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Optimal control of linear, stochastic systems with state and input constraints

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Abstract

In this paper we extend the work presented in the papers [1–3] where we considered optimal control of a linear, discrete time system subject to input constraints and stochastic disturbances. Here we basically look at the same problem but we additionally consider state constraints. We discuss several approaches for incorporating state constraints in a stochastic optimal control problem. We consider in particular a soft-constraint on the state constraints where constraint violation is punished by a hefty penalty in the cost function. Because of the stochastic nature of the problem, the penalty on the state constraint violation can not be made arbitrary high. We derive a condition on the growth of the state violation cost that has to be satisfied for the optimization problem to be solvable. This condition gives a link between the problem that we consider and the well known H_∞ control problem.

1 Introduction

In an industrial environment, the ability of the control system to efficiently deal with constraints in the process is of increasing importance. The reason is that the most profitable operation of the industrial plant is often obtained when a process is running at a constraint boundary (see [6]). It is often claimed that the increasing popularity of Model Predictive Control (MPC) in the industry stems from its capability to allow operation closer to constraint boundaries, when compared with conventional control techniques. When disturbances are acting on the plant which one aims to control, then it is evident that the better the control system is dealing with disturbances the closer to the constraint boundaries one can operate the plant.

When disturbances acting on the plant are stochastic, the classical MPC setting for which there exists a vast literature (see [4], [8], [7]) based on convex on-line optimization is faced with a difficulty. The difficulty with a stochastic disturbance in MPC is that the predicted behavior and the actual behavior of the plant can differ significantly. The standard, convex optimization in open loop does not take the difference between actual and predicted behavior of the plant into account. As a consequence, questions related to achievable performance can not be addressed properly, while the optimization criterion largely ignores the true characteristics of the plant. Hence the input is chosen on the basis of a criterion which does not reflect the true characteristic of the plant. Unfortunately, when a controller is designed in closed loop, constraints make a minimization of the expected value of the cost function over the horizon a very difficult optimization problem.

In the papers [1–3] we presented a stochastic disturbance rejection scheme for MPC based on a randomized algorithm which minimizes an empirical mean of the cost function. The optimization at each step is a closed loop optimization. Therefore it takes the effect of disturbances into account. Because we do not impose any a priori parameterization of the feedback laws over the horizon, the algorithm is computationally demanding but it gives a reliable measure of the achievable performance.

In this paper we extend our research further. The system that we consider is a linear, time invariant, discrete time system with constraints on the input and the state, subject to a stochastic disturbance. We pose our problem as an optimal control problem for the stochastic, constrained system with a cost function that is not necessarily quadratic and discuss possible approaches to the optimal control problem for the system with constraints on the state (section 2). Because of the stochastic nature of the problem, the penalty on the state constraint violation can not be made arbitrary high. We derive a condition on the growth of the state violation cost that has to be satisfied for the optimization problem to be solvable. This condition gives a link between the problem that we consider and the well known H_∞ control problem (section 3). We briefly describe our algorithm in section 4. In section 5 we present an example to illustrate the presented technique. Because of page limitations all the proofs have been omitted.

2 Problem formulation

We consider a linear, time-invariant plant subject to stochastic disturbances. The plant is described with the following state space model:

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$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + Ew(t) \\ z(t) &= C_z x(t) + D_z u(t) \end{aligned} \tag{1}$$

where u is the control input with $u(t) \in \mathbf{U} \subset \mathbb{R}^m$ and x is the state with $x(t) \in \mathbb{R}^n$. The set $\mathbf{U}$ is a closed, convex set which contains an open neighborhood of the origin. The second equation describes the controlled output z with $z(t) \in \mathbb{R}^p$. Finally, the disturbance w is a normally distributed stationary white noise stochastic process with mean 0 and covariance matrix $Q_w \in \mathbb{R}^{\ell\times\ell}$.

The system (1) is controlled by a *static feedback controller* i.e. at each t, the input $u(t)$ is a function of the state $x(t)$. The class of controllers Ψ that we consider is the set of continuous maps $\varphi : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbf{U}$ that map the origin of the state space into the zero input: $\varphi(0,t) = 0$ for all $t \in \mathbb{Z}_+$. In other words, we have

$$u(t) = \varphi(x(t), t)\,. \tag{2}$$

for some $\phi \in \Psi$. Starting at time $t = 0$, the state x and the output z are stochastic processes generated by (1) with the input (2).

