Optimal Approximation of Linear Operators: a Singular Value Decomposition Approach

Hardy B. Siahaan[1], Siep Weiland[1], Anton A. Stoorvogel[2,3]

[1]**Department of Electrical Engineering**
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
Email: h.siahaan@tue.nl, s.weiland@tue.nl

[2]**Department of Mathematics and Computing Science**
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
Email: a.a.stoorvogel@tue.nl

[3]**Department of Information Technology and Systems**
Delft University of Technology
P.O. Box 5031, 2600 GA Delft
The Netherlands
Email: a.a.stoorvogel@its.tudelft.nl

Abstract

The purpose of this paper is to propose a definition of a set of singular values and a singular value decomposition associated with a linear operator defined on arbitrary normed linear spaces. This generalizes the usual notion of singular values and singular value decompositions to operators defined on spaces equipped with the p-norm, where p is arbitrary. Basic properties of these generalized singular values are derived and the problem of optimal rank approximation of linear operators is investigated in this context. We give sufficient conditions for the existence of optimal rank approximants in the p-induced norm and discuss an application of generalized singular values for the identification of dynamical systems from data.

Keywords

optimal approximation, singular values, rank reduction, linear systems, optimal identification.

1 Introduction

Singular values and singular value decompositions are among the most important tools in linear algebra that have played a key role in systems analysis, control system design, model reduction, data compression, perturbation theory, signal analysis and many applications in numerical linear algebra. Unlike eigenvalues and eigenvalue decompositions, singular values and singular value decompositions provide structural information on the spacial distribution of mutually orthogonal amplification directions in the domain and co-domain of a linear map. As such, the singular value decomposition defines a numerically well conditioned basis for both the domain and the co-domain of a linear operator and is, in fact, the core numerical tool to implement basic algebraic concepts such as rank, null space, range, orthogonal complements, etc.

A basic algebraic treatment of singular values and their applications can be found in the standard works [2], [4]. In short, every matrix $M \in \mathbb{C}^{m\times n}$ admits a decomposition of the form

$$M = Y\Sigma X^* \tag{1.1}$$

where $X \in \mathbb{C}^{n\times n}$ and $Y \in \mathbb{C}^{m\times m}$ are orthogonal matrices and $\Sigma \in \mathbb{R}^{m\times n}$ is a matrix whose diagonal entries $(\Sigma)_{ii} = \sigma_i$, $i = 1, \ldots, \min(m, n)$, and which is zero elsewhere. Here, σ_i are non-negative real numbers, ordered according to $\sigma_1 \geq \ldots \geq \sigma_{\min(m,n)} \geq 0$ and called the *singular values* of M. The column vectors x_i of X and y_i of Y are the right and left *singular vectors* and equation (1.1) is referred to as a *singular value decomposition* of M. From (1.1) it follows that M allows a *diadic expansion* $M = \Sigma_{k=1}^{r}\sigma_i y_i x_i^*$, where $r = \operatorname{rank} M$.

The decomposition (1.1) proves useful for a wide variety of problems. It is the purpose of this paper to propose a generalization of this traditional notion of a singular value decomposition and to establish a number of its properties. In addition, we consider the approximation problem to find lower rank approximants M' of M which are optimal in that the error $M - M'$ has minimal induced norm when viewed as an operator on arbitrary normed spaces.

The paper is organized as follows. In section 2 we introduce singular values in a general fashion and establish some of its elementary properties. Problem formulations are collected in section 3. The main results on optimal rank approximations are given in section 4. An application on optimal system identification is discussed in section 5.

2 Generalized singular values

Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional vector spaces over the field of scalars $\mathbb{F}$. Let $n = \dim \mathcal{X}$ and $m = \dim \mathcal{Y}$ and define the p-norm of elements $x \in \mathcal{X}$ as

$$\|x\|_p := \begin{cases} \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} & \text{if } p < \infty \\ \max_{i=1,\ldots,n} |x_i| & \text{if } p = \infty \end{cases}$$

Here, x_i denotes the ith component of x. Let $(\mathcal{X}, \|\cdot\|_p)$ and $(\mathcal{Y}, \|\cdot\|_p)$ be normed linear vector spaces and let $M : \mathcal{X} \to \mathcal{Y}$ be a linear mapping. The *induced p-norm of M* is

$$\|M\|_{p\text{-ind}} := \sup_{0\neq x\in\mathcal{X}} \frac{\|Mx\|_p}{\|x\|_p}.$$

