa(G)$ and $\lambda(G)$ we denote the connectivity and the edge-connectivity of G, respectively. For a proper subset $X \subset V$ we denote by $G - X$ the induced subgraph $G[V \setminus X]$. We say that X is a *vertex cut* of G if $G - X$ has more components than G does. The *scattering number* $s(G)$ of a noncomplete connected graph G is defined by

$$s(G) = \max\{\omega(G - X) - |X| : X \subset V(G), \omega(G - X) > 1\},$$

where $\omega(G - X)$ stands for the number of components of $G - X$.

The notion of scattering number was introduced by Nash-Williams [10] but we found the first appearance in literature in a paper of Jung [45]. Jamrozik *et al.* [44] studied small maximal nonhamiltonian graphs by using this parameter. In [42], Hendry used the scattering number to study extremal nonhamiltonian graphs. It turned out that the concept of scattering number is more convenient than the closely related concept of toughness [42] for describing maximal and extremal nonhamiltonian graphs. In 1989, Ouyang *et al.* [51] for the first time proposed to use this parameter to measure the vulnerability of networks. They obtained some basic results on the scattering number, including a recursive algorithm for computing the scattering number of trees, and an analysis of the scattering number of the so-called Harary graphs [41]. Recently, several other papers on the hamiltonicity and computing the scattering number of some special classes of graphs appeared [34, 35, 43].

In Chapter 8, we give some results on the relationships between the scattering number and some other parameters of a graph. The following result characterizes the relationship between the scattering number and the connectivity of graphs.

Theorem 1.3.1
Let G be a noncomplete connected graph of order n. Then $2 - \kappa(G) \leq s(G) \leq n - 2\delta(G)$.

The inequalities in Theorem 1.3.1 are best possible and the following corollaries are immediate.

Corollary 1.3.2
Let G be a noncomplete connected graph of order n. Then
(1) $2-\kappa(G) \le s(G) \le n-2\kappa(G)$;
(2) $2-\lambda(G) \le s(G) \le n-2\lambda(G)$;
(3) $2-\delta(G) \le s(G) \le n-2\delta(G)$.

Corollary 1.3.3
Let G be a noncomplete connected graph of order n. Then

$$4-n \le s(G) \le n-2.$$

Corollary 1.3.4
Let G be a noncomplete connected graph of order n such that $2-\frac{n}{2} \le s(G) \le 0$. Then the length of a longest cycle is at least $4-2s(G)$.

Corollary 1.3.5
Let G be a noncomplete connected graph of order n such that $s(G) \le 2-\frac{n}{2}$. Then G is hamiltonian.

The toughness $t(G)$ of a graph G was defined by Chvátal in [17]. For a noncomplete connected graph G, the *toughness* of G is defined as

$$t(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\}.$$

In Chapter 8, we give the following two theorems on the relationship between the scattering number and toughness of graphs.

Theorem 1.3.6
Let G be a noncomplete connected graph such that $\kappa(G)=k$, $s(G)=s$ and $t(G)=t$. Then $t \ge \frac{k}{k+s}$.

Theorem 1.3.7
Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $t(G) = t$. Then $t \leq \frac{n-s}{n+s}$.

The results in Theorems 1.3.6 and 1.3.7 are best possible.

A graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. It was proved in [50] that the toughness of a claw-free graph G is $\frac{\kappa(G)}{2}$, where $\kappa(G)$ is the connectivity of G. For the scattering number of graphs, the following result is obvious.

Corollary 1.3.8
Let G be a claw-free graph of order n such that $\kappa(G) = k$. Then $2 - k \leq s(G) \leq \frac{n(2-k)}{2+k}$.

After the submission of the draft version of this thesis, H. J. Broersma and the author proved that for a connected claw-free graph G, $s(G) = 2 - \kappa(G)$.

The concept of integrity was introduced as a vulnerability parameter of graphs by Barefoot *et al.* in [3, 4]. For a noncomplete connected graph G, its *integrity* is defined as

$$I(G) = \min\{|X| + m(G-X) : X \subset V(G), \omega(G-X) > 1\},$$

where $m(G - X)$ is the order of a largest component of $G - X$.

In Chapter 8, we give the following two theorems on the relationship between the scattering number and integrity of graphs.

Theorem 1.3.9
Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $I(G) = I$. Then $I \geq 2\sqrt{n+s} - (s+1)$.

Theorem 1.3.10
Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $I(G) = I$. Then $I \leq n - s$.

The results in Theorems 1.3.9 and 1.3.10 are best possible.

The concept of tenacity was introduced in [18] by Cozzen, Moazzami & Stueckle as a vulnerability parameter of graphs. The *tenacity* of a noncomplete connected graph G is defined as

$$T(G) = \min\left\{\frac{|X| + m(G-X)}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\}.$$

In Chapter 8 we consider the relationship between the scattering number and the tenacity.

Theorem 1.3.11
Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $T(G) = T$. Then

$$T \geq \begin{cases} 1 - \dfrac{(s+1)^2}{4(n+s)} & s \geq 3, \\ 1 + \dfrac{2(1-s)}{n+s} & -1 \leq s \leq 2, \\ 1 + \dfrac{n-2-s}{4} & \text{otherwise.} \end{cases}$$

Theorem 1.3.12
Let G be a noncomplete connected graph of order n such that $\kappa(G) = k$, $s(G) = s$ and $T(G) = T$. Then $T \leq \dfrac{n+1}{s+k} - 1$.

The results in Theorems 1.3.11 and 1.3.12 are best possible.

Since it was proved by Kratsch *et al.* [47] that the problem of computing the scattering number of graphs is **NP**-hard, it is a very interesting problem to determine the scattering number of some special classes of graphs.

The *Cartesian product* of two graphs G_1 and G_2, denoted by $G_1 \times G_2$, is defined as follows: $V(G_1 \times G_2) = V(G_1) \times V(G_2)$, two vertices (u_1, u_2) and (v_1, v_2) are adjacent in $G_1 \times G_2$ if and only if $u_1 = v_1$ and $u_2v_2 \in E(G_2)$ or $u_1v_1 \in E(G_1)$ and $u_2 = v_2$. The Cartesian product of n graphs $G_1, G_2, \ldots, G_n$, denoted by $G_1 \times G_2 \times \cdots \times G_n$, is defined inductively as the Cartesian product of $G_1 \times G_2 \times \cdots \times G_{n-1}$ and G_n.

The Cartesian product $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ is called a *grid*. The Cartesian product of k copies of P_2, denoted by Q_k, is called a *k-hypercube*, or simply a hypercube. It is clear that hypercubes are grids. In Chapter 9, the scattering number of grids and of hypercubes is determined.

Theorem 1.3.13
Suppose that $n_1, n_2, \ldots, n_k$ are k integers not less than 2. Then
(1) $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = 1$, *when all n_i are odd;*
(2) $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = 0$, *otherwise.*

Corollary 1.3.14
The scattering number $s(Q_k)$ $(k \geq 2)$ of the hypercube Q_k is 0.

It is clear that the hypercube Q_k can also be defined as the Cartesian product of k copies of K_2. This suggests the following natural problem.

Problem 1.3.15
Determine the scattering number of $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_k}$.

We give a solution for Problem 1.3.15 for the case $k = 2$.

Theorem 1.3.16
$s(K_m \times K_n) = 4 - (m + n)$.

A graph $G = (V, E)$ is called a *split graph* (cf. [28, 37]) if its vertex set can be partitioned into an independent set I and a clique C.

Woeginger [56] proved that the toughness of split graphs can be computed in polynomial time. In Chapter 9 we prove that the scattering number of split graphs can be computed in polynomial time too.

Theorem 1.3.17
The scattering number of split graphs can be computed in polynomial time.

Part I

Paths and cycles in weighted graphs

Chapter 2

Heavy paths and cycles in weighted graphs

In this chapter, it is proved that for a 2-connected weighted graph, if every vertex has weighted degree at least d, then for every given vertex y, either y is contained in a cycle with weight at least $2d$ or every heaviest cycle is a Hamilton cycle. This result is a common generalization of Grötschel's theorem and Bondy-Fan's theorem assuring the existence of a cycle with weight at least $2d$ or a Hamilton cycle under the same conditions. Also, as a tool for proving this result, we first prove a result concerning heavy paths joining two specific vertices and passing through one given vertex.

2.1 Heavy paths in weighted graphs

We recall the following two theorems on the existence of long paths. It is easy to see that Theorem 2.1.2 generalizes Theorem 2.1.1.

Theorem 2.1.1 (Erdös & Gallai [24])
Let G be a 2-connected graph and d an integer. Let x and z be two distinct vertices of G. If $d(v) \geq d$ for all $v \in V(G) \backslash \{x, z\}$, then G contains an

(x, z)-path of length at least d.

Theorem 2.1.2 (Enomoto [22])
Let G be a 2-connected graph and d an integer. Let x and z be two distinct vertices of G. Suppose that $d(v) \geq d$ for all $v \in V(G) \backslash \{x, z\}$.

(1) *Then for every given vertex y of G, G contains an (x, y, z)-path of length at least d;*
(2) *If for some vertex $y \in V(G) \backslash \{x, z\}$, G contains no (x, y, z)-path of length more than d, then the component H_y of $G - x - z$ that contains y is isomorphic to K_{d-1} and $V(H_y) \subseteq N(x) \cap N(z)$. If $y \in \{x, z\}$, then the assertion holds for every component of $G - x - z$.*

Theorem 2.1.1 was generalized to weighted graphs by Bondy & Fan [12] as follows.

Theorem 2.1.3 (Bondy & Fan [12])
Let G be a 2-connected weighted graph and d a real number. Let x and z be two distinct vertices of G. If $d^w(v) \geq d$ for all $v \in V(G) \backslash \{x, z\}$, then G contains an (x, z)-path of weight at least d.

In this section, we combine the proof ideas from the proofs of Theorem 2.1.2 in [22] and of Theorem 2.1.3 in [12] to prove the following analogue of Theorem 2.1.2 for weighted graphs. This result also generalizes Theorem 2.1.3.

Theorem 2.1.4
Let G be a 2-connected weighted graph and d a real number. Let x and z be two distinct vertices of G. Suppose that $d^w(v) \geq d$ for all $v \in V(G) \backslash \{x, z\}$.

(1) *Then for every given vertex y of G, G contains an (x, y, z)-path of weight at least d.;*

(2) *If $w(e) > 0$ for all $e \in E(G)$ and for some vertex $y \in V(G) \setminus \{x, z\}$, G contains no (x, y, z)-path of weight more than d, then (a) the component H_y of $G - x - z$ that contains y is complete; (b) $V(H_y) \subseteq N(x) \cap N(z)$; (c) $w(xv) = \alpha_x, w(zv) = \alpha_z$ for all $v \in V(H_y)$ and $w(uv) = \beta_y$ for all $u, v \in V(H_y)$ so that $\alpha_x + \beta_y(|V(H_y)| - 1) + \alpha_z = d$. If $y \in \{x, z\}$, then the assertion holds for every component of $G - x - z$.*

Proof If $y \in \{x, z\}$, then the result in (1) follows from Theorem 2.1.3; The assertions in (2) can be proved by choosing any component of $G - x - z$ as H_y in the following proof. So we may assume that $y \notin \{x, z\}$.

Let $|V(G)| = n$. We use induction on n. If $n = 3$, let y be the third vertex other than x and z, then the path xyz is an (x, y, z)-path of weight $d^w(y) \geq d$.

Suppose now $n \geq 4$ and the theorem is true for all graphs on k vertices with $3 \leq k \leq n - 1$. Let $G' = G - z$ be the graph obtained by deleting z from G. We consider two cases:

Case 1 G' is 2-connected.

(1) Since G is 2-connected, we can choose $z' \in N(z) \setminus \{x\}$ such that

$$w(zz') = \max\{w(zv) : v \in N(z) \setminus \{x\}\}.$$

Then for all $v \in V(G') \setminus \{x\}$,

$$d^w_{G'}(v) = d^w(v) - w(zv) \geq d - w(zz').$$

By the induction hypothesis, for every given vertex $y \in V(G') \setminus \{x\}$, G' contains an (x, y, z')-path Q of weight at least $d - w(zz')$. Then the path $P = Qz'z$ is an (x, y, z)-path of weight at least d.

(2) If for some vertex $y \in V(G) \setminus \{x, z\}$, G contains no (x, y, z)-path of weight more than d, then the maximum weight of an (x, y, z')-path in G' is exactly $d' = d - w(zz')$. Moreover, by the induction hypothesis, G'

has the described structure. Let H'_y be the component of $G' - x - z'$ that contains y. (If $y = z'$, take any component of $G' - x - z'$ as H'_y). Thus, H'_y is complete, $V(H'_y) \subseteq N_{G'}(x) \cap N_{G'}(z')$ and G' is weighted so that

$$w(xv) = \alpha'_x, w(z'v) = \alpha'_{z'} \text{ for all } \ v \in V(H'_y)$$

and

$$w(uv) = \beta'_y \text{ for all } u, v \in V(H'_y),$$

where

$$\alpha'_x + \beta'_y(|V(H'_y)| - 1) + \alpha'_{z'} = d'.$$

If $v \in V(H'_y)$, then $d^w_{G'}(v) = d'$. Thus

$$w(zv) = d^w(v) - d^w_{G'}(v) \geq d - d' = w(zz').$$

Since $w(zz') > 0$, we have that $zv \in E(G)$. Moreover, by the choice of z', it is clear that $w(zv) = w(zz')$ for all $v \in V(H'_y)$. It follows that any vertex in $V(H'_y) \cup \{z\}$ could have been selected as the vertex z'. This implies that $\alpha'_{z'} = \beta'_y$.

Suppose that there exists another component H^* of $G' - x - z'$. By the induction hypothesis, there must then be an (x, z')-path of weight at least $d - w(zz')$ in $G[V(H^*) \cup \{x, z'\}]$. On the other hand, there is a (z, y, z')-path of weight $w(zz') + \beta'_y|V(H'_y)|$ in $G[V(H'_y) \cup \{z, z'\}]$. Combining these two paths, we get an (x, y, z)-path of weight at least $d + \beta'_y|V(H'_y)| > d$, which contradicts the assumption. Hence $G - x - z = G[V(H'_y) \cup \{z'\}]$ and

$$\begin{aligned} w(xz') &= d^w(z') - w(zz') - \beta'_y|V(H'_y)| \\ &\geq d - w(zz') - \beta'_y|V(H'_y)| = d' - \beta'_y|V(H'_y)| \\ &= \alpha'_x + \beta'_y(|V(H'_y)| - 1) + \alpha'_{z'} - \beta'_y|V(H'_y)| \\ &= \alpha'_x. \end{aligned}$$

Furthermore, by the assumption that G contains no (x, z)-path of weight more than d we know that $w(xz') \leq \alpha'_x$. So $xz' \in E(G)$ and $w(xz') = \alpha'_x$. Now let H_y denote the component of $G - x - z$ that contains y and set $\alpha_z = w(zz')$, $\alpha_x = \alpha'_x$ and $\beta_y = \beta'_y$. Then H_y is complete, $V(H_y) \subseteq N(x) \cap N(z)$ and G is weighted so that

$$w(xv) = \alpha_x, w(zv) = \alpha_z \text{ for all } v \in V(H_y)$$

and

$$w(uv) = \beta_y \text{ for all } u, v \in V(H_y),$$

where

$$\alpha_x + \beta_y(|V(H_y)| - 1) + \alpha_z = d.$$

Case 2 G' is not 2-connected.

(1) Since G is 2-connected, G' must be connected. We shall frequently make use of the following Claim.

Claim Suppose B is an end-block of G' and b is the unique cut-vertex of G' contained in B. Let B' be the subgraph of G induced by $V(B) \bigcup \{z\}$. Then for any given vertex y of B', B' contains a (b, y, z)-path P' of weight at least d.

Proof If $zb \in E(G)$, then B' is 2-connected and for all $v \in V(B') \backslash \{b, z\}$, we have

$$d^w_{B'}(v) = d^w(v) \geq d.$$

By the induction hypothesis, for any given vertex y of B', B' contains a (b, y, z)-path P' of weight at least d.

If $zb \notin E(G)$, add zb to B' and set $w(zb) = 0$. Applying the induction hypothesis to the resulting graph, we know that for any given vertex y

of B', the resulting graph contains a (b, y, z)-path of weight at least d. If $d > 0$, then $P' \neq zb$, since $w(zb) = 0$. If $d = 0$, then we can choose P' in B' such that $P' \neq zb$, since all we need is that $w(P') \geq d$. This shows that we always have a (b, y, z)-path P' in B' of weight at least d. ∎

Case 2.1 y is contained in a block of G' with two or more cut-vertices.

Choose an end-block B in G' with cut-vertex b such that there is an (x, y, b)-path Q in $G' - (B - b)$. Let B' be the subgraph of G induced by $V(B) \cup \{z\}$. By the above Claim, we have that there is a (b, z)-path P' in B' of weight at least d. Combining these two paths Q and P', we get an (x, y, z)-path of weight at least d.

Case 2.2 y is contained in an end-block B of G' with a cut-vertex b and $x \notin V(B)$.

Let B' be the subgraph of G induced by $V(B) \cup \{z\}$. It is easy to see that there exists an (x, b)-path Q in $G' - (B - b)$. By the above Claim we have that there is a (b, y, z)-path P' in B' of weight at least d. Combining these two paths Q and P', we get an (x, y, z)-path of weight at least d.

Case 2.3. y and x are contained in an end-block B_1 of G'.

If x is the unique cut-vertex of B_1, let B'_1 be the subgraph of G induced by $V(B_1) \cup \{z\}$. Then from the above Claim we know that there is an (x, y, z)-path P'_1 in B'_1 of weight at least d. Otherwise, since G' has at least two distinct end-blocks, we can choose an end-block B_2 in G' other than B_1. Let b_2 be the unique cut-vertex of G' contained in B_2 and B'_2 be the subgraph of G induced by $V(B_2) \cup \{z\}$. Then there is a (b_2, z)-path P'_2 in B'_2 of weight at least d by the above Claim, and there is also an (x, y, b_2)-path Q in $G' - (B_2 - b_2)$. Combining these two paths Q and P'_2, we get an (x, y, z)-path of weight at least d.

(2) From the above proof, we need only consider the case in which y

is contained in an end-block B_1 of G' with x as its unique cut-vertex. In this case, the result follows from the induction hypothesis by considering the graph $G[V(B_1) \cup \{z\}]$. ■

2.2 Heavy cycles in weighted graphs

There are many results on the existence of long cycles. The following two theorems are well-known.

Theorem 2.2.1 (Dirac [20])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a cycle of length at least $2d$ or a Hamilton cycle.

Theorem 2.2.2 (Grötschel [38])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then for every given vertex y of G, G contains either a y-cycle of length at least $2d$ or a Hamilton cycle.

It is clear that Theorem 2.2.2 is a generalization of Theorem 2.2.1.

Bondy & Fan generalized Theorem 2.2.1 to weighted graphs as follows:

Theorem 2.2.3 (Bondy & Fan [12])
Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geq d$ for every vertex v in G, then either G contains a cycle of weight at least $2d$ or every optimal cycle is a Hamilton cycle.

The aim of this section is to give a generalization of Theorem 2.2.2 to weighted graphs. This is obtained by using Theorem 2.1.4 in a similar way as in [12] Theorem 2.1.3 is used to obtain Theorem 2.2.3.