We consider a linear, time invariant system that is subject to a stochastic disturbance, with state constraints and a constrained input. It is well known, that a constrained input limits our ability to control the linear plant. To approach this in a more formal way, consider the system:

$$x(k+1) = Ax(k) + Bu(k) \qquad u(k) \in \mathbf{U}. \tag{3}$$

Suppose that at time $t = 0$ system (3) has an initial state $x_0 \in \mathbb{R}^n$. Further, suppose that the state x is generated by (3) with input (2). If there exists a controller $\varphi \in \Psi$ such that:

$$x(t) \to 0 \quad \text{as} \quad t \to \infty$$

we say that the state x_0 is a *null controllable point* in the state space. All null controllable points define a set in the state space which is known as the *recoverable set*, here denoted as $\underline{\mathbf{X}}$. In general, the recoverable set is a subset of the state space. If $\mathbf{U}$ is bounded then the recoverable set contains all points in the state space if and only if the matrix pair (A, B) is stabilizable and all eigenvalues of the system matrix A lie on or inside the unit circle. The recoverable set is the whole state space if and only if the system (3) is *globally asymptotically stabilizable* (see [9] for details when $\mathbf{U}$ is bounded). For the case that $\mathbf{U}$ is unbounded, the characterization when the recoverable set is equal to the whole state space is a bit more complex but relatively straightforward. In any case, the following assumption is natural when one deals with the stabilization of a linear system, subject to input constraints and unbounded disturbances.

Assumption 1 *The system* (1) *is globally asymptotically stabilizable. As a consequence* $\underline{\mathbf{X}} = \mathbb{R}^n$.

Next, suppose that constraints on the state x define a convex, closed set $\mathbf{X} \subseteq \mathbb{R}^n$ that contains the origin in its interior. The output z is used to measure performance. Our objective is to control plant (1) from an initial state to the origin in such a way that the size of the controlled output z is as small as possible while $x(t) \in \mathbf{X}$ and $u(t) \in \mathbf{U}$ for all t. Our performance measure, usually called the cost, is a convex function of the output z. A number of efficient algorithms exist that can be used to minimize a convex function. However, the dynamic structure of the problem makes this optimization far from trivial. The controlled output z is a stochastic process because it depends on the stochastic disturbance w. It is necessary to consider the expected value of the size of the output z, otherwise our performance measure would be stochastic. Thus, we consider the following performance measure for system (1):

$$P(x_0, \varphi) := \lim_{T\to\infty} \mathbb{E}\,\frac{1}{T}\sum_{t=0}^{T} g\,(z(t)) \tag{4}$$

where:

$$\begin{aligned} x(t+1) &= Ax(t) + B\varphi(x(t),t) + Ew(t) \\ z(t) &= C_z x(t) + D_z\varphi(x(t),t) \end{aligned}$$

and where $\mathbb{E}$ denotes expectation. A function $g : \mathbb{R}^p \to \mathbb{R}_+$ is a strictly convex function with $g(0) = 0$. Note that z is an affine function in x and u and therefore the composite function g is strictly convex in x and u.

A straightforward approach to the design of an optimal controller for the system (1) with constraints on the state, based on the performance measure (4) is stated in the following problem formulation.

Problem 1 *Suppose that at time* $t = 0$ *system* (1) *has an initial condition* $x(0) = x_0$, $x_0 \in \mathbf{X}$. *Under assumption 1, find an optimal controller* $\varphi^* \in \Psi$ *such that:*

$$x(t) \in \mathbf{X}, \qquad u(t) \in \mathbf{U} \tag{5}$$

for all $t \in \mathbb{Z}_+$ *and*

$$P(x_0, \varphi^*) \leq P(x_0, \varphi)$$

for all other controllers $\varphi \in \Psi$ *which guarantee* (5). *In addition, determine the optimal cost* $P(x_0, \varphi^*)$.

Problem 1 is a stochastic, optimal control problem with constraints on the state and the input. No constraint violation is allowed and the problem resembles what is known as the *hard constraint* approach. One difficulty with the problem 1 is that the set of admissible initial conditions

$$\Big\{x_0 \in \mathbf{X} : \text{there exist } \varphi \text{ such that (4) is finite and } x(t) \in \mathbf{X},\ u(t) \in \mathbf{U} \text{ for all } t \in \mathbb{Z}_+\Big\} \tag{6}$$

is almost always empty when w is unbounded (normally distributed w is a typical example). An empty set of admissible initial conditions implies that problem 1 is unsolvable.