Throughout, the notation $\mathcal{L} \subseteq \mathcal{X}$ is understood to mean that $\mathcal{L}$ is a *linear subspace* of $\mathcal{X}$. If $\mathcal{L} \subseteq \mathcal{X}$, then $M|_\mathcal{L}$ denotes the restriction of M to $\mathcal{L}$, i.e., $M|_\mathcal{L} : \mathcal{L} \to \mathcal{Y}$ is defined as $M|_\mathcal{L} x = Mx$ for $x \in \mathcal{L}$.

Definition 2.1. *The* p-norm induced singular values *of M are the numbers*

$$\sigma_k^{(p)} := \inf_{\substack{\mathcal{L}\subseteq\mathcal{X},\\ \dim\mathcal{L}\geq n-k+1}} \sup_{0\neq x\in\mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p} \tag{2.2}$$

where k runs from 1 till n. The set of these numbers is denoted by $\sigma^{(p)}(M)$.

Note that the induced p-norm singular values are non-negative real numbers. For $k = 1, \ldots, n$ we will also be interested in the arguments of the infimum in (2.2). For this purpose, define

$$\mathbb{L}_k^{(p)} := \{\mathcal{L} \subseteq \mathcal{X} \mid \dim \mathcal{L} \geq n-k+1 \text{ and } \sup_{0 \neq x \in \mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p} = \sigma_k^{(p)}\}. \tag{2.3}$$

Note that $\mathbb{L}_k^{(p)}$ is non-empty for all k and all p and that $\mathbb{L}_1^{(p)} = \mathcal{X}$ for all p. Whenever p is understood from the context we omit the superscript (p) and write σ_k, $\sigma(M)$ and $\mathbb{L}_k$. It is easy to see that

$$\begin{aligned} \sigma_1^{(p)} &= \|M\|_{p\text{-ind}} \\ \sigma_k^{(p)} &= \|M|_{\mathcal{L}_k}\|_{p\text{-ind}} \\ \sigma_n^{(p)} &= \inf_{0 \neq x \in \mathcal{X}} \frac{\|Mx\|_p}{\|x\|_p} \end{aligned}$$

where $\mathcal{L}_k \in \mathbb{L}_k^{(p)}$ and $k = 1, \ldots, n$. Some elementary results pertaining to the p-norm induced singular values are summarized in the following Proposition.

Proposition 2.1. *For all $p \in [1, \infty]$ there holds*

1. $\sigma_1^{(p)} \geq \sigma_2^{(p)} \geq \ldots \geq \sigma_n^{(p)} \geq 0$.
2. $\operatorname{rank}(M) = r < n$ *if and only if* $\sigma_{r+1}^{(p)} = \cdots = \sigma_n^{(p)} = 0$.
3. $\operatorname{rank}(M) = n$ *if and only if* $\sigma_n^{(p)} > 0$.
4. $\sigma_1^{(\infty)} \geq \sigma_1^{(p)}$.

Proof. Fix $p \in [1, \infty]$ and let $\mathbb{S}_k := \{\mathcal{L} \subseteq \mathcal{X} \mid \dim \mathcal{L} \geq n-k+1\}$.

1. Obviously $\sigma_k \geq 0$ for all $k = 1, \ldots, n$. Since $\mathbb{S}_k \subseteq \mathbb{S}_{k+1}$ it is immediate that

$$\begin{aligned} \sigma_k &= \inf_{\mathcal{L} \in \mathbb{S}_k} \sup_{0 \neq x \in \mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p} \\ &\geq \inf_{\mathcal{L} \in \mathbb{S}_{k+1}} \sup_{0 \neq x \in \mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p} \\ &= \sigma_{k+1}. \end{aligned}$$

2. Suppose $\operatorname{rank}(M) = r < n$ and define $\mathcal{K} := \ker M$. Then $\dim \mathcal{K} = n - r$ so that $\mathcal{K} \in \mathbb{S}_k$ for $k = r+1, \ldots, n$. But then

$$\begin{aligned} \sigma_k &= \inf_{\mathcal{L} \in \mathbb{S}_k} \sup_{0 \neq x \in \mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p} \leq \sup_{0 \neq x \in \mathcal{K}} \frac{\|Mx\|_p}{\|x\|_p} \\ &= 0 \end{aligned}$$

for $k = r+1, \ldots, n$. Since, $\sigma_k \geq 0$ (statement 1), it follows that $\sigma_{r+1} = \cdots = \sigma_n = 0$.