Theorem 2.2.4
Let G be a 2-connected weighted graph and d a real number. If $d^w(v) \geq d$ for every vertex v in G, then for every given vertex y of G, either G contains a y-cycle of weight at least $2d$ or every heaviest cycle in G is a Hamilton cycle.

This theorem also generalizes Theorem 2.2.3.

Before proving the above theorem, we need the following result.

Theorem 2.2.5
Let C be a heaviest cycle in a weighted graph G. Suppose that there is an (x, y, z)-path P in $G - C$ such that $|N_C(x)| \geq 1, |N_C(z)| \geq 1$ and $|N_C(x) \cup N_C(z)| \geq 2$. Define

$$X = N_C(x) \backslash N_C(z), Z = N_C(z) \backslash N_C(x) \text{ and } Y = N_C(x) \cap N_C(z).$$

If $|Y| = 1$ and either $X = \emptyset$ or $Z = \emptyset$, then there exists a y-cycle C' in G such that

$$w(C') \geq \frac{w(C)}{2} + \min\{d_C^w(x), d_C^w(z)\} + w(P).$$

Otherwise, there exist $l \geq 4$ y-cycles $C_1, C_2, \ldots, C_l$ in G such that

$$\sum_{i=1}^{l} w(C_i) \geq (l-2)w(C) + 2d_Y^w(x) + 2d_Y^w(z) + 4d_X^w(x) + 4d_Z^w(z) + lw(P).$$

Proof If $|Y| = 1$ and either $X = \emptyset$ or $Z = \emptyset$, we have two cases. In the case $|Y| = 1$ and $X = \emptyset$, we can assume that $Y = \{a_1\}$. Without loss of generality, we suppose that the segment $C[a_2, a_1]$ is of weight at least $w(C)/2$. So the cycle $C' = xPza_2C[a_2, a_1]a_1x$ is a y-cycle of weight

$$\begin{aligned} w(C') &\geq \frac{w(C)}{2} + w(xa_1) + w(za_2) + w(P) \\ &\geq \frac{w(C)}{2} + \min\{d_C^w(x), d_C^w(z)\} + w(P). \end{aligned}$$

The case $|Y| = 1$ and $Z = \emptyset$ can be discussed by the same argument.

Otherwise, let $A = X \cup Y \cup Z$ and suppose that $A = \{a_1, a_2, \ldots, a_k\}$, where a_i are in order around C. For each pair of vertices (a_i, a_{i+1}), we shall construct two new cycles from C by replacing the segment $C[a_i, a_{i+1}]$ with two (a_i, a_{i+1})- paths. These two paths are defined according to four cases:

(1) $a_i, a_{i+1} \in Y$. The two paths are

$$a_i x P z a_{i+1} \text{ and } a_i z P x a_{i+1}.$$

(2) $a_i \in Y$ and $a_{i+1} \in X$ or Z. The two paths are

$$a_i z P x a_{i+1} \text{ and } a_i x a_{i+1}, \text{ or } a_i x P z a_{i+1} \text{ and } a_i z a_{i+1}.$$

If $a_{i+1} \in Y$ and $a_i \in X$ or Z, the paths are defined in the same way.

(3) $a_i \in X$ and $a_{i+1} \in Z$ or $a_i \in Z$ and $a_{i+1} \in X$. The two paths are two copies of

$$a_i x P z a_{i+1} \text{ or } a_i z P x a_{i+1}.$$

(4) $a_i, a_{i+1} \in X$ or $a_i, a_{i+1} \in Z$. The two paths are two copies of

$$a_i x a_{i+1} \text{ or } a_i z a_{i+1}.$$

In each case, we have defined two paths to replace the segment $C[a_i, a_{i+1}]$ and hence formed two cycles . Since there are k pairs of vertices (a_i, a_{i+1}) $(i = 1, \ldots, k)$, we obtain $2k$ cycles. In these cycles, every edge of C is traversed $2k - 2$ times; every edge from x or z to Y is traversed twice, every edge from x to X is traversed four times and, similarly, every edge from z to Z is traversed four times. Now suppose that the path P is traversed l times (we determine l later). Then the weight sum of these $2k$ cycles is

$$2(k-1)w(C) + 2d_Y^w(x) + 2d_Y^w(z) + 4d_X^w(x) + 4d_Z^w(z) + lw(P).$$

Without loss of generality, we can denote the l cycles which pass through the path P (and also pass through the vertex y) by $C_1, C_2, \ldots, C_l$. Since C is a heaviest cycle, those $2k-l$ cycles other than $C_1, C_2, \ldots, C_l$ have weight at most $w(C)$. Hence we get the following inequality.

$$\sum_{i=1}^{l} w(C_i) \geq (l-2)w(C) + 2d_Y^w(x) + 2d_Y^w(z) + 4d_X^w(x) + 4d_Z^w(z) + lw(P).$$

Now we determine l. If $|Y| \geq 2$, then it is not difficult to see that $l \geq 2|Y|$; if $|Y| = 1, X \neq \emptyset$, and $Z \neq \emptyset$, then $l \geq 4$; if $|Y| = 0$, then noting that $|N_C(x)| \geq 1$ and $|N_C(z)| \geq 1$, we have that $X \neq \emptyset$ and $Z \neq \emptyset$, and $l \geq 4$. Therefore for all the cases we have that $l \geq 4$. ■

Proof of Theorem 2.2.4 Suppose that there exists a heaviest cycle C in G which is not a Hamilton cycle. From Theorem 2.2.3 we have that $w(C) \geq 2d$. If y is contained in the cycle C, then we are done. Otherwise, let H be the component of $G - C$ which contains y. We consider two cases:

Case 1 H is nonseparable.

Case 1.1 $V(H) = \{y\}$.

Suppose that $N_C(y) = \{a_1, a_2, \ldots, a_k\}$ $(k \geq 2)$, where a_i are in order around C. For each pair of vertices (a_i, a_{i+1}), we shall construct a y-cycle C_i from C by replacing the segments $C[a_i, a_{i+1}]$ with the path $a_i y a_{i+1}$. Since there are k pairs of vertices (a_i, a_{i+1}) $(i = 1, 2, \ldots, k)$, we obtain k cycles, and,

$$\sum_{i=1}^{k} w(C_i) = (k-1)w(C) + 2d_C^w(y)$$
$$\geq 2(k-1)d + 2d$$
$$= 2kd.$$

Then among these k cycles there must be a y-cycle C' with weight at least $2d$.

Case 1.2 $|V(H)| \geq 2$.

Choose distinct vertices x and z in H such that

(1) $|N_C(x)| \geq 1, |N_C(z)| \geq 1$, and

(2) $d_C^w(x) \geq d_C^w(z) \geq d_C^w(v)$ for all $v \in V(H)\backslash\{x, z\}$.

Case 1.2.1 $|N_C(x) \cup N_C(z)| \geq 2$.

By the choice of x and z, we have

$$d_H^w(v) = d^w(v) - d_C^w(v) \geq \max\{0, d - d_C^w(z)\} \text{ for all } v \in V(H)\backslash\{x\}.$$

If $|V(H)| = 2$, it is easy to find an (x, y, z)-path P in H of weight at least $\max\{0, d - d_C^w(z)\}$. Otherwise, applying Theorem 2.1.4 to H, we can choose an (x, y, z)-path P in H such that

$$w(P) \geq \max\{0, d - d_C^w(z)\}.$$

Now denote $N_C(x)\backslash N_C(z)$, $N_C(x) \cap N_C(z)$ and $N_C(z)\backslash N_C(x)$ by X, Y and Z, respectively. If $|Y| = 1$ and $X = \emptyset$ or $Z = \emptyset$, then by Theorem 2.2.5 we know that there is a y-cycle C' in G such that

$$w(C') \geq \frac{w(C)}{2} + \min\{d_C^w(x), d_C^w(z)\} + w(P) \geq 2d.$$

Otherwise, from Theorem 2.2.5 we know that G contains $l \geq 4$ y-cycles $C_1, C_2, \ldots, C_l$ such that

$$\begin{aligned}\sum_{i=1}^{l} w(C_i) &\geq (l-2)w(C) + 2d_Y^w(x) + 2d_Y^w(z) + 4d_X^w(x) + 4d_Z^w(z) + lw(P) \\ &= (l-2)w(C) + 2d_C^w(x) + 2d_C^w(z) + 2d_X^w(x) + 2d_Z^w(z) + lw(P) \\ &\geq (l-2)w(C) + 4d_C^w(z) + l\max\{0, d - d_C^w(z)\} \\ &\geq 2ld.\end{aligned}$$

Then among these l y-cycles in G there must be one with weight at least $2d$.

Case 1.2.2 $N_C(x) = N_C(z) = \{a\}$.

Since G is 2-connected, there exists a vertex $b \in V(C)\backslash\{a\}$ which is adjacent to some vertex $u \in V(H)\backslash\{x, z\}$. By the choice of x and z, we have

$$d_H^w(v) = d^w(v) - d_C^w(v) \geq d - d_C^w(x) \text{ for all } v \in V(H).$$

Applying Theorem 2.1.4 to H, we have an (x, y, u)-path Q in H of weight

$$w(Q) \geq d - d_C^w(x) = d - w(xa).$$

Then the path $axQub$ is of weight at least d. It is easy to see that we can form a y-cycle of weight at least $2d$.

Case 2 H is separable.

Case 2.1 y is contained in a block of H with two or more cut-vertices.

Let B_1 and B_2 be two distinct end-blocks of H, and let b_i be the unique cut-vertex of H contained in $B_i (i = 1, 2)$. For $i = 1, 2$, we choose $x_i \in V(B_i)\backslash\{b_i\}$ such that

(1) $|N_C(x_i)| \geq 1$, and

(2) $d^w_C(x_i) \geq d^w_C(v)$ for all $v \in V(B_i)\backslash\{b_i\}$.

It follows that

$$d^w_{B_i}(v) = d^w(v) - d^w_C(v) \geq \max\{0, d - d^w_C(x_i)\}$$

for all $v \in V(B_i)\backslash\{b_i\}, (i = 1, 2)$. Applying Theorem 2.1.4 to B_i we obtain an (x_i, b_i)-path P_i in B_i of weight

$$w(P_i) \geq \max\{0, d - d^w_C(x_i)\}.$$

If $|N_C(x_1) \cup N_C(x_2)| \geq 2$, then let P be an (x_1, y, x_2) -path in H of maximum weight. Then

$$w(P) \geq w(P_1) + w(P_2) \geq \max\{0, d - \min\{d^w_C(x_1), d^w_C(x_2)\}\}.$$

Denote $N_C(x_1)\backslash N_C(x_2)$, $N_C(x_2)\backslash N_C(x_1)$ and $N_C(x_1) \cap N_C(x_2)$ by X_1, X_2 and Y, respectively. If $|Y| = 1$ and $X_1 = \emptyset$ or $X_2 = \emptyset$, then by Theorem 2.2.5 we know that there is a y-cycle C' in G such that

$$w(C') \geq \frac{w(C)}{2} + \min\{d^w_C(x_1), d^w_C(x_2)\} + w(P) \geq 2d.$$

Otherwise, from Theorem 2.2.5 we know that G contains $l (l \geq 4)$ y-cycles $C_1, C_2, \ldots, C_l$ such that

$$\begin{aligned}\sum_{i=1}^{l} w(C_i) &\geq (l-2)w(C) + 2d^w_Y(x_1) + 2d^w_Y(x_2) \\ &\quad + 4d^w_{X_1}(x_1) + 4d^w_{X_2}(x_2) + lw(P) \\ &\geq 2(l-2)d + 4\min\{d^w_C(x_1), d^w_C(x_2)\} \\ &\quad + l\max\{0, d - \min\{d^w_C(x_1), d^w_C(x_2)\}\} \\ &\geq 2ld.\end{aligned}$$

So, among these l y-cycles there must be one with weight at least $2d$.

If $N_C(x_1) = N_C(x_2) = \{a\}$, then there exists a vertex $b \in V(C)\backslash\{a\}$ which is adjacent to some vertex $u \in V(H)\backslash\{x_1, x_2\}$. As $(V(B_1)\backslash\{b_1\}) \cap (V(B-2)\backslash\{b_2\}) = \emptyset$, u can not belong to both $V(B_1)\backslash\{b_1\}$ and $V(B_2) \backslash\{b_2\}$. Without loss of generality, we suppose that $u \notin V(B_2)\backslash\{b_2\}$. Considering the path joining u and the vertices of a (b_1, y, b_2)-path in H, we obtain a (u, y, b_1)-path or a (u, y, b_2)-path in H. Without loss of generality, we can suppose the existence of a (u, y, b_2)-path Q. Since y is not in $V(B_2)\backslash\{b_2\}$, Q is a path in $H - (B_2 - b_2)$. So the path $P = buQb_2P_2x_2a$ is of weight

$$w(P) \geq w(P_2) + w(x_2a) \geq d.$$

Therefore, it is easy to construct a y-cycle of weight at least $2d$.

Case 2.2 y is contained in an end-block B_1 of H.

Choose another end-block B_2 of H and let b_i be the unique cut-vertex of H contained in B_i $(i = 1, 2)$. For $i = 1, 2$, choose $x_i \in V(B_i)\backslash\{b_i\}$ such that

(1) $|N_C(x_i)| \geq 1$, and

(2) $d^w_C(x_i) \geq d^w_C(v)$ for all $v \in V(B_i)\backslash\{b_i\}$.

Applying Theorem 2.1.4 to B_1 and B_2, we obtain an (x_1, y, b_1)-path P_1 in B_1 of weight at least $\max\{0, d - d^w_C(x_1)\}$, and an (x_2, b_2)-path P_2 in B_2 of weight at least $\max\{0, d - d^w_C(x_2)\}$. It is also easy to know that there is a (b_1, b_2)-path Q in $H - (B_1 - b_1) - (B_2 - b_2)$. So the path $P = P_1QP_2$ is an (x_1, y, x_2)-path with weight

$$w(P) \geq w(P_1) + w(P_2) \geq \max\{0, d - \min\{d^w_C(x_1), d^w_C(x_2)\}\}.$$

If $|N_C(x_1) \cup N_C(x_2)| \geq 2$, similar to the argument in Case 2.1, we can get a y-cycle of weight at least $2d$.

If $N_C(x_1) = N_C(x_2) = \{a\}$, there exists a vertex $b \in V(C)\backslash\{a\}$ which is adjacent to some vertex $u \in V(H)\backslash\{x_1, x_2\}$.

If $u \in V(B_1)$ and $u = b_1$, the path $bb_1P_1x_1a$ is of weight at least d. If $u \neq b_1$, we can choose a (u, y, b_2)-path Q, then the path $P = buQb_2P_2x_2a$ is of weight at least d. So in both cases we can form a y-cycle of weight at least $2d$.

If $u \notin V(B_1)$, we can choose a (b_1, u)-path Q in $H - (B_1 - b_1)$, and therefore the path $P = ax_1P_1b_1Qub$ is of weight at least d. It is easy to form a y-cycle with weight at least $2d$. ■

Chapter 3

A Fan type condition for heavy cycles in weighted graphs

In this chapter, we prove the following result: Suppose G is a 2-connected weighted graph which satisfies the following conditions: 1. $\max\{d^w(x), d^w(y) | d(x, y) = 2\} \geq s/2$; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$; 3. In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least s. This generalizes a theorem of Fan on the existence of long cycles in unweighted graphs to weighted graphs. Although Conditions 2 and 3 are very strong, we show that we cannot omit them in the above result.

3.1 Introduction

The following two theorems on the existence of long cycles in graphs are well-known.

Theorem 3.1.1 (Dirac [20])
let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a Hamilton cycle or a cycle of length at least $2d$.

Theorem 3.1.2 (Pósa [53])
Let G be a 2-connected graph such that $d(u) + d(v) \geq c$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c.

Theorem 3.1.2 generalizes Theorem 3.1.1, and it was further generalized by Fan [26] as follows.

Theorem 3.1.3 (Fan [26])
Let G be a 2-connected graph such that $\max\{d(x), d(y) | d(x, y) = 2\} \geq c/2$. Then G contains either a Hamilton cycle or a cycle of length at least c.

As we pointed out in Chapter 2, Theorem 3.1.1 was generalized by Bondy & Fan in [12]. Later, Bondy *et al.* [11] gave a generalization of Theorem 3.1.2 to weighted graphs.

Theorem 3.1.4 (Bondy *et al.* [11])
Let G be a 2-connected weighted graph such that $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of weight at least s.

A natural question is whether Theorem 3.1.3 also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 3.1.5
Let G be a 2-connected weighted graph such that $\max\{d^w(x), d^w(y) | d(x, y) = 2\} \geq s/2$. Is it true that G contains either a Hamilton cycle or a cycle of weight at least s?

Unfortunately, the answer to the question in Problem 3.1.5 is negative. This can be shown by the 2-connected graph in Figure 3.1.1. In this graph, if we assign weight 1 to the edge v_2v_3, weight 7 to v_4v_6 and v_7v_9, and weight 5 to all the remaining edges, then it is easy to check that $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq 22$, whereas the graph contains no Hamilton cycle and the heaviest cycle of the graph is of weight 40.

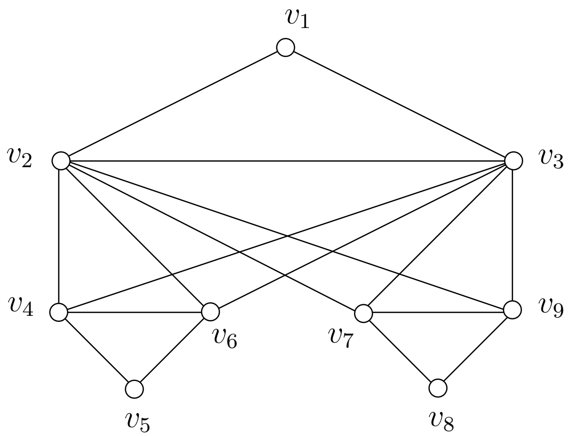

Figure 3.1.1

Let $G = (V, E)$ be a weighted graph with weight function $w : E \to \mathbb{R}$. Suppose that there exists a function $w' : V \to \mathbb{R}$ such that, for every edge uv of G,

$$w(uv) = \frac{w'(u) + w'(v)}{2}.$$

Then we say that the edge weight function w is *induced* (by the vertex weight function w'). If w' can be chosen in such a way that $w'(v) > 0$ for all $v \in V$, then we call w *positive-induced.* If we regard an unweighted graph as a weighted graph with weight 1 on each edge, then it is positive-induced. The answer to the question of Problem 3.1.5 is negative even when the edge weight function of the graph is supposed to be positive-induced. This can also be shown by the graph in Figure 3.1.1. If we assign weight 4 to the edges v_4v_5, v_5v_6, v_7v_8 and v_8v_9, and weight 5 to all the other edges, then the resulting weighted graph is still a counter-example to Problem 3.1.5, and the weight function is positive-induced. We leave the details to the reader.

So, if one wants to generalize Theorem 3.1.3 to weighted graphs, some extra conditions must be added. In this paper, we prove the following analogue of Theorem 3.1.3 for weighted graphs, which also generalizes Theorem 3.1.3.

Theorem 3.1.6
Let G be a 2-connected weighted graph which satisfies the following conditions:
1. $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq s/2$;
2. $w(xz) = w(yz)$ *for every vertex* $z \in N(x) \cap N(y)$ *with* $d(x,y) = 2$;
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least s.

We postpone the proof of Theorem 3.1.6 to the next section.