In the case that even though the disturbance is bounded, the disturbances can still be quite large, the set of admissible initial conditions (6) can be very small, which is too restrictive in many practical applications.

In this paper, we propose an approach for dealing with constraints on the state of stochastic systems. An optimal controller should control the plant optimally with respect to the performance measure (4) while keeping the state in the set $\mathbf{X}$ "as much as possible". When the state is in the set $\mathbf{X}$ the performance measure (4) determines the performance. When there is a high probability of a constraint violation, the performance of the system is determined by an additional cost that will penalize the constraint violation. In this way, we have two different regimes in which an optimal controller should work: minimizing the performance measure (4) when there is no constraint violation and minimizing the probability of constraint violation when the state is close to the boundary of the set $\mathbf{X}$. This idea is of course not new. For instance the paper [5] explicitly incorporates the probability of contraint constraints in its problem formulation. But in that paper over the prediction horizon, the authors use an open loop input signal while we design a closed loop controller over the prediction horizon and we can therefore better respond to the disturbance signal and in this way avoid a contraint violation.

In our setting, state constraints are incorporated by an additional cost that will penalize constraint violation.

Definition 1 *The constraint violation cost is a convex function* $h : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$ *with* $h(x) = 0$ *for all* $x \in \mathbf{X}$.

The state x depends on the stochastic disturbance w and is therefore stochastic itself. We consider the expected value of the constraint violation cost. The performance measure (4) and the expected value of the constraint violation cost are added in the cost function to reflect both requirements:

$$J(x_0, \varphi) \ := \ \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T}\Big\{g\big(z(t)\big) \ + \ h\big(x(t)\big)\Big\}. \quad (7)$$

Consider the following optimization problem:

Problem 2 *Given an initial condition* $x_0 \in \mathbf{X}$, *find an optimal controller* $\tilde{\varphi} \in \Psi$ *such that*

$$J(x_0, \tilde{\varphi}) \leq J(x_0, \varphi)$$

for all $\varphi \in \Psi$. *In addition, determine the* optimal cost *given by:*

$$V(x_0) := \inf_{\varphi} J(x_0, \varphi). \quad (8)$$

The optimization problem 2 is an optimal control problem of a linear discrete time system subject to stochastic disturbances and only constraints on the input. The constraints on the state have after all been incorporated implicitly by the modified cost function. The same modification for input constraints is not natural since a controller can always avoid constraint violation of the input signal which is not the case for state constraints.

The constraint violation cost h introduces an additional degree of freedom in the design of an optimal controller for the system (1). It determines the strategy in dealing with the state constraints. A choice:

$$h(x) = 0, \qquad x \in \mathbb{R}^n$$

would imply an optimal control problem without constraints on the state. Setting h to be:

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{X} \\ \infty & \text{if} \quad x \notin \mathbf{X} \end{cases} \quad (9)$$

makes problem 2 identical to problem 1 i.e. the hard constraints approach. In between these two extreme cases there are a large number of choices to tailor the cost (7) for the application at hand. Note however, that any choice that will make the constraint violation cost infinite in some point even for large x will make the set of admissible initial conditions (6) almost always empty. The following assumption is therefore necessary:

Assumption 2 *The constraint violation cost* h *is a finite-valued convex function i.e.* $h : \mathbb{R}^n \to \mathbb{R}_+$ *with* $h(x) = 0$ *for all* $x \in \mathbf{X}$, *instead of* $h : \mathbb{R}^n \to \mathbb{R}_+ \cup \{\infty\}$.

Assumption 2 is not very restrictive, simply because the growth of the constraint violation cost h can be made almost arbitrary large with assumption 2 satisfied. For example, consider the constraint violation cost that satisfies assumption 2 and has an exponential growth away from the boundary of $\mathbf{X}$:

$$h(x) = \begin{cases} 0 & \text{if} \quad x \in \mathbf{X} \\ e^{\gamma \|x\|} & \text{if} \quad x \notin \mathbf{X} \end{cases} \quad (10)$$

With γ large enough (10) can be made arbitrary large. Having a large γ will mean a tighter control with respect to the state constraints and is therefore an advantage. In general, when w is unbounded, there is an interesting question to be answered: How large γ can be made such that the optimization problem 2 still yields a finite cost? The answer to the question above can be deduced from the inequality given in the result presented in the following section. The inequality relates the growth of the constraint violation cost h and the decay of the probability density function of the disturbance. It gives an additional insight in the optimization problem (2) and, surprinsigly, relates finiteness of $V(x_0)$ to an H_∞ type condition.