3. Since M is linear and since $\mathbb{S}_n$ consists of one dimensional subspaces of $\mathcal{X}$ we have that $\sigma_n = \inf_{0\neq x \in \mathcal{X}} \frac{|Mx\|_p}{\|x\|_p}$ is strictly larger than zero if M has rank n.

4. It is shown in the Appendix C.2.16 of [1] that $\|M\|_{\infty\text{-ind}} \geq \|M\|_{p\text{-ind}}$. This is equivalent to $\sigma_1^{(\infty)} \geq \sigma_1^{(p)}$.

□

3 Problem formulations

In this section we consider a number of problems where the p-norm induced singular values play a natural role.

3.1 Rank deficiency

An important application of singular values stems from the numerical difficulty to determine the rank of a matrix M. In particular, for situations where M is near rank deficient, a numerically reliable calculation of $\mathrm{rank}(M)$ is sensitive to errors. Most numerical implementations to determine $\mathrm{rank}(M)$ calculate the *numerical rank*, defined as

$$r' = \mathrm{rank}(M, \varepsilon) := \min_{\|M-M'\|_{p\text{-ind}} \leq \varepsilon} \mathrm{rank}(M')$$

where $\varepsilon > 0$ is an accuracy level. In fact, this problem is a special case of the *optimal rank approximation problem*, which we formulate next.

3.2 Optimal rank approximation

Let $(\mathcal{X}, \|\cdot\|_p)$ and $(\mathcal{Y}, \|\cdot\|_p)$ be finite dimensional normed linear vector spaces of dimension n and m, respectively, and let $M : \mathcal{X} \to \mathcal{Y}$ be a linear mapping of rank r. Consider the problem of approximating M by a linear map $M' : \mathcal{X} \to \mathcal{Y}$ of rank at most k $(k < r)$, such that the p-induced norm

$$\|M - M'\|_{p\text{-ind}}$$

is minimal. We refer to this problem as the *optimal rank approximation problem* and to solutions M' as *optimal rank k approximants*.

3.3 Optimal system identification

Consider the problem to model the (real scalar valued) observed time series $\tilde{w}(t)$, $t = 0, \ldots N$, by an auto-regressive linear model of the form

$$\sum_{i=0}^{n} x_i w(t+i) = 0$$

where $x_i \in \mathbb{R}$ are the model coefficients and $n \geq 0$ is the model order. Let $x = (x_0, \ldots, x_n)^\top$ denote the model coefficient vector and define the *misfit* between model x and the data $\tilde{w}$ by

$$\mu(x, \tilde{w}) := \frac{\|e\|_p}{\|x\|_p}$$

where e is the vector of *residuals* $e(t) = \sum_{i=0}^n x_i \tilde{w}(t+i)$, $t = 0, \ldots N-n$. Given $\tilde{w}$, $n \geq 0$ and $\varepsilon \geq 0$, the *identification problem* amounts to finding all model coefficient vectors $x \in \mathbb{R}^{n+1}$ which have a guaranteed misfit in that $\mu(x, \tilde{w}) \leq \varepsilon$, i.e., we wish to characterize all models that can not be distinguished if one tolerates a misfit level ε. Note that this set may be empty. A model $x^* \in \mathbb{R}^{n+1}$ is said to be *optimal* if it minimizes the misfit $\mu(\cdot, \tilde{w})$.

Note the importance and interpretation of this problem for different values of p. The usual phrasing of this problem is in a stochastic context where the variance of e is to be minimized. This is equivalent of setting $p = 2$. Less conventional is the case where $p = \infty$. Solutions of the identification problem then have guaranteed upperbounds on the *amplitude* of their residuals, which seems of considerable interest for many applications in modeling.

We remark that the assumption that $\tilde{w}(t)$ and x_i are scalar valued has been made to simplify exposition only. Multivariable generalizations of this identification problem can be incorporated in a straightforward way. See Section 5 below.