It should be noted that neither of the last two conditions of Theorem 3.1.6 can be dropped. This can be shown by the graph in Figure 3.1.1. If we assign weights to edges as we did in the first counter-example to Problem 3.1.5, then the graph satisfies Conditions 1 and 2 of Theorem 3.1.6, but not Condition 3. On the other hand, if we assign weight 2 to the edges v_4v_5 and v_8v_9, weight 2.5 to v_5v_6 and v_7v_8, and weight 5 to all the other edges, then it is easy to check that $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq 17$, whereas the graph contains no Hamilton cycle and the heaviest cycle of the graph is of weight 30. So the new graph satisfies Conditions 1 and 3 of Theorem 3.1.6, but not Condition 2. This graph is also a counter-example to Problem 3.1.5.

As remarked before Conditions 2 and 3 are very strong. In fact one easily checks that Condition 2 implies that all weights are the same in the case of connected triangle-free graphs. Similarly Conditions 2 and 3 imply, e.g., that the edges incident with a vertex v with a noncomplete neighborhood all have the same weight; if v has a complete neighborhood there is more freedom for the distribution of weights among its incident edges.

3.2 Proof of Theorem 3.1.6

Let G be a 2-connected weighted graph satisfying the conditions of Theorem 3.1.6. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least s.

As Bondy *et al.* did in [12], we choose a path $P = v_1v_2\cdots v_p$ in G such that
(a) P is as long as possible;
(b) $w(P)$ is as large as possible, subject to (a);
(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).
From the choice of P, we can immediately see that $N(v_1)\cup N(v_p) \subseteq V(P)$.

Claim 1 *There exists no cycle of length p.*

Proof Suppose there exists a cycle C of length p. Since G contains no Hamilton cycle and G is connected, we can find a vertex $u \in V(G)\backslash V(C)$ and a path Q from u to a vertex $v \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). ■

Claim 2 $v_1v_p \notin E(G)$.

Proof If $v_1v_p \in E(G)$, then we can find a cycle $C = v_1v_2\cdots v_pv_1$ of length p, contradicting Claim 1. ■

Claim 3 *If* $v_i \in N(v_1)$*, then* $v_{i-1} \notin N(v_p)$*.*

Proof Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1 v_i v_{i+1} \cdots v_p v_{i-1} v_{i-2} \cdots v_1$ of length p, again contradicting Claim 1. ■

Now we consider two cases:

Case 1 $d^w(v_1) + d^w(v_p) < c$.

Without loss of generality, we can assume that $d^w(v_1) < c/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Claim 4 $v_1 v_i \in E(G)$ *for all* i *with* $3 \leq i \leq k$*.*

Proof Suppose that $v_1 v_{k-1} \notin E(G)$, hence $d(v_1, v_{k-1}) = 2$. From Condition 2 of the theorem, we know that $w(v_1 v_k) = w(v_{k-1} v_k)$. Then $v_{k-1} v_{k-2} \cdots v_1 v_k \cdots v_p$ is another longest path with the same weight as P. By the maximality of $d^w(v_1) + d^w(v_p)$, we have $d^w(v_{k-1}) \leq d^w(v_1) < c/2$. It follows from Condition 1 of the theorem that $d(v_1, v_{k-1}) \neq 2$, a contradiction. Thus, we conclude that $v_1 v_{k-1} \in E(G)$. If $k = 3$, we are done; otherwise, repeating the above arguments, we can obtain that $v_1 v_i \in E(G)$ for all i with $3 \leq i \leq k$. ■

Case 1.1 $w(v_1 v_{i-1}) = w(v_1 v_i) = w(v_{i-1} v_i) = w^*$ for all i with $3 \leq i \leq k$.

Claim 5 $d^w(v_i) \leq d^w(v_1)$ *for all* i *with* $2 \leq i \leq k-1$*.*

Proof Suppose that $d^w(v_j) > d^w(v_1)$ for some j with $2 \leq j \leq k-1$. Since $w(v_1 v_{j+1}) = w(v_j v_{j+1})$ and $v_1 v_{j+1} \in E(G)$ by Claim 4, $v_j v_{j-1} \cdots v_1 v_{j+1} v_{j+2} \cdots v_p$ is another longest path with the same weight as P. Then $d^w(v_j) + d^w(v_p) > d^w(v_1) + d^w(v_p)$, which contradicts the maximality of

$d^w(v_1) + d^w(v_p)$ in (c). ■

Claim 6 $d^w(v_{k+1}) > d^w(v_1)$.

Proof Note that $v_1v_{k+1} \notin E(G)$ by the choice of v_k, and the path $v_1v_kv_{k+1}$ is of length 2, so $d(v_1, v_{k+1}) = 2$. Using Condition 1 of the theorem we know that $\max\{d^w(v_1), d^w(v_{k+1})\} \geq c/2$. Since $d^w(v_1) < c/2$, we must have $d^w(v_{k+1}) \geq c/2 > d^w(v_1)$. ■

For every i with $2 \leq i \leq k-1$, v_i cannot be adjacent to any vertex outside P. Otherwise, there will be a path of length p, contradicting the choice of P in (a). Since G is 2-connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$. Choose $v_jv_s \in E(G)$ such that $j < k < s$ and s is as large as possible.

Case 1.1.1 $s \geq k+2$ (see Figure 3.2.1).

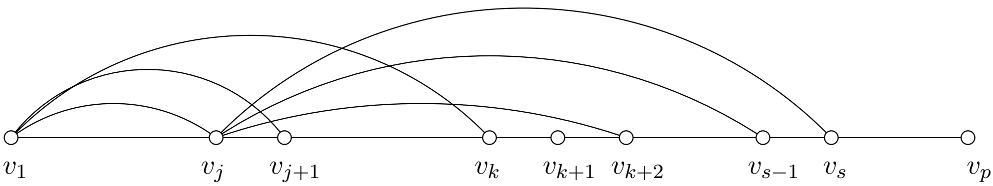

Figure 3.2.1

First note that $d(v_1, v_s) = 2$ by the choice of v_k. This implies that $w(v_jv_s) = w(v_1v_j) = w^*$. We can prove that $v_jv_{s-1} \in E(G)$. Otherwise, from Condition 2 of the theorem we have $w(v_{s-1}v_s) = w(v_jv_s) = w^*$. Then the path $v_{s-1}v_{s-2}\cdots v_{j+1}v_1\cdots v_jv_s\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we know that $d^w(v_{s-1}) \leq d^w(v_1) < c/2$. On the other hand, from Condition 1 of the theorem and $d(v_j, v_{s-1}) = 2$ we then get $d^w(v_j) \geq c/2 > d^w(v_1)$, contradicting Claim 5. So, we must have $v_jv_{s-1} \in E(G)$. If $s-1 > j+1$, we have another longest path $v_{s-2}v_{s-3}\cdots\ v_{j+1}v_1\cdots v_jv_{s-1}\cdots v_p$. Repeating the process above, we obtain that $v_jv_{s-2} \in E(G)$. Consequently, it is not difficult to prove that $v_jv_i \in E(G)$ and $w(v_jv_i) = w(v_1v_j) = w^*$ for all i with $k+1 \leq i \leq s$. Using Condition 3 we also have that $w(v_{i-1}v_i) = w^*$ for all i with $k+1 \leq i \leq s$.

In particular, $v_jv_{k+2} \in E(G)$ since $s \geq k+2$. This means that there is another longest path $v_{k+1}v_k \cdots v_{j+1}v_1 \cdots v_jv_{k+2} \cdots v_s \cdots v_p$ with the same weight as P. It follows from the choice of P in (c) that $d^w(v_{k+1}) \leq d^w(v_1)$, contradicting Claim 6.

Case 1.1.2 $s = k+1$ (see Figure 3.2.2).

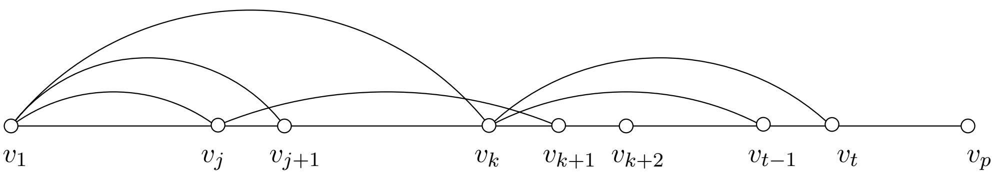

Figure 3.2.2

First, note that $v_kv_{k-1} \cdots v_{j+1}v_1 \cdots v_jv_{k+1} \cdots v_p$ is another longest path with the same weight as P, and so by the choice of P in (c) we have $d^w(v_k) \leq d^w(v_1) < c/2$.

By Claim 3 we may assume that $k+1 < p$. From the 2-connectedness of G and the choice of v_s, there must be an edge $v_kv_t \in E(G)$ such that $t \geq k+2$. From Condition 2 of the theorem, we have $w(v_kv_t) = w(v_1v_k) = w^*$. We can prove that $v_kv_{t-1} \in E(G)$. Otherwise, $d(v_k, v_{t-1}) = 2$. This implies that $w(v_{t-1}v_t) = w(v_kv_t) = w(v_1v_k) = w^*$. So, the path $v_{t-1}v_{t-2} \cdots v_{k+1}v_j \cdots v_1 \; v_{j+1} \cdots v_kv_t \cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), $d(v_{t-1}) \leq d^w(v_1) < c/2$. On the other hand, we have $\max\{d^w(v_k), d^w(v_{t-1})\} \geq c/2$ by the fact that $d(v_k, v_{t-1}) = 2$, a contradiction. With the same argument as before, we can prove that $v_kv_i \in E(G)$ and $w(v_{i-1}v_i) = w(v_kv_i) = w(v_1v_k) = w^*$

for all i with $k+1 \leq i \leq t$.

In particular, $v_k v_{k+2} \in E(G)$ since $t \geq k+2$. Hence, there is another longest path $v_{k+1}v_j \cdots v_1 v_{j+1} \cdots v_k v_{k+2} \cdots v_t \cdots v_p$ with the same weight as P. This implies that $d^w(v_{k+1}) \leq d^w(v_1) < c/2$, contradicting Claim 6.

This completes the proof of Case 1.1.

Case 1.2 There is some vertex v_i with $3 \leq i \leq k$ such that $w(v_1 v_{i-1})$, $w(v_1 v_i)$ and $w(v_{i-1}v_i)$ are all different.

In this case, choose vertex v_j such that $w(v_1 v_{j-1})$,$w(v_1 v_j)$ and $w(v_{j-1}v_j)$ are all different, and j is as large as possible. Denote the weight of $v_1 v_j$, $v_{j-1}v_j$ and $v_1 v_{j-1}$ by w_1, w_2 and w_3, respectively. It follows from Condition 3 that $w(v_{j-1}v_j) = w_2 \neq w_1 = w(v_j v_{j+1})$, and from Condition 2 of the theorem that $v_{j-1}v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1}v_{j+1}$ is different from the weight w_1 of the edge $v_{j+1}v_{j+2}$ since there is a triangle $v_1 v_{j-1} v_{j+1} v_1$ and $w(v_1 v_{j-1}) = w_3 \neq w_1 = w(v_1 v_{j+1})$. With the same argument, we can prove that $v_{j-1}v_i \in E(G)$ for all i with $j \leq i \leq k+1$. By the choice of v_k, we have that $w(v_{j-1}v_{k+1}) = w_3$.

If $v_k v_{k+2} \in E(G)$, then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_k v_{k+2}) = w(v_1 v_k) = w_1$. From $w(v_k v_{k+1}) = w(v_k v_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1}v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ since the two edges $v_{j-1}v_{k+1}$ and $v_{k+1}v_{k+2}$ have different weights. Again, by the fact that $d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1}v_{k+2}) = w(v_1 v_{j-1}) = w_3$. This leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ in which $w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3$ and $w(v_{k+1}v_{k+2}) = w_1$, contradicting Condition 3 of the theorem.

If $v_k v_{k+2} \notin E(G)$, then $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1}v_{k+2}) = w(v_k v_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ and $w(v_{j-1}v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ which is impossible by Condition 3 of the theorem.

Case 2 $d^w(v_1) + d^w(v_p) \geq c$.

Similar to the proof of Theorem 4 of [11], we will prove that G contains a cycle of weight at least s.

Claim 7 *If* $v_i \in N(v_1)$, *then* $w(v_{i-1}v_i) \geq w(v_1v_i)$. *If* $v_j \in N(v_p)$, *then* $w(v_jv_{j+1}) \geq w(v_jv_p)$.

Proof If $v_i \in N(v_1)$, the path $P' = v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ has the same length as P. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1v_i)$. The second assertion can be proved similarly. ■

Since G is 2-connected, by Lemma 1 of [9], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) P_k has end-vertices x_k and y_k, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;

(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on P.

By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let C be the cycle the edge set of which is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1)\backslash\{y_1\}$ and $v_j \in N(v_p)\backslash\{x_m\}$ that $v_{i-1}v_i$, $v_jv_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_jv_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1y_1$ and $P_m = x_mv_p$. Using Claim 7, this shows that

$$\begin{aligned} w(C) \geq & \sum_{v_i \in N(v_1)\backslash\{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p)\backslash\{x_m\}} w(v_jv_{j+1}) \\ & + w(v_1y_1) + w(x_my_p) \\ \geq & \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p) \\ = & d^w(v_1) + d^w(v_p) \geq c. \end{aligned}$$

■

3.3 Remarks

All the counter-examples to Problem 3.1.5 we presented in Section 3.1 have connectivity 2. We posed the following research problem in [57].

Problem 3.3.1
If G is a 3-connected weighted graph such that $\max\{d^w(x), d^w(y) | d(x,y) = 2\} \geq s/2$*, is it true that G contains either a Hamilton cycle or a cycle of weight at least s?*

The answer to this problem is also negative according to [23]. There are some examples presented by Enomoto show that: For every $k > 2$ and $\varepsilon > 0$, there is a graph G with suitable weight function satisfying

(1) G is k-connected;

(2) G is nonhamiltonian; and

(3) the weight of the heaviest cycles is less than $\varepsilon\mu$, where $\mu = \min\{\max \{d^w(u), d^w(v) | d(u,v) = 2\}\}$.

A graph is called a *claw* if it is isomorphic to the complete bipartite graph $K_{1,3}$. The graph $K_{1,3} + e$ (e is an edge) is called a *modified claw.* In [6], Theorem 3.1.3 was generalized as follows.

Theorem 3.3.2 (BEDROSSIAN *et al.* [6])
Let G be a 2-connected graph. If $\max\{d(u), d(v)\} \geq \frac{c}{2}$ *for every pair of nonadjacent vertices u and v, which are vertices of an induced claw or an induced modified claw of G, then G contains either a Hamilton cycle or a cycle of length at least c.*

This theorem was generalized to weighted graphs by FUJISAWA [31].

Theorem 3.3.3 (FUJISAWA [31])
Let G be a 2-connected weighted graph which satisfies the following conditions:

1. *For each induced claw and each induced modified claw of G, all its nonadjacent pairs of vertices x and y satisfy* $\max\{d^w(x), d^w(y)\} \geq \frac{s}{2}$;
2. *For each induced claw and each induced modified claw of G, all of its edges have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least s.

Chapter 4

A σ_3 type condition for heavy cycles in weighted graphs

In this chapter, we prove the following result: Suppose G is a 2-connected weighted graph which satisfies the following conditions: 1. The weighted degree sum of every three independent vertices is at least m; 2. $w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$; 3. In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight. Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$. This generalizes a theorem of Fournier & Fraisse on the existence of long cycles in k-connected unweighted graphs for the case $k = 2$. Our proof of the above result also suggests a new proof to the theorem of Fournier & Fraisse for the case $k = 2$.

4.1 Introduction

There have been many results on the existence of long cycles in graphs. The following three theorems are well-known.

Theorem 4.1.1 (Dirac [20])
Let G be a 2-connected graph such that $\delta(G) \geq r$. Then G contains either a Hamilton cycle or a cycle of length at least $2r$.

Theorem 4.1.2 (Pósa [53])
Let G be a 2-connected graph such that $\sigma_2(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of length at least s.

Theorem 4.1.3 (Fournier & Fraisse [29])
Let G be a k-connected graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}(G) \geq m$. Then G contains either a Hamilton cycle or a cycle of length at least $2m/(k+1)$.

It is easy to see that Theorem 4.1.2 generalizes Theorem 4.1.1, and Theorem 4.1.3 in turn generalizes Theorem 4.1.2.

Theorems 4.1.1 and 4.1.2 were generalized to weighted graphs by the following two theorems, respectively.

Theorem 4.1.4 (Bondy & Fan [12])
Let G be 2-connected weighted graph such that $\delta^w(G) \geq r$. Then either G contains a cycle of weight at least $2r$ or every heaviest cycle is a Hamilton cycle.

Theorem 4.1.5 (Bondy *et al.* [11])
Let G be a 2-connected weighted graph such that $\sigma_2^w(G) \geq s$. Then G contains either a Hamilton cycle or a cycle of weight at least s.

A natural question is whether Theorem 4.1.3 also admits an analogous generalization for weighted graphs. This leads to the following problem.

Problem 4.1.6
Let G be a k-connected weighted graph where $2 \leq k < \alpha(G)$, such that $\sigma_{k+1}^w(G) \geq m$. Is it true that G contains either a Hamilton cycle or a

cycle of weight at least $2m/(k+1)$?

If the answer to the question of this problem is positive, then the result would be best possible and it would generalize Theorem 4.1.3 and Theorem 4.1.5.

It seems very difficult to settle this problem, even for the case $k = 2$. In the next section, we prove that the answer to the case $k = 2$ of Problem 4.1.6 is positive if we add some extra conditions. These extra conditions were motivated by the result we obtained in Chapter 3. Our result is an analogue and also a generalization of Theorem 4.1.3 to weighted graphs for the case $k = 2$.

Theorem 4.1.7
Let G be a 2-connected weighted graph which satisfies the following conditions:

1. *The weighted degree sum of every three independent vertices is at least m;*
2. *$w(xz) = w(yz)$ for every vertex $z \in N(x) \cap N(y)$ with $d(x, y) = 2$;*
3. *In every triangle T of G, either all edges of T have different weights or all edges of T have the same weight.*

Then G contains either a Hamilton cycle or a cycle of weight at least $2m/3$.

4.2 Proof of Theorem 4.1.7

Let G be a 2-connected weighted graph satisfying the conditions of Theorem 4.1.7. Suppose that G does not contain a Hamilton cycle. Then it suffices to prove that G contains a cycle of weight at least $2m/3$.

The set-up of the proof is the same as that in the proof of Theorem 3.1.6. We repeat the first part for convenience.

Choose a path $P = v_1 v_2 \cdots v_p$ in G such that

(a) P is as long as possible;
(b) $w(P)$ is as large as possible, subject to (a);
(c) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a) and (b).
From the choice of P, we can immediately see, that $N(v_1) \cup N(v_p) \subseteq V(P)$.

Claim 1 *There exists no cycle of length p.*

Proof Suppose there exists a cycle C of length p. Since G contains no Hamilton cycle and G is connected, we can find a vertex $u \in V(G)\backslash V(C)$ and a path Q from u to a vertex $v \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). ■

Claim 2 $v_1 v_p \notin E(G)$.