3 Main result

Before presenting the result, we rewrite the cost (7) in more compact form as:

$$J(x,u) := \lim_{T\to\infty} \mathbb{E}\frac{1}{T}\sum_{t=0}^{T} j\big(x(t),u(t)\big)$$

where x is the state of the system generated recursively by (1) with the disturbance w and the input u given by (2) starting at $t = 0$ with an initial condition $x_0 \in \mathbf{X}$. The function j is defined as:

$$j\big(x,u\big) := g\big(x,u\big) + h(x) \quad x \in \mathbb{R}^n \quad u \in \mathbf{U}. \tag{11}$$

Consider the class $\Theta(R)$ of functions $\theta : \mathbb{R}^{n+m} \to \mathbb{R}$ for which there exist nonzero polynomials q and p such that:

$$q(x,u)e^{\|x\|_R^2} \le \theta(x,u) \le p(x,u)e^{\|x\|_R^2}$$

for all $x \in \mathbb{R}^n$ and $u \in \mathbf{U}$ where $\|x\|_R^2 := \langle x, Rx\rangle$ with $R \in \mathbb{R}^{n\times n}$. Functions $\theta \in \Theta$ have a so called "Polynomial - Exponential Growth".

A solvability condition for problem 2 is stated in the following theorem and relates the question of a finite cost to an H_∞ control problem. We first define the following set:

$$\mathbf{U}_c = \left\{u \in \mathbb{R}^m \mid \exists\{u_n\}_{n=1}^{\infty} \subset \mathbf{U}, \lim_{n\to\infty}\frac{u_n}{n} = u\right\}$$

Since $0 \in \mathbf{U}$ and $\mathbf{U}$ is convex it is easily checked that $\mathbf{U}_c$ is a cone. We define Ψ_c as the set of continuous maps $\varphi : \mathbb{R}^n \times \mathbb{Z}_+ \to \mathbf{U}_c$ that map the origin of the state space into the zero input

Theorem 1 *Consider problem 2 with $w \in \mathcal{N}(0, Q_w)$ and an initial condition $x_0 \in \mathbf{X}$. In addition to assumptions 1 and 2 assume the following:*

$$j(x,u) \in \Theta(R) \quad \textit{for all} \quad x \in \mathbb{R}^n \textit{ and } u \in \mathbf{U}. \tag{12}$$

Then, $V(x_0) < \infty$ if there exists a feedback controller $\varphi \in \Psi_c$ such that:

$$\sum_{t=0}^{\infty} \|x(t)\|_R^2 - \frac{1}{2}\|w(t)\|_{Q_w^{-1}}^2 \le 0 \quad \textit{with} \quad x(0) = x_0. \tag{13}$$

Condition (13) points out a connection between optimal control of stochastic systems with cost functions that have at most an exponential growth and the well known H_∞ optimal problem. In [10] the connection between the optimization that arises in certain class of risk-sensitivity problems and the H_∞ control problem is studied. Here, we are dealing with the stochastic, optimal control problem for the linear systems subject to constraints on the state and the (hard) constraint on the input.

We note that the above theorem includes some interesting special cases. In case $\mathbf{U}$ is bounded the set $\mathbf{U}_c$ contains only zero and the criterion (13) needs to be satisfied for the open loop system. Another special case is the case that $\mathbf{U}_c$ is a subspace which we obtain when for instance the set $\mathbf{U}$ is symmetric around 0. In this case (13) is equivalent to a classical H_∞ control problem and therefore easily verified.

The solvability condition presented in theorem 1 is derived under assumption (12) so it can be applied in the case when the constraint set $\mathbf{X}$ is a convex and bounded set. For the case when $\mathbf{X}$ is convex but unbounded it can be expected that the solvability condition (13) is conservative.