4 Optimal rank approximation

If $p = 2$, the optimal rank optimization problem is well understood and has a simple solution. Indeed, let (1.1) be a singular value decomposition of M and, in the notation of Section 1, set $M_k := \sum_{i=1}^k \sigma_i x_i y_i^*$. Then $\operatorname{rank} M_k \leq k$ and

$$\min_{\operatorname{rank}(M') \leq k} \|M - M'\|_{\text{2-ind}} = \|M - M_k\|_{\text{2-ind}} = \sigma_{k+1}^{(2)}$$

which shows that M_k is an optimal rank k approximant of M. In particular, any truncation of the diadic expansion of M defines an optimal lower rank approximant of M. Optimal rank k approximants are by no means unique. Indeed, if δ_i, $i = 1, \ldots, k$, satisfy $|\delta_i| \leq \sigma_{k+1}$ then

$$M_k' := \sum_{i=1}^k (\sigma_i + \delta_i) y_i x_i^* \tag{4.4}$$

satisfies $\|M - M_k'\|_{\text{2-ind}} = \sigma_{k+1}^{(2)}$ and is therefore also an optimal rank k approximant of M.

4.1 A lower bound on the error

If $p \neq 2$, the problem is more difficult. We first establish a lower bound on the mismatch between a matrix M and its lower rank approximations in the p-induced norm. We then derive a sufficient condition for which this lower bound becomes sharp. Finally, we show that optimal rank $n-1$ approximants always attain this lower bound. Throughout this section, $\mathcal{X}$ and $\mathcal{Y}$ will be finite dimensional vector spaces of dimension n and m, respectively.

Proposition 4.1. *Let $M : \mathcal{X} \to \mathcal{Y}$ have rank r and let $M_k : \mathcal{X} \to \mathcal{Y}$ have rank at most k with $k < r$. Then*

$$\|M - M_k\|_{p\text{-}ind} \geq \sigma_{k+1}^{(p)}.$$

Proof. Let $\mathcal{K}_k = \ker M_k$. Then $\dim \mathcal{K}_k \geq n - k$ and we note that

$$\begin{aligned}\|M - M_k\|_{p\text{-ind}} &= \sup_{0 \neq x \in \mathcal{X}} \frac{\|(M - M_k)x\|_p}{\|x\|_p} \\ &\geq \sup_{0 \neq x \in \mathcal{K}_k} \frac{\|(M - M_k)x\|_p}{\|x\|_p} \\ &= \sup_{0 \neq x \in \mathcal{K}_k} \frac{\|Mx\|_p}{\|x\|_p}\end{aligned}$$

Since $\dim \mathcal{K}_k \geq n - k$, it follows that

$$\sup_{0 \neq x \in \mathcal{K}_k} \frac{\|Mx\|_p}{\|x\|_p} \geq \inf_{\substack{\mathcal{L} \in \mathcal{X} \\ \dim \mathcal{L} \geq n-k}} \sup_{0 \neq x \in \mathcal{L}} \frac{\|Mx\|_p}{\|x\|_p}$$

which shows that $\|M - M_k\|_{p\text{-ind}} \geq \sigma_{k+1}^{(p)}$. □

A natural question is whether the lower bound in Proposition 4.1 can actually be attained for a rank k matrix M_k. To answer this question, recall that two subspaces $\mathcal{L}'$ and $\mathcal{L}''$ of $\mathcal{X}$ are said to be *complementary* if $\mathcal{L}' \cap \mathcal{L}'' = \{0\}$ and $\mathcal{L}' + \mathcal{L}'' = \mathcal{X}$. If $(\mathcal{L}', \mathcal{L}'')$ is a complementary pair, every $x \in \mathcal{X}$ admits a unique decomposition $x = x' + x''$ with $x' \in \mathcal{L}'$ and $x'' \in \mathcal{L}''$. In that case, we write $x' = \Pi_{\mathcal{L}'|\mathcal{L}''} x$ and $x'' = \Pi_{\mathcal{L}''|\mathcal{L}'} x$ where $\Pi_{\mathcal{L}'|\mathcal{L}''} : \mathcal{X} \to \mathcal{L}'$ and $\Pi_{\mathcal{L}''|\mathcal{L}'} : \mathcal{X} \to \mathcal{L}''$ define the natural projections on $\mathcal{L}'$ along $\mathcal{L}''$ and on $\mathcal{L}''$ along $\mathcal{L}'$, respectively.