Proof If $v_1 v_p \in E(G)$, then we can find a cycle $C = v_1 v_2 \cdots v_p v_1$ of length p, contradicting Claim 1. ■

Claim 3 *If $v_i \in N(v_1)$, then $v_{i-1} \notin N(v_p)$.*

Proof Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$. Then we can form a cycle $C = v_1 v_i v_{i+1} \cdots v_p v_{i-1} v_{i-2} \cdots v_1$ of length p, again contradicting Claim 1. ■

Claim 4 *If $v_i \in N(v_1)$, then $w(v_{i-1}v_i) \geq w(v_1 v_i)$. If $v_j \in N(v_p)$, then $w(v_j v_{j+1}) \geq w(v_j v_p)$.*

Proof If $v_i \in N(v_1)$, the path $P' = v_{i-1} v_{i-2} \cdots v_1 v_i \cdots v_p$ has the same length as P. So, because of (b), we must have $w(P) \geq w(P')$, hence $w(v_{i-1}v_i) \geq w(v_1 v_i)$. The second assertion can be proved similarly. ■

Since G is 2-connected, by Lemma 1 of [9], there is a sequence of internally disjoint paths $P_1, P_2, \ldots, P_m$ such that

(1) P_k has end-vertices x_k and y_k, and $V(P_k) \cap V(P) = \{x_k, y_k\}$ for $k = 1, 2, \ldots, m$;

(2) $v_1 = x_1 < x_2 < y_1 \leq x_3 < y_2 \leq x_4 < \cdots < y_{m-2} \leq x_m < y_{m-1} < y_m = v_p$, where the inequalities denote the order of the vertices on P.

By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$ for $k = 1, 2, \ldots, m$, and let C be the cycle whose edge set is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 3 we have for all $v_i \in N(v_1) \backslash \{y_1\}$ and $v_j \in N(v_p) \backslash \{x_m\}$ that $v_{i-1}v_i$, $v_jv_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_jv_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1y_1$ and $P_m = x_mv_p$. Using Claim 4, this shows that

$$\begin{aligned} w(C) \geq & \sum_{v_i \in N(v_1) \backslash \{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p) \backslash \{x_m\}} w(v_jv_{j+1}) \\ & + w(v_1y_1) + w(x_my_p) \\ \geq & \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p) \\ = & \, d^w(v_1) + d^w(v_p). \end{aligned}$$

Without loss of generality, we can assume that $d^w(v_1) \leq w(C)/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Now we consider two cases.

Case 1 There exists a vertex $v_i \in V(P)$ such that $v_1v_i \in E(G)$ but $v_1v_{i-1} \notin E(G)$ for some i with $3 \leq i \leq k$.

By Claim 3 we know that $v_{i-1}v_p \notin E(G)$, so the three vertices v_1, v_{i-1} and v_p are independent. From Condition 2 of the theorem and the fact $d(v_1, v_{i-1}) = 2$ we know that $v_{i-1}v_{i-2}\cdots v_1v_i\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we have $d^w(v_{i-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1) + d^w(v_p) \leq w(C)$, we have $d^w(v_1) + d^w(v_{i-1}) + d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2 $v_1v_i \in E(G)$ for all i with $3 \leq i \leq k$.

Case 2.1 $w(v_1v_{i-1}) = w(v_1v_i) = w(v_{i-1}v_i) = w^*$ for all i with $3 \leq i \leq k$.

For every i with $2 \leq i \leq k-1$, v_i can not be adjacent to any vertex outside P. Otherwise, there will be a path of length p, contradicting the choice of P in (a). Since G is 2-connected, there must be an edge $v_jv_s \in E(G)$ with $j < k < s$. Choose $v_jv_s \in E(G)$ such that $j < k < s$ and s is as large as possible. From Claim 3 we have $s < p$.

Case 2.1.1 $s \geq k+2$.

By the choice of v_k we know that $v_1v_{s-1} \notin E(G)$. If $v_{s-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_{s-1}v_p\cdots v_sv_j\cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{s-1} and v_p are independent. By the choice of v_k, we have $d(v_1, v_s) = 2$. If $v_jv_{s-1} \in E(G)$, then $d(v_1, v_{s-1}) = 2$. Then it follows from Condition 2 of the theorem that $w(v_jv_{s-1}) = w(v_jv_s) = w(v_1v_j) = w^*$, and from Condition 3 of the theorem we get $w(v_{s-1}v_s) = w^*$. If $v_jv_{s-1} \notin E(G)$, then

$d(v_jv_{s-1}) = 2$. This implies that $w(v_{s-1}v_s) = w(v_jv_s) = w^*$. Thus, in both cases the path $v_{s-1}v_{s-2}\cdots v_{j+1}v_1\cdots v_jv_s\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), we know that $d^w(v_{s-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1)+d^w(v_p) \leq w(C)$, we have $d^w(v_1)+d^w(v_{s-1})+d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

Case 2.1.2 $s = k+1$.

By Claim 3 we may assume that $k+1 < p$. From the 2-connectedness of G and the choice of v_s, there must be an edge $v_kv_t \in E(G)$ such that $t \geq k+2$. By the choice of v_k, we know that $v_1v_{t-1} \notin E(G)$. On the other hand, if $v_{t-1}v_p \in E(G)$, then we can form a cycle $v_1v_{j+1}\cdots v_kv_t\cdots v_pv_{t-1}\cdots v_{k+1}v_j\cdots v_1$ of length p, contradicting Claim 1. So, the three vertices v_1, v_{t-1} and v_p are independent.

If $v_kv_{t-1} \in E(G)$, then from Condition 2 of the theorem we have $w(v_kv_{t-1}) = w(v_kv_t) = w(v_1v_k) = w^*$, and from Condition 3 of the theorem, the edge $v_{t-1}v_t$ has weight w^*. If $v_kv_{t-1} \notin E(G)$, then from Condition 2 of the theorem we also get $w(v_{t-1}v_t) = w^*$. Thus, in both cases the path $v_{t-1}v_{t-2}\cdots v_{k+1}v_j\cdots v_1\ v_{j+1}\cdots v_kv_t\cdots v_p$ is another longest path with the same weight as P. By the choice of P in (c), $d(v_{t-1}) \leq d^w(v_1) \leq w(C)/2$. With $d^w(v_1)+d^w(v_p) \leq w(C)$, we have $d^w(v_1)+d^w(v_{t-1})+d^w(v_p) \leq 3w(C)/2$. It follows from Condition 1 of the theorem that the weight of the cycle C is at least $2m/3$.

This completes the proof of Case 2.1.

Case 2.2 There is some vertex v_i with $3 \leq i \leq k$ such that $w(v_1v_{i-1})$, $w(v_1v_i)$ and $w(v_{i-1}v_i)$ are all different.

In this case, choose vertex v_j such that $w(v_1v_{j-1})$, $w(v_1v_j)$ and $w(v_{j-1}v_j)$ are all different, and j is as large as possible. Denote the weight of v_1v_j, $v_{j-1}v_j$ and v_1v_{j-1} by w_1, w_2 and w_3, respectively. It follows from

Condition 3 (or from Condition 2 if $j = k$) that $w(v_{j-1}v_j) = w_2 \neq w_1 = w(v_jv_{j+1})$, and from Condition 2 of the theorem that $v_{j-1}v_{j+1} \in E(G)$. If $j < k$, then the weight of the edge $v_{j-1}v_{j+1}$ is different from the weight w_1 of the edge $v_{j+1}v_{j+2}$ since there is a triangle $v_1v_{j-1}v_{j+1}v_1$ and $w(v_1v_{j-1}) = w_3 \neq w_1 = w(v_1v_{j+1})$. With the same argument, we can prove that $v_{j-1}v_i \in E(G)$ for all i with $j \leq i \leq k+1$. By the choice of v_k, we have that $w(v_{j-1}v_{k+1}) = w_3$.

Suppose first that $v_kv_{k+2} \in E(G)$. Then $d(v_1, v_{k+2}) = 2$. This shows that $w(v_kv_{k+2}) = w(v_1v_k) = w_1$. From $w(v_kv_{k+1}) = w(v_kv_{k+2}) = w_1$ and Condition 3 of the theorem we know that $w(v_{k+1}v_{k+2}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ since the two edges $v_{j-1}v_{k+1}$ and $v_{k+1}v_{k+2}$ have different weights. Again, by the fact $d(v_1, v_{k+2}) = 2$, we obtain that $w(v_{j-1}v_{k+2}) = w(v_1v_{j-1}) = w_3$. This leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ in which $w(v_{j-1}v_{k+1}) = w(v_{j-1}v_{k+2}) = w_3$ and $w(v_{k+1}v_{k+2}) = w_1$, contradicting Condition 3 of the theorem.

Hence $v_kv_{k+2} \notin E(G)$. Thus we have $d(v_k, v_{k+2}) = 2$. This implies that $w(v_{k+1}v_{k+2}) = w(v_kv_{k+1}) = w_1$. Therefore, there must be an edge $v_{j-1}v_{k+2} \in E(G)$ and $w(v_{j-1}v_{k+2}) = w_3$. This also leads to a triangle $v_{j-1}v_{k+1}v_{k+2}v_{j-1}$ which is impossible by Condition 3 of the theorem. ■

4.3 Remarks

The proof of Theorem 4.1.3 in [29] is very complicated. It is clear that our proof of Theorem 4.1.7 provides a simpler proof for Theorem 4.1.3 in the case $k = 2$. We do not know whether the extra conditions in Theorem 4.1.7 are necessary. The results in [57] indicate that for some generalizations of long cycle results to weighted graphs one cannot avoid such additional conditions. We do not believe that there is an analogous generalization of Theorem 4.1.3 for the case $k \neq 2$.

Chapter 5

Long paths and cycles in weighted graphs

In this chapter we prove: If G is a 2-connected weighted graph such that $\max\{d^w(u), d^w(v)\} \geq s/2$ for every pair of vertices u and v in G with $d(u,v) = 2$, then either G is hamiltonian or G contains a longest path P such that the weighted degree sum of the two end-vertices of P is at least s. This generalizes a theorem of Qiao on long paths in unweighted graphs. At the same time, it is proved that a 2-connected weighted graph G with $d(u) + d(v) \geq c$ and $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v contains either a Hamilton cycle or a cycle of length at least c and of weight at least s. This is a generalization of an earlier result of Bondy *et al.* on the existence of heavy cycles in weighted graphs.

5.1 Long paths in weighted graphs

The following result on the existence of long paths in graphs is due to Qiao [54].

Theorem 5.1.1 (Qiao [54])
Let G be a 2-connected graph such that $\max\{d(u), d(v)\} \geq c/2$ *for every pair of vertices u and v in G with $d(u,v) = 2$. Then either G is hamiltonian or G contains a longest path P such that the degree sum of the two end-vertices of P is at least c.*

In this section we present a generalization of the above result to weighted graphs as follows.

Theorem 5.1.2
Let G be a 2-connected weighted graph such that $\max\{d^w(u), d^w(v)\} \geq s/2$ *for every pair of vertices u and v in G with $d(u,v) = 2$. Then either G is hamiltonian or G contains a longest path P such that the weighted degree sum of the two end-vertices of P is at least s.*

Proof Let G be a 2-connected weighted graph such that $\max\{d^w(u), d^w(v)\} \geq s/2$ for every pair of vertices u and v with $d(u,v) = 2$. Suppose that G is not hamiltonian, then it suffices to prove that G contains a longest path such that the weighted degree sum of the two end-vertices is at least s.

Choose a path $P = v_1 v_2 \cdots v_p$ in G such that

(a) P is as long as possible;

(b) $d^w(v_1) + d^w(v_p)$ is as large as possible, subject to (a).

From the choice of P, we can immediately see that $N(v_1) \cup N(v_p) \subseteq V(P)$.

Claim 1 *There exists no cycle with length p.*

Proof Suppose that there exists a cycle C with length p. Since G is connected but not hamiltonian, we can find a vertex $u \in V(G) \backslash V(C)$ and a path Q from u to v_j for some $v_j \in V(C)$, such that Q is internally disjoint from C. The subgraph $C \cup Q$ of G contains a path longer than P, contradicting the choice of P in (a). ■

Claim 2 $v_1v_p \notin E(G)$.

Proof If $v_1v_p \in E(G)$, then we can form a cycle $C = v_1v_2\cdots v_pv_1$ with length p, contradicting Claim 1. ■

Claim 3 *If* $v_i \in N(v_1)$*, then* $v_{i-1} \notin N(v_p)$.

Proof Suppose $v_i \in N(v_1)$ and $v_{i-1} \in N(v_p)$, Then we can form a cycle $C = v_1v_iv_{i+1}\cdots v_pv_{i-1}v_{i-2}\cdots v_1$ of length p, again contradicting Claim 1. ■

Now let us prove that $d^w(v_1) + d^w(v_p) \geq s$.

Suppose that $d^w(v_1) + d^w(v_p) < s$. Without loss of generality, we can assume that $d^w(v_1) < s/2$.

Since G is 2-connected, v_1 is adjacent to at least one vertex on P other than v_2. Choose $v_k \in N(v_1) \cap V(P)$ such that k is as large as possible. By Claim 2 it is clear that $3 \leq k \leq p-1$.

Claim 4 $v_1v_i \in E(G)$ *for all* i *such that* $3 \leq i \leq k$.

Proof Since $v_1v_k \in E(G)$, there is another longest path $v_{k-1}v_{k-2}\cdots v_1v_k \cdots v_p$. By the maximality of $d^w(v_1) + d^w(v_p)$, we have that $d^w(v_{k-1}) \leq d^w(v_1) < s/2$. Then $\max\{d^w(v_{k-1}), d^w(v_1)\} < s/2$. It follows from the condition of the theorem that $d(v_1, v_{k-1}) \neq 2$. However, $v_1v_kv_{k-1}$ is a path of length 2. Thus we must have that $v_1v_{k-1} \in E(G)$. If k=3, we are done; otherwise, repeating the process above, we obtain that $v_1v_i \in E(G)$ for all i such that $3 \leq i \leq k$. ■

Claim 5 $d^w(v_i) \leq d^w(v_1)$ *for all* i *such that* $2 \leq i \leq k-1$.

Proof Suppose that $d^w(v_j) > d^w(v_1)$ for some j such that $2 \leq j \leq k-1$. Since $v_1v_{j+1} \in E(G)$ by Claim 4, $v_jv_{j-1}\cdots v_1v_{j+1}\cdots v_k\cdots v_p$ is another longest path with $d^w(v_j) + d^w(v_p) > d^w(v_1) + d^w(v_p)$, which contradicts the maximality of $d^w(v_1) + d^w(v_p)$. ■

Claim 6 $d^w(v_{k+1}) > d^w(v_1)$.

Proof Note that $v_1v_{k+1} \notin E(G)$ by the choice of v_k and that the path $v_1v_kv_{k+1}$ is a path of length 2. We find that $d(v_1, v_{k+1}) = 2$. Using the condition of the theorem, we obtain that $\max\{d^w(v_1), d^w(v_{k+1})\} \geq s/2$. Since $d^w(v_1) < s/2$, we have that $d^w(v_{k+1}) \geq s/2 > d^w(v_1)$. ■

For every i such that $2 \leq i \leq k-1$, v_i cannot be adjacent to any vertex outside P; otherwise, there is a path with length at least p by Claim 4, contradicting the choice of P in (a). Since G is 2-connected, there must exist an edge $v_jv_r \in E(G)$ such that $j < k < r$. Choose $v_jv_r \in E(G)$ such that $j < k < r$ and r is as large as possible.

Now consider the following two cases:

Case 1 $r \geq k+2$ (see Figure 5.1.1).

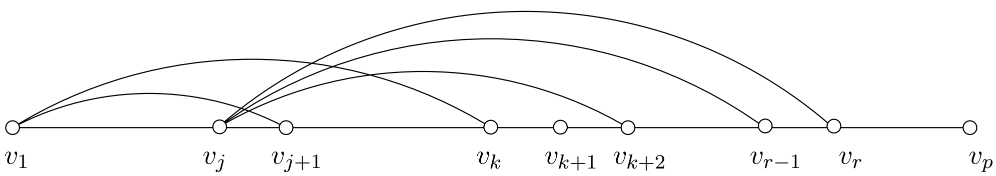

Figure 5.1.1

By Claim 4, $v_1v_{j+1} \in E(G)$ and so there is another longest path $v_{r-1}v_{r-2}\cdots v_{j+1}v_1\cdots v_jv_r\cdots v_p$. It follows from the maximality of $d^w(v_1)+d^w(v_p)$ that $d^w(v_{r-1}) \leq d^w(v_1) < s/2$. On the other hand, by Claim 5, we have that $d^w(v_j) \leq d^w(v_1) < s/2$. Consequently $\max\{d^w(v_{r-1}), d^w(v_j)\} < s/2$. Using the condition of the theorem, we have that $d(v_j, v_{r-1}) \neq 2$. Since $v_jv_rv_{r-1}$ is a path of length 2, we must have $v_jv_{r-1} \in E(G)$. If $r-1 > j+1$, we have another longest path $v_{r-2}v_{r-3}\cdots v_{j+1}v_1\cdots v_jv_{r-1}v_r\cdots v_p$.

Repeating the process, we obtain that $v_j v_{r-2} \in E(G)$. If $r-2 > j+1$, by repeating the same process again, we have that $v_j v_{r-3} \in E(G)$, and so on. Consequently we have that $v_j v_i \in E(G)$ for all i such that $j+1 \leq i \leq r$.

In particular, $v_j v_{k+2} \in E(G)$ since $r \geq k+2$. This means that there is a longest path $v_{k+1} v_k \cdots v_{j+1} v_1 \cdots v_j v_{k+2} \cdots v_p$ with $d^w(v_{k+1}) + d^w(v_p) > d^w(v_1) + d^w(v_p)$ by Claim 6. This is impossible by the maximality of $d^w(v_1) + d^w(v_p)$.

Case 2 $r = k+1$ (see Figure 5.1.2).

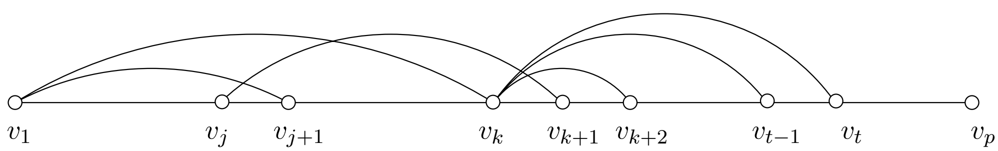

Figure 5.1.2

First, note that $v_k v_{k-1} \cdots v_{j+1} v_1 \cdots v_j v_{k+1} \cdots v_p$ is another longest path, and so by the maximality of $d^w(v_1) + d^w(v_p)$ we have that $d^w(v_k) \leq d^w(v_1) < s/2$.

By Claim 3 we may assume that $k+1 < p$. It follows from the 2-connectedness of G and the choice of v_r that there must be an edge $v_k v_t \in E(G)$ such that $t \geq k+2$. This implies that there is another longest path $v_{t-1} v_{t-2} \cdots v_{k+1} v_j \cdots v_1 v_{j+1} \cdots v_k v_t \cdots v_p$, and then by the maximality of $d^w(v_1) + d^w(v_p)$ we have that $d^w(v_{t-1}) \leq d^w(v_1) < s/2$. Together with $d^w(v_k) \leq d^w(v_1) < s/2$, this implies that $\max\{d^w(v_k), d^w(v_{t-1})\} < s/2$, and so by the condition of the theorem we have $d(v_k, v_{t-1}) neq 2$. Note that $v_k v_t v_{t-1}$ is a path of length 2. We obtain that $v_k v_{t-1} \in E(G)$. If $t-1 > k+1$, we repeat the process. By the same argument as before, we have $v_k v_i \in E(G)$ for all i such that $k+1 \leq i \leq t$.