By using the result presented in theorem 1 we can check solvability of the optimization problem 2 for a maximal growth of the constraint violation cost h, determined by R. That can be done by solving an H_∞ optimization problem. The optimization problem 2 is still a difficult problem to solve. A way to tackle the optimization problem 2 is to design a model predictive controller. The resulting controller will not be the optimal one for the optimization problem (2) but the approximation with the predictive controller can be arbitrary good, depending on the size of the control horizon. The simplification is due to the fact that we only have to consider a finite number of constraints and we can use dynamic programming with a finite number of steps.

The next section briefly points out how to solve this problem numerically.

4 Algorithm

As pointed out in [1–3] the standard, convex optimization in open loop that is prevailing in the MPC literature can not be applied when stochastic disturbances are considered, because it is not possible to reduce the variance of the state over the control horizon without the control in closed loop. Algorithms presented in previous papers are based on the computation of the empirical mean. With suitable modifications and extensions the same approach can be used to develop an algorithm that will solve the optimization problem 2. A controller that satisfies condition (13) can be used as a feasible initial point for the algorithm.

In model predictive control we basically have to optimize a finite horizon control problem:

$$J(x,u) := \mathbb{E}\sum_{t=0}^{T} j\big(x(t),u(t)\big)$$

However, we have to optimize in closed loop and this feedback obviously has to preserve the causality structure. Basically the problem is equivalent to solving:

$$\mathbb{E}_{w(0)} \inf_{u(0)} \mathbb{E}_{w(1)} \inf_{u(1)} \cdots \mathbb{E}_{w(T)} \inf_{u(T)} \sum_{t=0}^{T} j\big(x(t),u(t)\big)$$

where $\mathbb{E}_{w(i)}$ denotes the conditional expectation with respect to $w(i)$. In [2] it is clarified that by replacing the expectations in the above optimization by an emperical mean we can reduce the above problem to a finite-dimensional convex optimization problem. It is still a computationally intensive optimization but we can at least obtain an arbitrarily accurate estimate of the optimal cost. Due to space limitations, we are not able to describe the algorithm in detail.

5 An example

In this section we present an example in which we consider a "double integrator" system of the form:

$$\begin{aligned} x(k+1) &= \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(k) + \begin{pmatrix} 1 \\ 0 \end{pmatrix} w(k) \\ z(k) &= \begin{pmatrix} 0 & 0 \\ 0.7 & 0 \\ 0 & 0.7 \end{pmatrix} x(k) + \begin{pmatrix} 0.33 \\ 0 \\ 0 \end{pmatrix} u(k) \end{aligned} \tag{14}$$

The input is constrained by:

$$-0.5 \le u \le 0.5 \quad u \in \mathbb{R}.$$

The disturbance is a normally distributed random variable with zero mean and variance 0.2:

$$w \in \mathcal{N}(0, 0.2) \quad w \in \mathbb{R}.$$

The state x is parameterized as:

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

and we impose a constraint on the state:

$$x_2 \ge 0.$$

It is assumed that the system has an initial state:

$$x(0) = \begin{pmatrix} 0 \\ 10 \end{pmatrix}.$$

The task is to steer the system (14), subject to the stochastic disturbance, from the initial state to the origin with the constrained input while respecting constraint on the state. With this aim, we design a stochastic model predictive controller that is based on the disturbance sampling and utilizes approach to the control of stochastic systems subject to constraints on the state and the input described in problem 2. For a more detailled description of the design of such a controller we refer to [1–3]. We denote this controller as a **stochastic MPC** controller. For our cost function we choose:

$$g(z) = \|z\|^2 \quad z \in \mathbb{R}^3 \tag{15}$$

and as penalty for constraint violation we use:

$$h(x) = \begin{cases} 0 & \text{if } x_2 \ge 0 \\ e^{100x_2^2} - 1 & \text{if } x_2 < 0 \end{cases} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 \tag{16}$$

With functions g and h as above, the controller minimizes the expectation of the quadratic cost when the state is away from the constraint $x_2 > 0$. When the state is near or on the boundary of the constraint the exponential constraint violation cost h dominates and the main objective of the controller is to avoid a constraint violation.