The following theorem provides a sufficient condition under which the lower bound in Proposition 4.1 will be sharp.

Theorem 4.1. *Given M, define the sets $\mathbb{L}_k^{(p)}$ by (2.3). If there exist $\mathcal{L}' \in \mathbb{L}_{k+1}^{(p)}$ and $\mathcal{L}'' \subseteq \mathcal{X}$ such that*

1. *$(\mathcal{L}', \mathcal{L}'')$ are complementary and*

2. *$\|\Pi_{\mathcal{L}'|\mathcal{L}''}\|_{p\text{-}ind} \leq 1$*

then there exists $M_k : \mathcal{X} \to \mathcal{Y}$ of rank at most k such that

$$\|M - M_k\|_{p\text{-}ind} = \sigma_{k+1}^{(p)}.$$

In particular, M_k given by $M_k|_{\mathcal{L}'} = 0$ and $M_k|_{\mathcal{L}''} = M|_{\mathcal{L}''}$ is an optimal rank k approximant of M.

Proof. In view of Proposition 4.1, it suffices to show that M_k, as specified, has rank $\leq k$ and satisfies $\|M - M_k\|_{p\text{-ind}} \leq \sigma_{k+1}^{(p)}$. To see this, first note that $\dim \mathcal{L}' \geq n-k$ which means that $\dim \mathcal{L}'' \leq k$ so that $\operatorname{rank} M_k \leq k$. Second, observe that

$$\begin{aligned}\|M - M_k\|_{p\text{-ind}} &= \sup_{0 \neq x \in \mathcal{X}} \frac{\|Mx - M_k x\|_p}{\|x\|_p} = \sup_{\substack{x' \in \mathcal{L}'; x'' \in \mathcal{L}'' \\ x' + x'' \neq 0}} \frac{\|Mx'\|_p}{\|x' + x''\|_p} \leq \sup_{0 \neq x' \in \mathcal{L}'} \frac{\|Mx'\|_p}{\|x'\|_p} \\ &= \sup_{0 \neq x' \in \mathcal{L}'} \frac{\|Mx'\|_p}{\|x'\|_p} \\ &= \sigma_{k+1}^{(p)}.\end{aligned}$$

Here, we used in the third inequality that $\Pi_{\mathcal{L}'|\mathcal{L}''}$ is a contraction, i.e., $\|x'\|_p = \|\Pi_{\mathcal{L}'|\mathcal{L}''} x\|_p \leq \|x\|_p$. The last equality follows from the definition of $\mathbb{L}_{k+1}^{(p)}$. It follows that M_k is an optimal rank k approximant of M. □

The main issue of the above result is the existence of a subspace $\mathcal{L}''$, complementary to $\mathcal{L}' \in \mathbb{L}_{k+1}^{(p)}$ such that the projection $\Pi_{\mathcal{L}'|\mathcal{L}''}$ defines a contraction on $\mathcal{X}$. We will investigate these conditions for a number of special cases.

4.2 Nonexistence of contractive projection

Theorem 4.1 provides sufficient conditions for which the lower bound in Proposition 4.1 will be attained. These conditions will not always be satisfied. In fact, to see how strict these conditions are, consider the case where $n = 3$, p is even, $p \neq 2$, and $\mathcal{L}'$ is a two dimensional subspace of $\mathcal{X} = \mathbb{R}^3$, spanned by the non-zero vectors x and y, i.e. $\mathcal{L}' = \operatorname{span}(x, y)$. A subspace $\mathcal{L}''$ of $\mathcal{X}$ will satisfy the conditions 1 and 2 of Theorem 4.1 if and only if $\mathcal{L}'' = \operatorname{span}(z)$ with $z \neq 0$ such that

1. $\det(x, y, z) \neq 0$ and

2. $\sum_{i=1}^{3} (\alpha x_i + \beta y_i + \gamma z_i)^p - (\alpha x_i + \beta y_i)^p \geq 0$ for all $\alpha, \beta, \gamma \in \mathbb{R}$.