In particular, $v_kv_{k+2} \in E(G)$ since $t \geq k+2$. Hence, there is another longest path $v_{k+1}v_j \cdots v_1v_{j+1} \cdots v_kv_{k+2} \cdots v_t \cdots v_p$ with $d^w(v_{k+1}) + d^w(v_p) > d^w(v_1) + d^w(v_p)$ by Claim 6. This is contrary to the maximality of $d^w(v_1) + d^w(v_p)$. ■

5.2 Long cycles in weighted graphs

The following result on the existence of long cycles in graphs is well-known.

Theorem 5.2.1 (Pósa [53])
Let G be a 2-connected graph such that $d(u) + d(v) \geq c$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c.

In [11], Bondy *et al.* give a generalization of Theorem 5.2.1 to weighted graphs as follows:

Theorem 5.2.2 (Bondy *et al.* [11])
Let G be a 2-connected weighted graph such that $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of weight at least s.

The aim of this section is to give a further generalization of Theorem 5.2.1 to weighted graphs.

Theorem 5.2.3
Let G be a 2-connected weighted graph such that $d(u) + d(v) \geq c$ and $d^w(u) + d^w(v) \geq s$ for every pair of nonadjacent vertices u and v in G. Then G contains either a Hamilton cycle or a cycle of length at least c and weight at least s.

It is clear that Theorem 5.2.3 generalizes Theorem 5.2.2, and of course

Theorem 5.2.1.

Proof of Theorem 5.2.3 Let G be a 2-connected weighted graph such that $d(u)+d(v)\geq c$ and $d^w(u)+d^w(v)\geq s$ for every pair of nonadjacent vertices u and v in G. Suppose that G does not contain a Hamilton cycle. It suffices to prove that G contains a cycle of length at least c and weight at least s.

Choose a path $P=v_1v_2\cdots v_p$ in G such that

(a) P is as long as possible;

(b) $w(P)$ is as large as possible, subject to (a).

By the choice in (a) we immediately see that $N_G(u)\bigcup N_G(v)\subseteq V(P)$.

As in the proof of Theorem 5.1.2, it is easy to see that there is no cycle in G of length p. Therefore, we have

Claim 1 $v_1v_p\notin E(G)$.

Claim 2 *If* $v_i\in N_G(v_1)$, *then* $v_{i-1}\notin N_G(v_p)$.

By the maximality of P, we have

Claim 3 *If* $v_i\in N(v_1)$, *then* $w(v_{i-1}v_i)\geq w(v_1v_i)$. *If* $v_j\in N(v_p)$, *then* $w(v_jv_{j+1})\geq w(v_jv_p)$.

Since G is 2-connected, by Lemma 1 of [9], there is a sequence of internally disjoint paths $P_1, P_2,\cdots,P_m$ such that

(1) P_k has end-vertices x_k and y_k, and $V(P_k)\cap V(P)=\{x_k,y_k\}$ for $k=1,2,\ldots,m$;

(2) $v_1=x_1<x_2<y_1\leq x_3<y_2\leq x_4<\cdots<y_{m-2}\leq x_m<y_{m-1}<y_m=v_p$, where the inequalities denote the order of the vertices on P.

By Claim 2, we have $m \geq 2$. It is not difficult to see that we can choose these paths such that

(3) if $v_i \in N(v_1)$, then $v_i \in P[v_2, x_2] \cup P[y_1, x_3]$ for $m \geq 3$, or $v_i \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$;

(4) if $v_j \in N(v_p)$, then $v_j \in P[y_{m-2}, x_m] \cup P[y_{m-1}, v_{p-1}]$ for $m \geq 3$, or $v_j \in P[v_2, x_2] \cup P[y_1, v_{p-1}]$ for $m = 2$.

Now denote by C_k the cycle $P_k \cup P[x_k, y_k]$, $k = 1, \cdots, m$, and let C be the cycle the edge set of which is the symmetric difference of the edge sets of these cycles C_k. By (3), (4) and Claim 2 we have for all $v_i \in N(v_1) \backslash \{y_1\}$ and $v_j \in N(v_p) \backslash \{x_m\}$ that $v_{i-1}v_i, v_jv_{j+1} \in E(C)$ and $v_{i-1}v_i \neq v_jv_{j+1}$. Also note that since $N(v_1) \cup N(v_p) \subseteq V(P)$, we must have $P_1 = v_1y_1$ and $P_m = x_mv_p$.

Define $S = \{v_{i-1} | v_1v_i \in E(G)\}$ and $T = \{v_{j+1} | v_jv_p \in E(G)\}$. By Claim 2, the above discussion and the condition of the theorem, it is easy to see that the length of C is at least $|S| + |T| = d(v_1) + d(v_p) \geq c$.

On the other hand, from Claim 3 we have

$$
\begin{aligned}
w(C) \geq & \sum_{v_i \in N(v_1) \backslash \{y_1\}} w(v_{i-1}v_i) + \sum_{v_j \in N(v_p) \backslash \{x_m\}} w(v_jv_{j+1}) \\
& + w(v_1y_1) + w(x_mv_p) \\
\geq & \sum_{v_i \in N(v_1)} w(v_1v_i) + \sum_{v_j \in N(v_p)} w(v_jv_p) \\
= & \, d^w(v_1) + d^w(v_p) \geq s,
\end{aligned}
$$

which proves the theorem. ■

Part II

Paths and cycles in colored graphs

Chapter 6

Monochromatic or heterochromatic paths and cycles

In this chapter, we are concerned with monochromatic paths and cycles, and heterochromatic paths and cycles in colored graphs. Some sufficient conditions for the existence of (long) monochromatic paths and cycles, and those for the existence of long heterochromatic paths and cycles are presented.

6.1 Monochromatic paths and cycles

First, let us consider the problem under what conditions a colored graph contains a monochromatic path or a monochromatic cycle. It is clear that every colored graph contains at least one monochromatic path. Moreover, it is obvious that not every colored graph contains monochromatic cycles.

The *arboricity* $a(G)$ of a graph G is defined as the minimum number of edge-disjoint forests into which G can be decomposed. Clearly, it is also the minimum number of colors necessary to color the edges of G so that no cycle is monochromatic. So we have

Proposition 6.1.1
Let G be a colored graph. If $c(G) < a(G)$, then G contains at least one monochromatic cycle. ■

The arboricity $a(G)$ can be determined by applying the matroid partitioning algorithm of Edmonds [48]. In [52] Picard & Queyranne showed that this parameter can be determined in at most $O(n^4)$ operations, by using network flow methods. It is (almost) trivial to check whether a colored graph contains a monochromatic cycle: for each color class E_i check whether the induced subgraph $G[E_i]$ contains a cycle.

The following result on the existence of monochromatic paths and cycles with a prescribed length is obvious.

Proposition 6.1.2
Let G be a colored graph with color classes $E_1, E_2, \ldots, E_c$. Then G has a monochromatic path (cycle) of length at least l if and only if for some i with $1 \leq i \leq c$, the induced subgraph $G[E_i]$ has a path (cycle) of length at least l. ■

If we regard an uncolored graph G as a colored graph (G, C) for which all edges have the same color, then (G, C) contains a monochromatic path (cycle) of length at least l if and only if G contains a path (cycle) of length at least l. Since the problem of deciding whether there is a path (cycle) of length at least l in an (uncolored) graph is **NP**-complete, the problem of deciding whether there is a monochromatic path (cycle) of length at least l in a colored graph is also **NP**-complete.

There are many results on the existence of long paths and cycles in (uncolored) graphs. Here we list two of them, the first of which was already mentioned in the previous chapters.

Theorem 6.1.3 (Erdös & Gallai [24])
Let G be a graph of order n and size m. Then G contains a path of length at least $\frac{2m}{n}$.

Theorem 6.1.4 (Erdös & Gallai [24])
Let G be a graph of order n and size m such that $m \geq n$. Then G contains a cycle of length at least $\frac{2m}{n-1}$.

Using Theorems 6.1.2 to 6.1.4, it is not difficult to prove the following results:

Proposition 6.1.5
Suppose that G is a colored graph of order n and size m. Then G contains a monochromatic path of length at least $\frac{2m}{c(G)n}$. ■

Proposition 6.1.6
Suppose that G is a colored graph of order n and size m such that $m \geq c(G)n$. Then G contains a monochromatic cycle of length at least $\frac{2m}{c(G)(n-1)}$. ■

As it was shown in [24], Theorem 6.1.3 is best possible. Let pK_r denote the disjoint union of p copies of K_r. This graph has $n = pr$ vertices and $m = \frac{pr(r-1)}{2}$ edges. It is easy to check that $\frac{2m}{n} = r - 1$. On the other hand, pK_r contains no path of length greater than $r-1$. Of course, this graph also shows that the result in Proposition 6.1.5 in the case $c(G) = 1$ is best possible. This example can be extended to general cases to show the sharpness of the result in Proposition 6.1.5.

Let G and H be two colored graphs. The *colored Cartesian product* of G and H is the graph $G \times H$ with a coloring defined as follows: From the definition of the Cartesian product of graphs, to every vertex u of G, there corresponds a subgraph H_u of $G \times H$ such that H_u is isomorphic to

H. To each edge e of H_u, assign the color of the edge corresponding to e in H. Similarly, to every vertex v of H, there corresponds a subgraph G_v of $G \times H$ such that G_v is isomorphic to G. To each edge e of G_v, assign the color of the edge corresponding to e in G. The colored Cartesian product $G_1 \times G_2 \times \cdots \times G_k$ of $k \geq 2$ colored graphs $G_1, G_2 \cdots, G_k$ can be defined inductively.

Let G_i $(1 \leq i \leq c)$ be the colored graph K_r such that all the edges of G_i receive the same color i. By K_r^c we denote the colored Cartesian product $G_1 \times G_2 \times \cdots \times G_c$. It is not difficult to see that the colored graph K_r^c has $n = r^c$ vertices, $m = \dfrac{cr^c(r-1)}{2}$ edges and c colors. This implies that $\dfrac{2m}{cn} = r - 1$. On the other hand, the colored graph K_r^c has no monochromatic path of length greater than $r - 1$. This shows that the result in Proposition 6.1.5 is best possible. Clearly the disjoint union of some copies of the colored graph K_r^c defined above can also be used to show the sharpness of the result of Proposition 6.1.5.

Theorem 6.1.4 is also best possible. This can be shown by the graph $\Gamma_{p,r}$ defined as follows: The graph $\Gamma_{p,r}$ is a connected graph which has exactly $n = p(r-1) + 1$ vertices and each of the p blocks of it is a clique on r vertices. This graph has $m = \dfrac{pr(r-1)}{2}$ edges and clearly $\dfrac{2m}{n-1} = r$. On the other hand, it has no cycle of length greater than r. Of course this example also shows that the result of Proposition 6.1.6 is best possible in the case $c(G) = 1$.

Let G_{p_i} $(1 \leq i \leq c)$ be the colored graph $\Gamma_{p_i,r}$ such that all the edges of it receive the same color i. Denote by G the colored Cartesian product $G_{p_1} \times G_{p_2} \times \cdots \times G_{p_c}$. Then G has

$$n = \sum_{i=1}^{c} \sum_{1 \leq j_1 < j_2 < \cdots < j_i \leq c} p_{j_1} p_{j_2} \cdots p_{j_i} (r-1)^i + 1$$

vertices,

$$m = \frac{r(r-1)}{2}\sum_{i=1}^{c} i \sum_{1\le j_1<j_2<\cdots<j_i\le c} p_{j_1}p_{j_2}\cdots p_{j_i}(r-1)^{i-1}$$

edges, and c colors. Therefore,

$$\frac{2m}{c(n-1)} = \frac{2\times\frac{r(r-1)}{2}\sum_{i=1}^{c} i \sum_{1\le j_1<j_2<\cdots<j_i\le c} p_{j_1}p_{j_2}\cdots p_{j_i}(r-1)^{i-1}}{c(\sum_{i=1}^{c}\sum_{1\le j_1<j_2<\cdots<j_i\le c} p_{j_1}p_{j_2}\cdots p_{j_i}(r-1)^{i}+1-1)}$$
$$= r - \frac{\sum_{i=1}^{c-1}(c-i)\sum_{1\le j_1<j_2<\cdots<j_i\le c} p_{j_1}p_{j_2}\cdots p_{j_i}(r-1)^{i-1}}{c\sum_{i=1}^{c}\sum_{1\le j_1<j_2<\cdots<j_i\le c} p_{j_1}p_{j_2}\cdots p_{j_i}(r-1)^{i-1}}.$$

It is clear that $\lceil\frac{2m}{c(n-1)}\rceil = r$. On the other hand, the colored graph G contains no monochromatic cycle of length greater than r. This shows that the result in Proposition 6.1.6 is best possible.

6.2 Heterochromatic paths and cycles

If we regard an uncolored graph G as a colored graph (G, C) in which all edges have different colors, then G contains a path (cycle) of length at least l if and only if (G, C) contains a heterochromatic path (cycle) of length at least l. As we mentioned earlier, the problem of deciding whether there is a path (cycle) of length at least l in an (uncolored) graph is **NP**-complete. Therefore the problem of deciding whether there is a heterochromatic path (cycle) of length at least l in a colored graph is **NP**-complete, too. In this section we will consider under what conditions there is a heterochromatic path (cycle) with a prescribed length in a colored graph.

Let G be a colored graph. By selecting precisely one edge from each color class of G, we obtain a new colored graph G', such that all the edges of G' have different colors, and $c(G') = c(G)$. Using Theorems 6.1.3 and 6.1.4, it is easy to prove the following results.

Proposition 6.2.1
Let G be a colored graph of order n. Then G contains a heterochromatic path of length at least $\frac{2c(G)}{n}$. ■

Proposition 6.2.2
Let G be a colored graph of order n such that $c(G) \geq n$. Then G contains a heterochromatic cycle of length at least $\frac{2c(G)}{n-1}$. ■

Clearly Propositions 6.2.1 and 6.2.2 generalize Theorems 6.1.3 and 6.1.4, respectively.

Furthermore, we have the following two results on the existence of long heterochromatic paths.

Proposition 6.2.3
Let G be a colored graph and k an integer. Suppose that $d^c(v) \geq k$ for every vertex v of G. Then for every vertex z of G there exists a heterochromatic z-path of length at least $\lceil \frac{k+1}{2} \rceil$.

Proof Choose a longest heterochromatic z-path P with length l. Denote the other end-vertex of P as v. Then from the assumption of the proposition, we know that all incident edges of v with the other end not on P have colors also appearing in $E(P)$. Therefore, $d^c(v) \leq l+(l-1) = 2l-1$. On the other hand, $d^c(v) \geq k$, so we have $l \geq \lceil \frac{k+1}{2} \rceil$. ■

Proposition 6.2.4
Let G be a colored graph and s an integer. Suppose that $|CN(u) \cup CN(v)| \geq s > 1$ for every pair of vertices u and v of G. Then G

contains a heterochromatic path of length at least $\lceil \frac{s}{3} \rceil + 1$.

Proof Choose a longest heterochromatic path P with length l. Denote the end-vertices of P as u and v. Then from the assumption of the proposition, we know that all incident edges of u and v with the other end not on P have colors also appearing in $E(P)$. Therefore, $|CN(u) \cup CN(v)| \leq l + (l-1) + (l-2) = 3l - 3$. On the other hand, $|CN(u) \cup CN(v)| \geq s$, so we have $l \geq \lceil \frac{s}{3} \rceil + 1$. ■

In the following, we give some sufficient conditions for the existence of heterochromatic triangles or quadrilaterals.

Proposition 6.2.5
Let G be a colored graph of order $n \geq 4$, such that $|CN(u) \cup CN(v)| \geq n - 1$ for every pair of vertices u and v of G. Then G contains at least one heterochromatic triangle or one heterochromatic quadrilateral.

Proof If $|CN(u)| = n - 1$ for every vertex u of G, then $d(u) = n - 1$, and G is a complete graph. It is clear that every triangle of G is heterochromatic. So, we need only consider the case that there is some vertex $u \in V(G)$ with $|CN(u)| < n - 1$.

Suppose that G contains neither heterochromatic triangles nor heterochromatic quadrilaterals. Without loss of generality, we can assume that $V(G) = \{x_1, x_2, \ldots, x_k, u, v, y_{k+2}, y_{k+3}, \ldots, y_{n-2}, y_{n-1}\}$, $d^c(u) = k + 1 < n - 1$, $C(ux_i) = i$ for $i = 1, 2, \ldots, k$, $C(uv) = k + 1$ and $C(vy_j) = j$ for $j = k + 2, \ldots, n - 1$.

First, consider the vertex u and a vertex $x_i \in \{x_1, x_2, \ldots, x_k\}$. Since $CN(v) \cup (CN(x_i) \cap \{C(x_i x_1), C(x_i x_2), \ldots, C(x_i x_{i-1}), C(x_i x_{i+1}), \ldots, C(x_i x_k), C(x_i u)\}) \subseteq \{1, 2, \ldots, k + 1\}$ and $|CN(u) \cup CN(x_i)| \geq n - 1$, x_i must be adjacent to each vertex $y_j \in \{y_{k+2}, y_{k+3}, \ldots, y_{n-1}\}$, and $C(x_i y_j) = j$ by our assumption that G contains neither heterochromatic triangles nor heterochromatic quadrilaterals.

Now consider the two vertices u and y_{n-1}. Since $CN(u) \cup (CN(y_{n-1}) \cap \{C(y_{n-1}x_1),\ C(y_{n-1}x_2), \ldots, C(y_{n-1}x_k)\}) \subseteq \{1, 2, \ldots, k+1, n-1\}$ and $|CN(u) \cup CN(y_{n-1})| \geq n-1$, we have that $y_{n-1}y_j \in E(G)$ and $C(y_{n-1}y_j) = j$ for $j = k+2, \ldots, n-2$ by our assumption that G contains no heterochromatic triangles.

So, we have $CN(u) \cup (CN(y_{n-2}) \cap \{C(y_{n-2}x_1), C(y_{n-2}x_2), \ldots, C(y_{n-2}x_k), C(y_{n-2}u),\ C(y_{n-2}v), C(y_{n-2}y_{n-1})\}) \subseteq \{1, 2, \ldots, k,\ k+1, n-2\}$. Therefore, $|CN(u) \cup CN(y_{n-2})| \leq |\{1, 2, \ldots, k, k+1, n-2\}| + |\{y_{k+2}, y_{n+3}, \ldots, y_{n-3}\}| = (k+2) + (n-k-4) = n-2 < n-1$, a contradiction.

The proof of the result is complete. ■

Although the proofs of the results in Propositions 6.2.3 to 6.2.5 are easy, it can be shown that these results are best possible in the sense that there exist some graphs showing that they cannot be improved. However, we think that perhaps much stronger results are possible to obtain if one excludes some small counter-examples or simple classes of counter-examples. The proof techniques we applied here do not seem to be strong enough for obtaining such improvements. Maybe an approach using probabilistic proof techniques could yield such improvements.

The mentioned facts and our experiences indicate that the problems on paths and cycles in colored graphs are even harder than similar problems on paths and cycles in weighted graphs.

Chapter 7

Paths and cycles with few or many colors

In this chapter, we prove that the problem of finding a path (cycle) with as few different colors as possible in a colored graph is **NP**-hard. Several exact and approximation algorithms for finding a path with the fewest colors are provided. The complexity of the exact algorithms and the performance ratio of the approximation algorithms are analyzed. We also pose a conjecture on the existence of paths and cycles with many different colors.