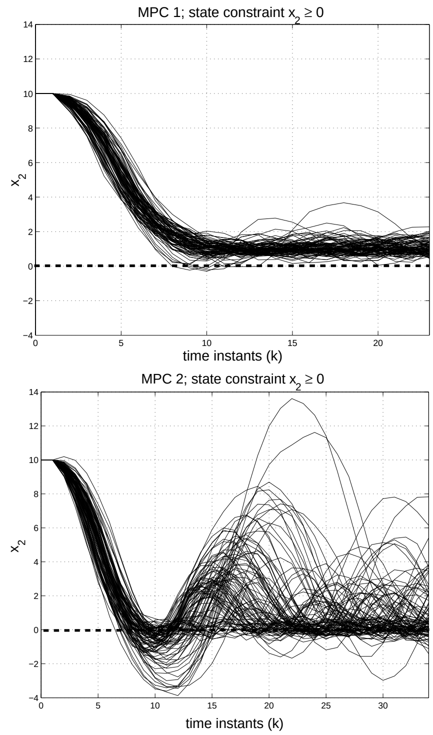

Figure 1: State trajectories for the "double integrator" example

As a reference, we use a "standard" model predictive control scheme that is based on the optimization in open loop, with the length of the control horizon $N = 5$. Note that both state and input constraints are explicitly incorporated but we assume that over the prediction horizon the disturbance is equal to its mean value. We denote this controller as a **standard MPC** controller. In Standard MPC controller the disturbance is assumed to be equal to its mean value over the control horizon. We perform two sets of simulations. In one of them the system (14) is controlled by Stochastic MPC controller and in another one with controller Standard MPC. In each set of simulations there is a 100 simulations, each one of them performed with the different realization of the disturbance w. The resulting trajectories of x_2 are plotted on figure 1. When the system is "far" from the constraint boundary, both controllers show similar performance. When the state of the system is near or on the boundary of the state constraint the standard MPC controller is not able to realistically predict a possibility of the constraint violation, because of the assumption that the disturbance in the "next time step" over the control horizon is equal to the mean value of the disturbance, in this case zero. A probability that w will be smaller than zero is high so for a large number of disturbance trajectories the state constraint is violated. On the contrary, Stochastic MPC controller computes the optimal map from the state to the input for a number of points in the state space. These points are determined by a stochastic sampling of the disturbance and therefore we can expect that the number of sample points is larger in regions where the state of the system is likely to be. The Stochastic MPC controller takes into an account a possibility of the constraint violation when the state of the system is near the boundary of the constraint. This leads to the more realistic “prediction” and the control strategy that respects the state constraints better.

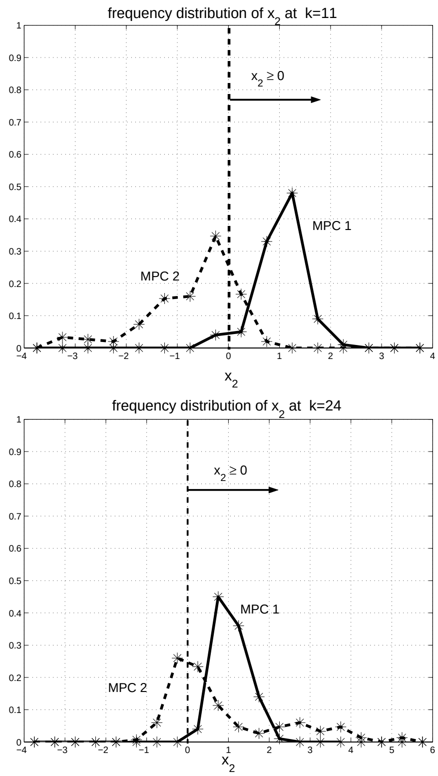

Figure 2: Frequency distributions of the state

To support that further, we compute a frequency distribution of trajectories at $k = 11$ (“overshoot” region) and the “steady state” region at the end of simulations. Results are shown in figure 2.

6 Conclusion

In this paper we extend the work from [1–3] by considering state constraints in addition to constraints on the input. An obvious way to extend the problem is in direction of so called hard constraints methodology. We show that this is not possible for unbounded disturbances. A typical example of such disturbances is well known Gaussian white noise. An alternative is to allow constraint violation but to keep a possibility of the state constraint violation small. That is accomplished by introducing an additional cost that penalizes constraint violation. In the approach described in this paper, it is natural to ask for a large penalty so that the state is kept within constraints as much as possible. However, the penalty can not be arbitrary large. We present a condition on the growth of the penalty function. That condition connects the optimization problem that we consider with the well known H_∞ control problem. We show how to check solvability of the optimization problem for an exponential growth of the state constraint violation cost.

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