7.1 Paths and cycles with few colors

If we regard an (uncolored) graph G as a colored graph (G, C) for which all edges have different colors, then a shortest path between two given vertices in G is a path between the two vertices with the fewest colors in the colored graph (G, C). It is well-known that the problem of finding a shortest path between two given vertices in a (weighted) graph can be solved efficiently. There are many polynomial-time algorithms to solve this problem. In this section, we will consider the complexity aspects of finding a path between two given vertices with the fewest colors in a colored graph.

Problem 7.1.1

INSTANCE: *Graph $G = (V, E)$ with a coloring $C : E \to \mathbb{N}$ and two given vertices s_0 and t_0, positive integer $K \leq c(G)$.*

QUESTION: *Is there a path P from s_0 to t_0 such that $c(P) \leq K$?*

According to [33], the following problem is **NP**-complete.

3-SATISFIABILITY (3-SAT)

INSTANCE: *Collection $C = \{C_1, C_2, \ldots, C_m\}$ of clauses on a finite set U of variables such that $|C_i| = 3$ for $1 \leq i \leq m$.*

QUESTION: *Is there a truth assignment for U that satisfies all the clauses in C?*

In this section, we use this result to show that Problem 7.1.1 is **NP**-complete, too.

Theorem 7.1.2

*Problem 7.1.1 is **NP**-complete.*

Proof It is easy to see that Problem 7.1.1 is in **NP**. One way to see this is to observe that a nondeterministic algorithm need only guess an (s_0, t_0)-path P in G, and check in linear time whether $c(P) \leq K$.

We shall now show that **3-SAT** can be polynomially transformed to Problem 7.1.1. Given a Boolean formula F consisting of m clauses $C_1, C_2, \ldots, C_m$ (with three literals per clause) and involving n variables $x_1, x_2, \ldots, x_n$, we shall construct a graph $G = (V, E)$ with a coloring $C : E \to \mathbb{N}$ and two vertices s_0 and t_0, such that G has an (s_0, t_0)-path P with $c(P) \leq n + 1$ if and only if F is satisfiable.

First, for the variable x_i $(1 \leq i \leq n)$, we construct a subgraph A_i of G, where $V(A_i) = \{s_{i-1}, u_{i1}, u_{i2}, s_i\}$ and $E(A_i) = \{s_{i-1}u_{i1}, u_{i1}s_i, s_{i-1}u_{i2},$

$u_{i2}s_i\}$. Assign a special color 0 to the edges $s_{i-1}u_{i1}$ and $s_{i-1}u_{i2}$, the color i to the edge $u_{i1}s_i$ and the color i' to the edge $u_{i2}s_i$ for $i = 1, 2, \ldots, n$. Then we get a colored graph $A = \cup_{i=1}^{n} A_i$. For the clause C_j $(1 \le j \le m)$, we construct a subgraph B_j of G, where $V(B_j) = \{t_{j-1}, v_{j1}, v_{j2}, v_{j3}, t_j\}$ and $E(B_j) = \{t_{j-1}v_{j1}, t_{j-1}v_{j2}, t_{j-1}v_{j3}, v_{j1}t_j, v_{j2}t_j, v_{j3}t_j\}$. For $j = 1, 2, \ldots, m$ and $k = 1, 2, 3$, assign the color 0 to the edge $t_{j-1}v_{jk}$, the color h to the edge $v_{jk}t_j$ if the kth literal of C_j is x_h and the color h' to the edge $v_{jk}t_j$ if the kth literal of C_j is $\bar{x}_h$. Then we get a colored graph $B = \cup_{j=1}^{m} B_j$. The colored graph G is obtained by connecting the two graphs A and B with an edge s_nt_m and coloring this edge with the color 0, see Figure 7.1.1. Clearly the construction of G can be accomplished in polynomial time.

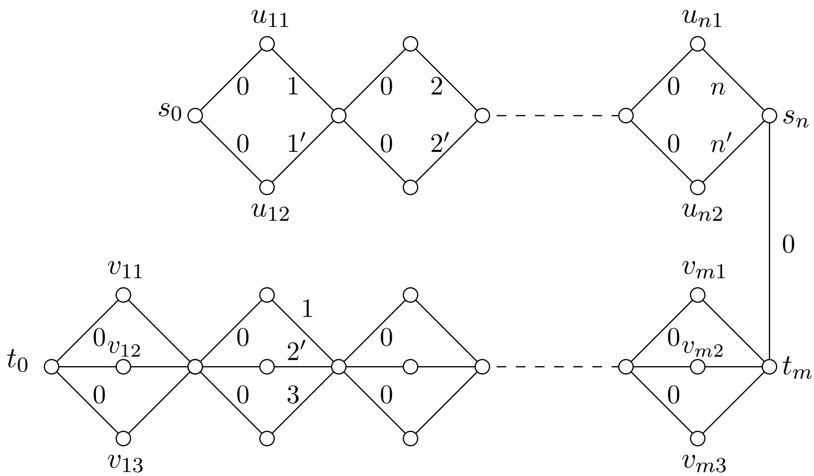

Figure 7.1.1: The graph in the proof of Theorem 7.1.2 in the case $C_2 = x_1\bar{x}_2x_3$.

It is not difficult to verify that there is an (s_0, t_0)-path P with $c(P) \le n + 1$ if and only if F is satisfiable. We leave the details to the reader. ■

The following consequence of Theorem 7.1.2 is immediate.

Corollary 7.1.3
*Finding a path with as few different colors as possible in a colored graph is **NP**-hard.* ■

Remark 7.1.4
Broersma & Li [16] proved that the problem of finding a spanning tree with as few colors as possible in a colored graph is **NP**-hard by using the minimum dominating set problem. It is not difficult to see that the graph G we constructed in the proof of Theorem 7.1.2 has a spanning tree with at most $n + 1$ colors if and only if F is satisfiable. So, our technique also provides a new proof to Broersma & Li's result.

The problem of finding a cycle with as few colors as possible in a colored graph is also **NP**-hard. We consider the following decision problem.

Problem 7.1.5

INSTANCE:	*2-connected graph $G = (V, E)$ with a coloring $C : E \rightarrow \mathbb{N}$ and a given vertex u, positive integer $K \leq c(G)$.*
QUESTION:	*Is there a cycle C passing through u such that $c(C) \leq K$?*

Theorem 7.1.6
*Problem 7.1.5 is **NP**-complete.*

Proof Let G a connected colored graph. Construct a 2-connected colored graph G' by adding a new vertex u, connecting u to every vertex v of G with an edge and assigning an extra color 0 to the new edges. Then G contains a path P such that $c(P) \leq K$ if and only if G' contains a cycle C passing through u such that $c(C) \leq K + 1$. It follows from Corollary 7.1.3 that Problem 7.1.5 is **NP**-complete. ■

Corollary 7.1.7
*Finding a cycle with as few colors as possible in a 2-connected colored graph is **NP**-hard.* ■

As we proved in Theorem 7.1.2, finding a path with as few colors as possible (minimum path) between two given vertices in a colored graph is **NP**-hard. However, if $c(G)$ is much smaller than $|V(G)|$, say, $c(G) = O(\log_2 |V(G)|)$, there will be some efficient algorithm for solving this problem.

One approach to finding a minimum path between two given vertices s_0 and t_0 is to check whether there is an (s_0, t_0)-path in the graphs $G[E_{i_1}]$ with $1 \le i_1 \le k$, $G[E_{i_1} \cup E_{i_2}]$ with $1 \le i_1, i_2 \le k$ and $i_1 \ne i_2$, $\dots$, and $G[E_{i_1} \cup E_{i_2} \cup \dots \cup E_{i_d}]$ with $1 \le i_1, i_2, \dots, i_d \le k$ and $i_p \ne i_q$ for $1 \le p \ne q \le k$, where $E_1, E_2, \dots, E_k$ are the color classes, $k = c(G)$, and d is the distance between s_0 and t_0. The complexity of such an algorithm is

$$\left[\binom{k}{1} + \binom{k}{2} + \dots + \binom{k}{\min\{d,k\}}\right] O(|V(G)|)$$
$$= \min\{O(k^d|V(G)|), O(2^k|V(G)|)\}.$$

Another approach is to transform the colored graph to a number of weighted graphs and use Dijkstra's Algorithm. Assign, e.g., the weights $1, |V(G)|, |V(G)|^2 \dots, |V(G)|^{k-1}$ to the edges of the colored graph such that all the edges with the same color get equal weights. There are $k!$ possibilities, so we get at most $k!$ weighted graphs. It is not difficult to see that a minimum path between two given vertices s_0 and t_0 can be found by determining the shortest paths in each of these weighted graphs. The complexity of such an algorithm is $O(k!|V(G)|^2)$.

It is also of interest to consider approximation algorithms for the minimum path problem. If we use a shortest path between two vertices as an approximate solution for a minimum path, the approximation ratio is $c(G)$. We can also design an approximation algorithm which is similar to Dijkstra's Algorithm for finding a shortest path.

Algorithm 7.1.8

Step 1. *Set* $C(s_0) = \emptyset$, $C(S_0) = 0$, $c(v) = \infty$ *for* $v \neq s_0$, $S_0 = \{s_0\}$ *and* $i = 0$.

Step 2. *For each* $v \in V \backslash S_i$, *replace* $C(v)$ *by* $C(u_i) \cup \{C(u_i v)\}$ *if* $c(v) > |C(u_i) \cup \{C(u_i v)\}|$ *and set* $c(v) = |C(v)|$. *Compute* $\min_{v \in V \backslash S_i} \{c(v)\}$ *and let* u_{i+1} *denote a vertex for which this minimum is attained. Set* $S_{i+1} = S_i \cup \{u_{i+1}\}$.

Step 3. *If* $u_{i+1} = t_0$, *stop. Otherwise, replace* i *by* $i+1$ *and go to Step 2.*

The approximation factor of this algorithm can get arbitrarily large. This can be shown by the graph in Figure 7.1.2.

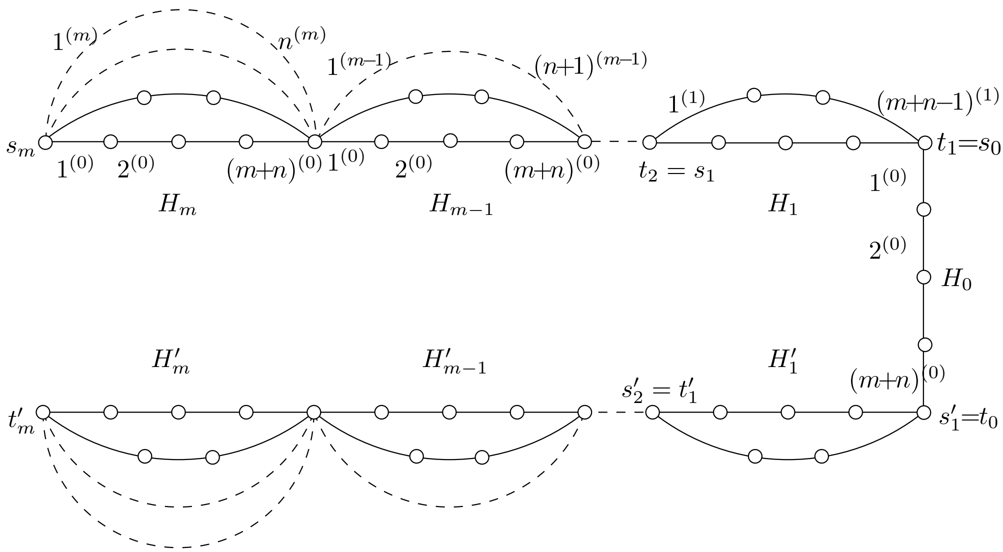

Figure 7.1.2

The graph in Figure 7.1.2 is constructed as follows. Let H_0 be an (s_0, t_0)-path P_0 with $m+n$ edges and assign the colors $1^{(0)}, 2^{(0)}, \ldots, (m+n)^{(0)}$ to the edges of P_0, respectively. For $1 \leq i \leq m$, the graph H_i is obtained from H_{i-1} by adding a new (s_{i-1}, t_{i-1})-path P_i with $m + n - i$

edges and assigning the colors $1^{(i)}, 2^{(i)}, \ldots, (m+n-i)^{(i)}$ to the edges of P_i, respectively. We denote the two vertices s_{i-1} and t_{i-1} of H_i by s_i and t_i, respectively. By denoting the vertices s_i by s'_i and t_i by t'_i, we get a new graph H'_i for each i with $1 \leq i \leq m$. The graph in Figure 7.1.2 is constructed from the graphs H_0, H_i and H'_i $(1 \leq i \leq m)$ by identifying the vertices t_0 with s'_1, the vertices s_{i-1} with t_i, and the vertices s'_i with t'_{i-1} for $1 \leq i \leq m$.

It is easy to see that the minimum path between s_m and t'_m in the graph H is of $m+n$ colors. Whereas we will get an approximate result $n(m+1) + \dfrac{m(m+1)}{2}$ if we apply the above algorithm to the graph H. So the approximation factor is

$$|1 - \frac{n(m+1) + \frac{m(m+1)}{2}}{m+n}| = \frac{mn + \frac{m(m+1)}{2}}{m+n} \to \infty,$$

when $m \to \infty$ in the case $n = 1$ or $n = m^{1+\epsilon}$ $(\epsilon > 0)$.

7.2 Paths and cycles with many colors

If we regard an uncolored graph as a colored graph in which all edges have different colors, then the number of colors of a subgraph is just the number of edges of it. It is well-known that the problem of finding a longest path or a longest cycle in a graph is **NP**-hard. Therefore, the problem of finding a path or a cycle with as many colors as possible in a colored graph are also **NP**-hard.

In the past decades, many sufficient conditions for the existence of long paths and cycles have been derived. The oldest result of this type is due to DIRAC.

Theorem 7.2.1 (DIRAC [20])
Let G be graph and d an integer. If $d(v) \geq d$ for every vertex v of G,

then G contains (1) a path of length at least d, and (2) a cycle of length at least $d+1$ if $d>1$.

It is an interesting problem to establish whether Theorem 7.2.1 admits a generalization to colored graphs. We pose the following conjecture.

Conjecture 7.2.2
Let G be a colored graph and d an integer. If $d^c(v) \geq d$ for every vertex v of G, then G contains (1) *a path with at least $d-1$ colors, and* (2) *a cycle with at least d colors if $d>1$.*

If the above conjecture is true, then it would be best possible. Let K_r be a complete graph of order $r \geq 4$. The γ-coloring of K_r when r is even is defined as a proper $(r-1)$-coloring of the edges of K_r. In the case that r is odd, the γ-coloring of K_r is defined as follows: first assign a γ-coloring to $K_r - v$ for some vertex v; then color the edges incident with v with $r-1$ colors which are all different from the colors of the γ-coloring of $K_r - v$. Let G be the disjoint union of some copies of the complete graph K_r. Assign colors to the edges of G such that each component of G receives a γ-coloring. Then it is easy to check that $d^\gamma(v) \geq r-1$ for every vertex v of G and G contains no path with more than $r-2$ colors and no cycle with more than $r-1$ colors. Let H be a colored graph on $n = k(r-1)+1$ vertices such that each of the k blocks of H is a complete graph on r vertices and has a γ-coloring. This graph also contains no cycle with more than $r-1$ colors, but $d^\gamma(v) \geq r-1$ for every vertex v of H.

By imposing a higher connectivity, the bound on the cycle length in Theorem 7.2.1 can be increased.

Theorem 7.2.3 (Dirac [20])
Let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v of G, then G contains either a Hamilton cycle or a cycle of

length at least $2d$*.*

Let $K_{n,n+1}$ be the complete bipartite graph with bipartition (X, Y) such that $|X| = n$ and $|Y| = n+1$. Assign a coloring to $K_{n,n+1}$ as follows: first color the graph $K_{n,n+1} - y$ for some vertex $y \in Y$ by a proper n-edge-coloring, then assign the same n colors of $K_{n,n+1}$ to the n edges incident to y, respectively. It is easy to show that $d^c(v) \geq n$ for each vertex v of $K_{n,n+1}$, but $K_{n,n+1}$ contains neither a Hamilton cycle nor a cycle with more than n colors. This shows that, different from Theorem 7.2.3, imposing a higher connectivity on the graphs in Conjecture 7.2.2 cannot guarantee the existence of cycles with more colors.

Part III

The scattering number of a graph

Chapter 8

Scattering number and other vulnerability parameters

In this chapter, the relationships between the scattering number and some other vulnerability parameters of graphs, namely the (edge-) connectivity, toughness, integrity and tenacity are studied.

8.1 Introduction

The *scattering number* $s(G)$ of a noncomplete connected graph G is defined by

$$s(G) = \max\{\omega(G - X) - |X| : X \subset V(G), \omega(G - X) > 1\},$$

where $\omega(G - X)$ stands for the number of components of $G - X$.

The notion of scattering number was introduced by Nash-Williams [10] but we found the first appearance in literature in a paper of JUNG [45]. JAMROZIK *et al.* [44] studied small maximal nonhamiltonian graphs by using this parameter. In [42], HENDRY used the scattering number to study extremal nonhamiltonian graphs. It turned out that the concept of scattering number is more convenient than the closely related concept of toughness [42] for describing maximal and extremal nonhamiltonian graphs. In 1989, Ouyang *et al.* [51] for the first time proposed to use this parameter to measure the vulnerability of networks. They obtained some basic results on the scattering number, including a recursive algorithm for computing the scattering number of trees, and an analysis of the scattering number of the so-called Harary graphs [41]. Recently, several other papers on the hamiltonicity and computing the scattering number of some special classes of graphs appeared [34, 35, 43].

Although most of the early results on the scattering number deal with paths and cycles in graphs, the concept itself is an appropriate parameter to measure the vulnerability of graphs. Unlike the connectivity measures, the scattering number is meant to incorporate not only the difficulty to break down the network but also the damage that has been caused.

Besides the connectivity, edge-connectivity and scattering number, there are several other vulnerability parameters of graphs. Among them, the toughness, integrity and tenacity are very similar to the scattering number. In this chapter, we are mainly concerned with the relationships between the scattering number and these vulnerability parameters. In Section 8.2, we consider its connection with connectivity, edge-connectivity and minimum degree. In Sections 8.3 to 8.5 we give an analysis of the relationships between the scattering number and the toughness, integrity and tenacity, respectively.

8.2 Scattering number and connectivity parameters

Among the most studied vulnerability parameters of graphs, are the connectivity and edge-connectivity. These measures determine in a certain sense the resistance of the graph to attacks to break down the graph by operations such as deletions of vertices or edges. From the definition of the scattering number, it is easy to see that this parameter represents, to some extent, a trade-off between the effort to break down the network and how badly the network is damaged. In this section we will derive some relationships between the scattering number and the connectivity parameters. We first present the following useful lemma.

Lemma 8.2.1 ([7])
Let G be a noncomplete connected graph of order n. Then $\kappa(G) \geq \max\{1, 2\delta(G) - n + 2\}$.

Theorem 8.2.2
Let G be a noncomplete connected graph of order n. Then $2 - \kappa(G) \leq s(G) \leq n - 2\delta(G)$.

Proof Suppose that X^* is a vertex cut of G such that $|X^*| = \kappa(G)$. Then

$$\begin{aligned} s(G) &= \max\{\omega(G - X) - |X| : X \subset V(G), \omega(G - X) > 1\} \\ &\geq \omega(G - X^*) - |X^*| \\ &\geq 2 - \kappa(G). \end{aligned}$$

Now let us prove that $s(G) \leq n - 2\delta(G)$. For simplicity, we denote $\kappa(G)$ and $\delta(G)$ by κ and δ, respectively.

Suppose that X is an arbitrary vertex cut of G and $|X| = x$, $\omega(G - X) = p$. Denote the p components of $G - X$ by $G_1, G_2, \ldots, G_p$, and set $|V(G_i)| = n_i$ $(1 \leq i \leq p)$. Then $\sum_{i=1}^{p} n_i = n - x$.

If $x \geq \delta$, then since the maximum possible value of p is $n - x$, we have that

$$\omega(G - X) - |X| \leq (n - x) - x = n - 2x \leq n - 2\delta.$$

If $x \leq \delta - 1$, then every component of $G - X$ has at least two vertices; to the contrary, suppose that there exists some j $(1 \leq j \leq p)$ such that $n_j = 1$. Denote the unique vertex of G_j by u. Then $d(u) \leq x < \delta$, a contradiction.

By similar arguments, $(n_i - 1) + x \geq \delta$ $(1 \leq i \leq p)$. Hence,

$$\begin{aligned} p\delta &\leq \sum_{i=1}^{p}((n_i - 1) + x) \\ &= \sum_{i=1}^{p} n_i + p(x-1) \\ &= (n - x) + p(x - 1). \end{aligned}$$

Therefore,

$$p \leq \frac{n-x}{\delta - x + 1}.$$

So we have

$$\omega(G - X) - |X| \leq \frac{n-x}{\delta - x + 1} - x.$$

Now set $s(x) = \dfrac{n-x}{\delta - x + 1} - x$. Then

$$\begin{aligned} s(G) &= \max\{\omega(G-X) - |X| : X \subset V(G), \omega(G - X) \geq 2\} \\ &\leq \max\{\max_{x \geq \delta}\{n - 2x\}, \max_{\kappa \leq x \leq \delta - 1} \lfloor s(x) \rfloor\} \\ &= \max\{n - 2\delta, \max_{\kappa \leq x \leq \delta - 1} \lfloor s(x) \rfloor\}. \end{aligned} \tag{1}$$

Consider

$$\begin{aligned} \Delta s(x) = s(x+1) - s(x) &= \frac{n-(x+1)}{\delta - x} - (x+1) - \frac{n-x}{\delta - x + 1} + x \\ &= \frac{-x^2 + (2\delta + 1)x - (\delta + 1)^2 + n}{(\delta - x)(\delta - x + 1)}. \end{aligned}$$

Since $x \leq \delta - 1$, we have $\delta - x \geq 1$ and $\delta - x + 1 \geq 2$. So, $(\delta - x)(\delta - x + 1) > 0$, and $\Delta s(x) \geq 0$ if and only if

$$x^2 - (2\delta + 1)x + (\delta + 1)^2 - n \leq 0.$$

The two roots of the equation $x^2 - (2\delta + 1)x + (\delta + 1)^2 - n = 0$ are

$$x_1 = \delta + \frac{1 - \sqrt{4n - 4\delta - 3}}{2}$$

and

$$x_2 = \delta + \frac{1 + \sqrt{4n - 4\delta - 3}}{2}.$$

It is easy to check that if $x \leq \lfloor x_1 \rfloor$ or $x \geq \lceil x_2 \rceil$, $s(x)$ is a decreasing function; on the interval $\lceil x_1 \rceil \leq x \leq \lfloor x_2 \rfloor$, $s(x)$ is an increasing function. It follows from $x_2 > \delta$, $x \leq \delta - 1$ and Lemma 8.2.1 that

$$\max_{\kappa \leq x \leq \delta - 1} \lfloor s(x) \rfloor \leq \max\{\lfloor s(\max\{1, 2\delta - n + 2\}) \rfloor, \lfloor s(\delta - 1) \rfloor\}.$$

Therefore, noting that $s(2\delta - n + 2) = n - 2\delta$, and using (1) we obtain

$$s(G) \leq \max\{n - 2\delta, \lfloor s(1) \rfloor, \lfloor s(\delta - 1) \rfloor\}.$$

Since $s(1) = \dfrac{n-1}{\delta} - 1 = \dfrac{n - 2\delta}{\delta} + 1 - \dfrac{1}{\delta}$ and $\delta \geq 1$, we have that $\lfloor s(1) \rfloor \leq n - 2\delta$.

Since $s(\delta - 1) = \dfrac{n - (\delta - 1)}{2} - (\delta - 1) = n - 2\delta + \dfrac{\delta + 2 - n}{2} + \dfrac{1}{2}$ and $\delta \leq n - 2$, we have that $\lfloor s(\delta - 1) \rfloor \leq n - 2\delta$.

Therefore, we conclude that

$$s(G) \leq n - 2\delta.$$

The proof of the theorem is complete. ∎

Remark 8.2.3
The lower bound for $s(G)$ in Theorem 8.2.2 is best possible. This can be shown by the graph $K_k + (K_1 \cup K_{n-k-1})$ with $n \geq k+2$. The upper bound for $s(G)$ in Theorem 8.2.2 is also best possible. This can be shown by the graph $K_\delta + (n-\delta)K_1$ with $n \geq \delta + 1$.

The following corollaries are immediate.

Corollary 8.2.4
Let G be a noncomplete connected graph of order n. Then

(1) $2 - \kappa(G) \leq s(G) \leq n - 2\kappa(G)$;

(2) $2 - \lambda(G) \leq s(G) \leq n - 2\lambda(G)$;

(3) $2 - \delta(G) \leq s(G) \leq n - 2\delta(G)$. ■

Corollary 8.2.5
Let G be a noncomplete connected graph of order n. Then $4-n \leq s(G) \leq n-2$. ■

The following result on cycles in graphs is well-known.

Theorem 8.2.6 (Dirac [20])
let G be a 2-connected graph and d an integer. If $d(v) \geq d$ for every vertex v in G, then G contains either a Hamilton cycle or a cycle of length at least $2d$.

With this result and Theorem 8.2.2, we have

Corollary 8.2.7
Let G be a noncomplete connected graph of order n such that $2 - \dfrac{n}{2} \leq s(G) \leq 0$. Then the length of a longest cycle is at least $4 - 2s(G)$. ■

Corollary 8.2.8
Let G be a noncomplete graph of order n such that $s(G) \leq 2 - \dfrac{n}{2}$. Then G is hamiltonian. ■

8.3 Scattering number and toughness

Let G be a noncomplete connected graph. The *toughness* $t(G)$ of G is defined as

$$t(G) = \min\left\{\frac{|X|}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\},$$

where $\omega(G-X)$ is the number of components of $G-X$.

The concept of toughness was introduced by Chvátal [17]. It is an extensively studied graph parameter, which is mainly due to the famous conjecture of Chvátal that there exists a t_0 such that every graph with $t(G) \geq t_0$ is hamiltonian. This parameter is also of particular interest because it is considered to be a reasonable measure for the vulnerability of graphs.

From the definitions of the scattering number and the toughness, it is clear that these two parameters are very similar. In this section, we discuss the relationship between these two parameters.

Theorem 8.3.1
Let G be a noncomplete connected graph such that $\kappa(G) = k$, $s(G) = s$ and $t(G) = t$. Then $t \geq \dfrac{k}{k+s}$.

Proof Let X be a vertex cut of G. Then from the the definition of $s(G)$ we know that $\omega(G-X) - |X| \leq s$. This implies that $|X| \geq \max\{k, \omega(G-X) - s\}$. Denote the order of G by n. Then $\omega(G-X) + |X| \leq n$. Combining this with the first inequality, we have $\omega(G-X) \leq \dfrac{n+s}{2}$. From the second inequality, we obtain that

$$\frac{|X|}{\omega(G-X)} \geq \frac{\max\{k, \omega(G-X) - s\}}{\omega(G-X)}.$$

If $2 \le \omega(G-X) \le k+s$, then

$$\frac{\max\{k, \omega(G-X)-s\}}{\omega(G-X)} = \frac{k}{\omega(G-X)} \ge \frac{k}{k+s};$$

If $k+s \le \omega(G-X) \le \frac{n+s}{2}$, then

$$\frac{\max\{k, \omega(G-X)-s\}}{\omega(G-X)} = \frac{\omega(G-X)-s}{\omega(G-X)} \ge \frac{k}{k+s}.$$

This means that for every vertex cut X of G we have

$$\frac{|X|}{\omega(G-X)} \ge \frac{k}{k+s}.$$

Therefore, the toughness $t(G) = t \ge \frac{k}{k+s}$. ■

Remark 8.3.2

The result in Theorem 8.3.1 is best possible. This can be shown by the graph $G = K_k + (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_{k+s}})$, where $n \ge 3$, $1 \le k \le n-2$, $2-k \le s \le n-2k$, $n_i \ge 1$ for $1 \le i \le k+s$ and $\sum_{i=1}^{k+s} n_i = n-k$.

Theorem 8.3.3

Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $t(G) = t$. Then $t \le \frac{n-s}{n+s}$.

Proof Let X be a vertex cut of G. Then from the definition of $t(G)$ we know that $|X| \ge t\omega(G-X)$. At the same time, combining this with the fact $\omega(G-X) + |X| \le n$ we have that $\omega(G-X) \le \frac{n}{1+t}$. Therefore,

$$\begin{aligned}\omega(G-X) - |X| &\le \omega(G-X) - t\omega(G-X)\\ &= \omega(G-X)(1-t)\\ &\le \frac{n(1-t)}{1+t}.\end{aligned}$$

Since X is an arbitrary vertex cut of G, we have that $s(G) = s \leq \frac{n(1-t)}{1+t}$, i.e., $t \leq \frac{n-s}{n+s}$. ■

Remark 8.3.4

The result in Theorem 8.3.3 is best possible. This can be shown by the graph $G = K_{nt/(1+t)} + \frac{n}{1+t}K_1$, where $n \geq 3$, $\frac{1}{n-1} \leq t \leq \frac{n-2}{2}$, and both $\frac{nt}{1+t}$ and $\frac{n}{1+t}$ are integers.

A graph is called *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. Claw-free graphs have been studied extensively in the last two decades [27]. It was proved in [50] that the toughness of a claw-free graph G is $\frac{\kappa(G)}{2}$, where $\kappa(G)$ is the connectivity of G. From Theorems 8.3.1 and 8.3.3, the following result on the scattering number of claw-free graphs is obvious.

Corollary 8.3.5

Let G be a claw-free graph of order n such that $\kappa(G) = k$. Then $2 - k \leq s(G) \leq \frac{n(2-k)}{2+k}$. ■

Computing the toughness or the scattering number of a graph are both known to be **NP**-hard problems [5, 46]. Since it is clear that computing the toughness of a claw-free graph is polynomial (computing $\kappa(G)$ for an arbitrary graph is polynomial). A natural question is: Can we compute the scattering number of claw-free graphs in polynomial time? After the submission of the draft version of this thesis, H. J. Broersma and the author proved that for a connected claw-free graph G, $s(G) = 2 - \kappa(G)$, implying that computing $s(G)$ for claw-free graphs can be done in polynomial time. In the next chapter we consider this question for several other classes of graphs.

8.4 Scattering number and integrity

The concept of integrity was introduced by Barefoot *et al.* [3, 4] as a vulnerability measure of a graph. The *integrity* of a noncomplete connected graph G is defined as

$$I(G) = \min\{|X| + m(G - X) : X \subset V(G), \omega(G - X) > 1\},$$

where $m(G - X)$ is the order of a largest component of $G - X$.

In this section we consider the relationship between the scattering number and the integrity of graphs.

Theorem 8.4.1
Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $I(G) = I$. Then $I \geq 2\sqrt{n+s} - (s+1)$.

Proof Let X be a vertex cut of G. Then from the definition of $s(G)$ we know that $\omega(G - X) - |X| \leq s$. It is easy to see that $m(G - X) \geq \dfrac{n - |X|}{\omega(G - X)}$. Therefore,

$$\begin{aligned}
|X| + m(G - X) &\geq |X| + \frac{n - |X|}{\omega(G - X)} \\
&= \frac{n + (\omega(G - X) - 1)|X|}{\omega(G - X)} \\
&\geq \frac{n + (\omega(G - X) - 1)(\omega(G - X) - s)}{\omega(G - X)} \\
&= \frac{n + s}{\omega(G - X)} + \omega(G - X) - (s + 1) \\
&\geq 2\sqrt{n + s} - (s + 1).
\end{aligned}$$

Since X is an arbitrary vertex cut of G, it follows from the definition of $I(G)$ that $I \geq 2\sqrt{n + s} - (s + 1)$. ∎

Remark 8.4.2
The result in Theorem 8.4.1 is best possible. This can be shown by the

graph $G = K_{\sqrt{n+s}-s} + \sqrt{n+s}K_{\sqrt{n+s}-1}$, where $n \geq 3$, $4-n \leq s \leq n-2$ and $\sqrt{n+s}$ is an integer. It is easy to check that $s(G) = s$ and $I(G) = 2\sqrt{n+s} - (s+1)$.

Theorem 8.4.3
Let G be a connected noncomplete graph of order n such that $s(G) = s$ and $I(G) = I$. Then $I \leq n - s$.

Proof Let X be a vertex cut of G. Then from the definition of $I(G)$ we know that $|X| + m(G-X) \geq I$. Therefore,

$$\begin{aligned}\omega(G-X) &\leq n - |X| - m(G-X) + 1 \\ &= n + 1 - (|X| + m(G-X)) \\ &\leq n + 1 - I.\end{aligned}$$

On the other hand, since $|X| \geq 1$, we have

$$\omega(G-X) - |X| \leq n - I.$$

By the definition of $s(G)$, we know $s(G) = s \leq n - I$. This implies that $I \leq n - s$. ■

Remark 8.4.4
The result in Theorem 8.4.3 is best possible. This can be shown by the graph $G = K_1 + ((n-m-1)K_1 \cup K_m)$. It is easy to check that $s(G) = n - (m+1)$ and $I(G) = m + 1$.

8.5 Scattering number and tenacity

The concept of tenacity was introduced by Cozzen, Moazzami & Stueckle [18] as a vulnerability parameter of graphs. The *tenacity* of a noncomplete connected graph G is defined as

$$T(G) = \min\left\{\frac{|X| + m(G-X)}{\omega(G-X)} : X \subset V(G), \omega(G-X) > 1\right\}.$$

In this section we consider the relationship between the scattering number and the tenacity.

Theorem 8.5.1

Let G be a noncomplete connected graph of order n such that $s(G) = s$ and $T(G) = T$. Then

$$T \geq \begin{cases} 1 - \dfrac{(s+1)^2}{4(n+s)} & s \geq 3, \\ 1 + \dfrac{2(1-s)}{n+s} & -1 \leq s \leq 2, \\ 1 + \dfrac{n-2-s}{4} & \textit{otherwise}. \end{cases}$$

Proof Let X be a vertex cut of G. Then from the definition of $s(G)$ we know that $\omega(G-X) - |X| \leq s$. It is not difficult to see that $m(G-X) \geq \dfrac{n-|X|}{\omega(G-X)}$. Therefore,

$$\begin{aligned} \frac{|X| + m(G-X)}{\omega(G-X)} &\geq \frac{|X| + \dfrac{n-|X|}{\omega(G-X)}}{\omega(G-X)} \\ &= \frac{(\omega(G-X)-1)|X| + n}{(\omega(G-X))^2} \\ &\geq \frac{(\omega(G-X)-1)(\omega(G-X)-s) + n}{(\omega(G-X))^2} \\ &= \frac{(\omega(G-X))^2 - (s+1)\omega(G-X) + n + s}{(\omega(G-X))^2} \\ &= 1 + \frac{n+s-(s+1)\omega(G-X)}{(\omega(G-X))^2}. \end{aligned}$$

It is also easy to see that $2 \leq \omega(G-X) \leq \dfrac{n+s}{2}$. Now let us determine the minimum value of $\dfrac{n+s-(s+1)\omega(G-X)}{(\omega(G-X))^2}$ when $2 \leq \omega(G-X) \leq \dfrac{n+s}{2}$. Set $f(u) = \dfrac{n+s-(s+1)u}{u^2}$. Then $f'(u) = \dfrac{(s+1)u - 2(n+s)}{u^3}$.

The root of $f'(u) = 0$ is $u = \frac{2(n+s)}{s+1}$ when $s+1 \neq 0$. Since $f''(u) = \frac{6(n+s) - 2(s+1)u}{u^4}$, we have $f''(\frac{2(n+s)}{s+1}) = \frac{(s+1)^4}{8(n+s)^3}$ in the case $s+1 \neq 0$. It follows from Corollary 8.2.4 that $n+s > 0$. Thus, we have $f''(\frac{2(n+s)}{s+1}) > 0$ when $s+1 \neq 0$. We distinguish three cases.

Case 1 $s+1 > 0$.

The function $f(u)$ attains its minimum when $u = \frac{2(n+s)}{s+1}$. It is not difficult to see that $\frac{2(n+s)}{s+1} \geq 2$. On the other hand, $\frac{2(n+s)}{s+1} \leq \frac{n+s}{2}$ if $s \geq 3$, and $\frac{2(n+s)}{s+1} > \frac{n+s}{2}$ if $s < 3$. Therefore,

$$\min_{2 \leq u \leq \frac{n+s}{2}} f(u) = \begin{cases} f(\frac{2(n+s)}{s+1}) & s \geq 3, \\ f(\frac{n+s}{2}) & -1 \leq s \leq 2. \end{cases}$$

$$= \begin{cases} -\frac{(s+1)^2}{4(n+s)} & s \geq 3, \\ \frac{2(1-s)}{n+s} & -1 \leq s \leq 2. \end{cases}$$

Case 2 $s+1 = 0$.

In this case, $f(u) = \frac{n+s}{u^2}$ and it is a decreasing function when $u > 0$. Therefore,

$$\min_{2 \leq u \leq \frac{n+s}{2}} f(u) = f(\frac{n+s}{2}) = \frac{4}{n+s} = \frac{2(1-s)}{n+s}.$$

Case 3 $s+1 < 0$.

In this case $\frac{2(n+s)}{s+1} < 0$ and $f(u)$ is an increasing function when $u \geq \frac{2(n+s)}{s+1}$. Therefore,

$$\min_{2 \leq u \leq \frac{n+s}{2}} f(u) = f(2) = \frac{n-s-2}{4}.$$

From all the discussions above we deduce that

$$\frac{|X|+m(G-X)}{\omega(G-X)} \geq \begin{cases} 1-\dfrac{(s+1)^2}{4(n+s)} & s \geq 3, \\ 1+\dfrac{2(1-s)}{n+s} & -1 \leq s \leq 2, \\ 1+\dfrac{n-s-2}{4} & \textit{otherwise.} \end{cases}$$

Noting that X is an arbitrary vertex cut of G, it follows from the definition of tenacity that

$$T \geq \begin{cases} 1-\dfrac{(s+1)^2}{4(n+s)} & s \geq 3, \\ 1+\dfrac{2(1-s)}{n+s} & -1 \leq s \leq 2, \\ 1+\dfrac{n-s-2}{4} & \textit{otherwise.} \end{cases}$$

■

Remark 8.5.2
The result in Theorem 8.5.1 is best possible. This can be shown by the graph $G_1 = K_{\frac{2(n+s)}{s+1}-s} + \frac{2(n+s)}{s+1} K_{\frac{s-1}{2}}$ such that $3 \leq s \leq n-2$, and both $\dfrac{2(n+s)}{s+1}$ and $\dfrac{s-1}{2}$ are integers; by the graph $G_2 = K_{\frac{n-s}{2}} + \frac{n+s}{2} K_1$ such that $-1 \leq s \leq 2$, and both $\dfrac{n+s}{2}$ and $\dfrac{n-s}{2}$ are integers; and by the graph $G_3 = K_{2-s} + 2K_{\frac{n+s-2}{2}}$ such that $4-n \leq s < -1$ and $\dfrac{n+s-2}{2}$ is an integer.

Theorem 8.5.3
Let G be a noncomplete connected graph of order n such that $\kappa(G) = k$, $s(G) = s$ and $T(G) = T$. Then

$$T \leq \frac{n+1}{s+k} - 1.$$

Proof Let X be a vertex cut of G. Then from the definition of $T(G)$ we know that

$$T \leq \frac{|X| + m(G-X)}{\omega(G-X)}.$$

On the other hand, it is not difficult to see that $|X| + m(G-X) \leq n + 1 - \omega(G-X)$. Therefore,

$$T \leq \frac{n+1-\omega(G-X)}{\omega(G-X)},$$

which implies that $\omega(G-X) \leq \dfrac{n+1}{T+1}$. With the fact $|X| \geq k$, we have

$$\omega(G-X) - |X| \leq \frac{n+1}{T+1} - k.$$

Since X is an arbitrary vertex cut of G, it follows from the definition of $s(G)$ that $s \leq \dfrac{n+1}{T+1} - k$. From Theorem 8.2.2 (a) we know that $s + k \geq 2$. Therefore, we have $T \leq \dfrac{n+1}{s+k} - 1$. ■

Remark 8.5.4

The result in Theorem 8.5.3 is best possible. This can be shown by the graph $G = K_k + (\frac{n-T}{T+1}K_1 \cup K_{\frac{(n+1)T}{T+1}-k})$ such that $1 \leq k \leq n-1$, and $\dfrac{n-T}{T+1}$ is an integer.

Chapter 9

The scattering number of some special classes of graphs

In this chapter, we deal with the complexity aspect of computing the scattering number for some special classes of graphs. The scattering number of grids and of the Cartesian products of two complete graphs is determined. At the same time we prove that the scattering number of split graphs can be computed in polynomial time.

9.1 Introduction

As we pointed out in the previous chapter, the concept of scattering number can be used to describe the vulnerability of graphs. So, it is of prime importance to determine the scattering number of a graph when this parameter is used to measure the vulnerability of networks. It was proved by KRATSCH *et al.* [47] that the problem of computing the scattering number of graphs is **NP**-hard. So it is a very interesting problem to determine (the complexity of computing) the scattering number for some special classes of graphs.

In Section 9.2 we determine the scattering number of grids, and that of hypercubes as a special case. Section 9.3 is devoted to determining the scattering number of the Cartesian products of two complete graphs. In Section 9.4 we prove that the scattering number of split graphs can be computed in polynomial time.

9.2 The scattering numbers of grids and hypercubes

It is well-known that the Cartesian products of graphs, like hypercubes, grids and tori are highly recommended for the design of interconnection networks in multiprocessor computing systems. Hence, there is a large amount of literature containing the study of the vulnerability of these graphs; see [2] and [55]. The aim of this section is to determine the scattering number of grids, and that of hypercubes as a special case.

Theorem 9.2.1
Suppose that $n_1, n_2, \ldots, n_k$ are k integers not less than 2. Then
(1) $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = 1$, *when all n_i are odd;*
(2) $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) = 0$, *otherwise.*

We need the following two lemmas, the first one of which is obvious.

Lemma 9.2.2
Let H be a spanning subgraph of a noncomplete connected graph G. Then $s(H) \geq s(G)$. ■

Lemma 9.2.3 (Ouyang *et al.* [51])
$s(P_n) = 1$ *for* $n \geq 3$; $s(C_n) = 0$ *for* $n \geq 4$; $s(K_{p,q}) = |p-q|$ *for* $p+q \geq 3$.

Proof of Theorem 9.2.1 First note that, if G is a bipartite graph with bipartition (A, B) and H is a bipartite graph with bipartition (C, D), then $G \times H$ is a bipartite graph with bipartition $((A \times C) \cup (B \times D), (A \times D) \cup (B \times C))$. Therefore, if all n_i are odd, we have

$$P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} \subseteq K_{\frac{n_1 n_2 \cdots n_k - 1}{2}, \frac{n_1 n_2 \cdots n_k + 1}{2}}.$$

From Lemmas 9.2.2 and 9.2.3, we know that

$$s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \geq s(K_{\frac{n_1 n_2 \cdots n_k - 1}{2}, \frac{n_1 n_2 \cdots n_k + 1}{2}}) = 1.$$

On the other hand, it is easy to see that if a graph G has a Hamilton path, then $G \times P_n$ also has a Hamilton path. So, the grid $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ has a Hamilton path P. It follows from Lemmas 9.2.2 and 9.2.3 that $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \leq s(P) = 1$, which completes the proof of statement (1).

If some n_i is even, then we have

$$P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k} \subseteq K_{\frac{n_1 n_2 \cdots n_k}{2}, \frac{n_1 n_2 \cdots n_k}{2}}.$$

Hence, from Lemmas 9.2.2 and 9.2.3 we know that

$$s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \geq s(K_{\frac{n_1 n_2 \cdots n_k}{2}, \frac{n_1 n_2 \cdots n_k}{2}}) = 0.$$

It has been proved by several authors that in this case $P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}$ has a Hamilton cycle. We add a proof here for convenience.

Without loss of generality, we can assume that n_1 is even. When $k = 2$, it is easy to find a Hamilton cycle in $P_{n_1} \times P_{n_2}$. Suppose that for $m \geq 2$, the graph $Gr_m = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m}$ has a Hamilton cycle. Consider $Gr_{m+1} = P_{n_1} \times P_{n_2} \times \cdots \times P_{n_m} \times P_{n_{m+1}} = Gr_m \times P_{n_{m+1}}$. Obviously, Gr_{m+1} contains n_{m+1} disjoint subgraphs, each isomorphic to Gr_m. Denote these graphs by $G_1, G_2, \ldots, G_{n_{m+1}}$, and the bijection between G_i and G_j by θ_{ij}. It follows from the definition of the Cartesian product of graphs that there is an edge between each vertex x of G_i and its corresponding vertex $\theta_{ij}(x)$ in G_j for $j = i + 1$ and $1 \leq i \leq n_m$. Note that since

$G_1, G_2, \ldots, G_{n_{m+1}}$ are all isomorphic to G_{r_m}, by the induction hypothesis we can find a Hamilton cycle H_i in G_i $(i = 1, 2, \ldots, n_{m+1})$ such that $H_1, H_2, \ldots, H_{n_{m+1}}$ are also pairwise isomorphic.

Now let us construct a Hamilton cycle in $Gr_{n_{m+1}}$. First, select an edge u_1v_1 on H_1, and set $H_{12} = H_1 \cup H_2 + \{u_1\theta_{12}(u_1), v_1\theta_{12}(v_1)\} - \{u_1v_1, \theta_{12}(u_1)\theta_{12}(v_1)\}$. Then H_{12} is a cycle containing all the vertices of G_1 and G_2. Next, select an edge u_2v_2 in H_2 other than the edge $\theta_{12}(u_1)\theta_{12}(v_1)$, and set $H_{123} = H_{12} \cup H_3 + \{u_2\theta_{23}(u_2), v_2\theta_{23}(v_2)\} - \{u_2v_2, \theta_{23}(u_2)\theta_{23}(v_2)\}$. Then H_{123} is a cycle containing all the vertices of G_1, G_2 and G_3. Repeating the process above, we will finally obtain a cycle H containing all the vertices of $G_1, G_2, \ldots, G_{n_{m+1}}$, which is a Hamilton cycle of Gr_{m+1}.

It follows from Lemmas 9.2.2 and 9.2.3 that $s(P_{n_1} \times P_{n_2} \times \cdots \times P_{n_k}) \leq s(H) = 0$, which completes the proof of statement (2). ■

Since Q_k is isomorphic to the Cartesian product of k copies of P_2, the next consequence of Theorem 9.2.1 is obvious.

Corollary 9.2.4
The scattering number $s(Q_k)$ of the hypercube Q_k $(k \geq 2)$ is 0. ■

Given a graph G on n vertices with scattering number s, it was proved in [51] that the minimum number of edges of G is $\lceil \frac{n(2-s)}{2} \rceil$ when $4-n \leq s \leq 0$, and $n-1$ when $1 \leq s \leq n-2$. Note that the order of the hypercube Q_k is 2^k and the size of Q_k is $k2^{k-1}$. So, hypercubes are not the minimal graphs with respect to the scattering number.

9.3 The scattering number of $K_m \times K_n$

The hypercube Q_k is isomorphic to the Cartesian product of k copies of K_2. In this section, we consider the following

Problem 9.3.1
Determine the scattering number of $K_{n_1} \times K_{n_2} \times \cdots \times K_{n_k}$.

We give an answer to Problem 9.3.1 for the case $k = 2$. The general case seems to be complicated.

Theorem 9.3.2
$s(K_m \times K_n) = 4 - (m + n)$.

Proof Let X be a vertex cut of the graph $G = K_m \times K_n$ such that $s(G) = \omega(G - X) - |X|$. Set $k = \omega(G - X)$, then X must be minimal with $\omega(G - X) = k$. The vertex set of G is of the form $V \times W$, where $|V| = m$ and $|W| = n$. From the minimality of X, we know that the vertex set of the jth component of $G - X$ is $V_j \times W_j$ with $V_j \subset V$ and $W_j \subset W$. Furthermore, $V_i \cap V_j = \emptyset$ and $W_i \cap W_j = \emptyset$ when $i \neq j$. Therefore, we have

$$|X| = mn - \sum_{i=1}^{k} m_i n_i, \tag{1}$$

where $m_i = |V_i|$ and $n_i = |W_i|$ $(i = 1, 2 \cdots k)$.

Now let us determine the minimum value of $mn - \sum_{i=1}^{k} m_i n_i$ in (1), i.e., solve the following nonlinear integer programming problem

$$\max f(M, N) = \sum_{i=1}^{k} m_i n_i$$

$$s.t \begin{cases} \sum_{i=1}^{k} m_i = m; \sum_{i=1}^{k} n_i = n; \\ M = (m_1, m_2, \ldots, m_k); \\ N = (n_1, n_2, \ldots, n_k); \\ m_i, n_i \ (\geq 1) \text{ are integers for } \ i = 1, \ldots, k. \end{cases}$$

Suppose that $(M^0, N^0) = (m_1^0, m_2^0, \ldots, m_k^0, n_1^0, n_2^0, \ldots, n_k^0)$ is an arbitrary feasible solution of the above nonlinear programming problem. By the symmetry of m_i and m_j, we may assume that $m_1^0 \le m_2^0 \le \cdots \le m_k^0$. Let n_j^0 be the first number larger than 1 among $n_1^0, n_2^0, \ldots, n_k^0$. Construct a new feasible solution

$$(M^1, N^1) = (m_1^0, m_2^0, \ldots, m_k^0, \underbrace{1, 1, \ldots, 1}_{j}, n_{j+1}^0 + n_j^0 - 1, n_{j+2}^0, \ldots, n_k^0).$$

It is clear that $f(M^1, N^1) \ge f(M^0, N^0)$. Repeating the above process, we can finally get a feasible solution

$$(M', N') = (m_1^0, m_2^0, \ldots, m_k^0, \underbrace{1, 1, \ldots, 1}_{k-1}, n - k + 1).$$

Now by using the nondecreasing value of the new n-vector, applying a similar process with respect to M', we get a feasible solution

$$(M'', N'') = (\underbrace{1, 1, \ldots, 1}_{k-1}, m - k + 1, \underbrace{1, 1, \ldots, 1}_{k-1}, n - k + 1).$$

Since (M^0, N^0) is an arbitrary feasible solution, we know that (M'', N'') is optimal. That is to say, the right hand side of (1) is minimized by $m_1 = m_2 = \cdots = m_{k-1} = 1, m_k = m - k + 1$ and $n_1 = n_2 = \cdots = n_{k-1} = 1, n_k = n - k + 1$, or a symmetric solution. Thus, we have

$$\begin{aligned} |X| &\ge mn - (k-1) - (m-k+1)(n-k+1) \\ &= (k-1)(m+n-k). \end{aligned}$$

This implies that

$$\begin{aligned} s(G) &= \omega(G - X) - |X| \\ &\le k - (k-1)(m+n-k) \\ &= k^2 - (m+n)k + m + n. \end{aligned}$$

It is easy to see that $g(k) = k^2-(m+n)k+m+n$ is a decreasing function when $k \leq \dfrac{m+n}{2}$. Moreover, it follows from $\sum_{i=1}^{k} m_i = m$ and $\sum_{i=1}^{k} n_i = n$ that $k \leq (m+n)/2$. Combining this with the fact $k \geq 2$ we obtain that

$$s(G) \leq k^2 - (m+n)k + m + n \leq g(2) = 4 - (m+n).$$

On the other hand, $\kappa(G) = m+n-2$ and $s(G) \geq 2-(m+n-2) = 4-(m+n)$ by Theorem 8.2.2. Thus we conclude that $s(G) = s(K_m \times K_n) = 4-(m+n)$. ■

9.4 The scattering number of split graphs

A graph $G = (V, E)$ is called a *split graph* (cf. [28, 37]) if its vertex set can be partitioned into an independent set I and a clique C. Usually, a split graph is denoted by $G = (C, I, E)$.

In [56], Woeginger proved that the toughness of split graphs can be computed in polynomial time. As we shown in Section 8.3, the definition of scattering number is very similar to that of toughness, and these two parameters are closely related. This leads us to consider the problem of computing the scattering number of split graphs. In this section we will show that the scattering number of split graphs can be computed in polynomial time, too. The proof is a minor modification of the proof given in [56]. We present it here just for the sake of completeness.

Theorem 9.4.1
The scattering number of split graphs can be computed in polynomial time.

Proof Let $G = (C, I, E)$ be a noncomplete connected split graph with n vertices, and let s be an integer with $4-n \leq s \leq n-2$. Our goal is to decide whether there exists a vertex cut X^* of G such that $\omega(G-X^*) > |X^*|+s$. In case such a vertex cut X^* does exist, one may assume without loss of generality, that $X^* \subseteq C$: otherwise, replace X^* by $X^* \cap C$. This does not increase $|X^*|$ and cannot decrease $\omega(G - X^*)$.

If $X^* = C$, then $\omega(G - X^*) = |I|$ holds, and this case is trivial to check. Otherwise, $X^* \subsetneqq C$ holds. Then $\omega(G - X^*)$ equals the number of vertices $v \in I$ with $N(v) \subseteq X^*$ plus one. Hence, the problem boils down to deciding whether there exists a proper subset U of I such that

$$|N(U)| + s < |U| + 1,$$

or in other words, to decide whether there exists a proper subset $U \subset I$ for which

$$f(U) := |N(U)| - |U| < 1 - s. \tag{2}$$

Observe that $f(U)$ is a *submodular* function $2^I \longrightarrow \mathbb{R}$, i.e., that $f(A) + f(B) \geq f(A \cup B) + f(A \cap B)$ holds for all $A, B \subseteq I$ (cf. e.g., [39]). Therefore, the minimum of $f(U)$ can be determined in polynomial time by the ellipsoid method [39] or by Cunningham's combinatorial algorithm [19] (Although the running time of Cunningham's algorithm is only pseudopolynomial in the value of $f(U)$, it is strictly polynomial in our case: the values of $f(U)$ are polynomially bounded in $|V|$). Consequently, one can decide in polynomial time whether there exists an set U that fulfills inequality (2).

It follows from Corollary 8.2.4 that the scattering number of a non-complete connected graph on n vertices is an integer between $4 - n$ and $n - 2$. Therefore, for a split graph G of order n, one can enumerate all these numbers in polynomial time and check whether $s(G) \leq s$. ■

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Index

Summary

This thesis contains results on paths and cycles in graphs and on a more or less related topic, the vulnerability of graphs. In the first part of the thesis, Chapters 2 through 5, we concentrate on paths and cycles in weighted graphs. A number of sufficient conditions are presented for graphs to contain paths and cycles with certain properties, in particular heavy (or long) paths and cycles. The second part of the thesis, Chapters 6 and 7, contains some basic results on paths and cycles in (edge-) colored graphs. In the third part of the thesis, Chapters 8 and 9, we focus on a graph parameter which can be used to analyze the vulnerability of graphs, i.e., the scattering number of graphs.

In Chapter 1 we present a general introduction to the topics of the thesis and give an overview of the main results we obtained, together with some connections with older results. Specific terminology and notation can be found just before each of the three topics.

In Chapter 2 we provide a sufficient condition for a weighted graph to contain heavy cycles passing through a given vertex or Hamilton cycles. This result is a common generalization of Grötschel's theorem and Bondy and Fan's theorem assuring the existence of heavy cycles or Hamilton cycles in weighted graphs with the same condition. Also, as a tool for proving this result, we show a result concerning heavy paths joining two specific vertices and passing through one given vertex.

In Chapter 3 we give a generalization of a well-known result of Fan on the existence of long cycles in unweighted graphs. It is shown that to generalize Fan's theorem to weighted graphs, some extra conditions must be added.

In Chapter 4 we give a σ_3 type condition for heavy cycles in weighted graphs. Our result generalizes a theorem of Fournier and Fraisse on long cycles in k-connected unweighted graphs in the case $k = 2$.

In Chapter 5 we give two theorems on long paths and cycles in weighted graphs. These results generalize a result of Qiao on the existence of long paths in unweighted graphs and extend an earlier result of Bondy and Fan on the existence of heavy cycles in weighted graphs.

In Chapter 6 we start a new topic which has been less studied, namely paths and cycles in colored graphs. Some sufficient conditions for the existence of (long) monochromatic paths and cycles, and those for the existence of long heterochromatic paths and cycles are presented.

In Chapter 7 we prove that the problem of finding a path (cycle) with the fewest colors in a colored graph is **NP**-hard. Several exact and approximation algorithms for finding a path with the fewest colors are provided. The complexity of the exact algorithms and the performance ratio of the approximation algorithms are analyzed. We also pose a conjecture on the existence of paths and cycles with many different colors.

In Chapter 8 the relationships between the scattering number and some other vulnerability parameters of graphs, namely the (edge-) connectivity, toughness, integrity and tenacity are studied.

In Chapter 9 we deal with the complexity aspect of computing the scattering number for some special classes of graphs. The scattering number of grids and of the Cartesian products of two complete graphs are determined. We also prove that the scattering number of split graphs can be computed in polynomial time.

Curriculum Vitae

Shenggui Zhang was born on October 27, 1968, in Zizhou County of Shaanxi Province, P.R. China. From 1976 until 1986 he attended primary and middle school in his hometown. In September 1986, he started to study pure mathematics at Shaanxi Normal University in Xi'an. After receiving his Bachelor's degree in July 1990, he became a graduate student at Northwestern Polytechnical University. He specialized in operations research and completed his Master's degree thesis, entitled 'The Breaktivity of Graphs', under the supervision of Professor Ziguo Wang. In April 1993, he received his Master's degree and began his job as a teacher at Northwestern Polytechnical University. He has been teaching mathematics and doing research on graph theory at this university since then.

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