In particular, the latter condition implies that

$$\sum_{i=1}^{3} x_i^{p-1} z_i = 0$$
$$\sum_{i=1}^{3} y_i^{p-1} z_i = 0$$
$$\sum_{i=1}^{3} (x_1 + y_i)^{p-1} z_i = 0$$

which yields (generically) that $z_1 = z_2 = z_3 = 0$; i.e. $z = 0$. Hence there does not exist a complementary subspace $\mathcal{L}''$ such that the projection $\Pi_{\mathcal{L}'|\mathcal{L}''}$ is contractive.

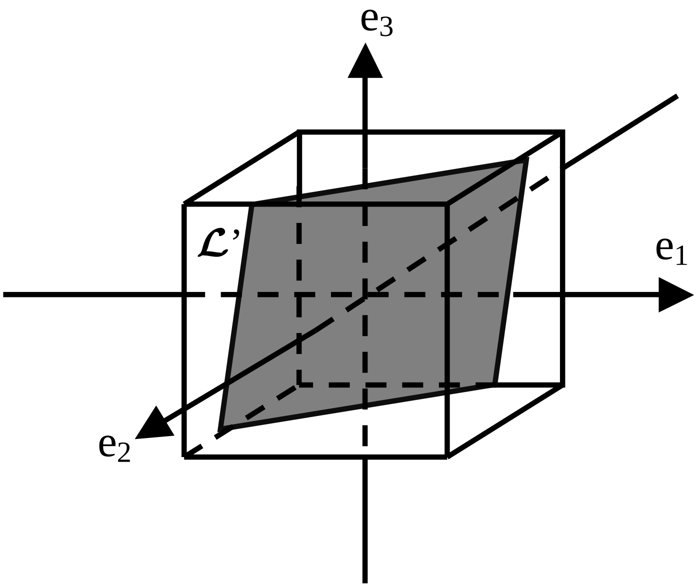

Figure 1: Nonexistence of contractive projection for the case $p = \infty$

Alternatively, we can give a geometric argument to show that the conditions in Theorem 4.1 do not need to be satisfied. Indeed, let $n = 3$, $p = \infty$ and let $\mathcal{L}'$ be given by the two-dimensional subspace indicated in Figure 1. Then it is easily seen that the projection of the unit-ball in $(\mathbb{R}^3, \|\cdot\|_\infty)$ on $\mathcal{L}'$ along any complementary subspace $\mathcal{L}''$ of $\mathcal{L}'$ is not contractive.

4.3 The case $p = 2$ and arbitrary k

If $p = 2$, $\mathcal{X}$ becomes a Hilbert space with the natural inner product $\langle \cdot, \cdot \rangle$. For every subspace $\mathcal{L} \subseteq \mathcal{X}$, its orthogonal complement $\mathcal{L}^\perp := \{x \in \mathcal{X} \mid \langle x, y \rangle = 0 \text{ for all } y \in \mathcal{L}\}$ is complementary to $\mathcal{L}$ and the orthogonal projection $\Pi_{\mathcal{L}|\mathcal{L}^\perp}$ is obviously a contraction. Hence, optimal rank k approximants always exist in this case and are given by the expression (4.4). This case is well understood and can be found in many text books (e.g. [2, 4]).

4.4 The case $k = n - 1$ and arbitrary p

Let p be arbitrary, suppose that $n = \operatorname{rank} M$ and consider the optimal rank approximation problem with $k = n - 1$. The set $\mathbb{L}_n^{(p)}$ then consists of subspaces of dimension ≥ 1 only. Let $\mathcal{L}' \in \mathbb{L}_n^{(p)}$ be a one dimensional subspace and let $x' \in \mathcal{L}'$ be a nonzero element. Then $\mathcal{L}' = \operatorname{span}(x')$. The following lemma is easily seen.

Lemma 4.1. *If $\mathcal{L}' = \operatorname{span}(x')$ for a nonzero $x' \in \mathcal{X}$ then $\mathcal{L}'' \subseteq \mathcal{X}$ will be complimentary to $\mathcal{L}'$ if and only if*

$$\mathcal{L}'' = \{x'' \in \mathcal{X} \mid \langle w, x'' \rangle = 0\} \tag{4.5}$$

where $w \in \mathcal{X}$ is a nonzero vector such that $\langle w, x' \rangle \neq 0$.

Hence, Lemma 4.1 provides a parametrization of all complements of a given one-dimensional subspace spanned by a nonzero vector $x' \in \mathcal{X}$. In order to characterize complementary suspaces $(\mathcal{L}', \mathcal{L}'')$ for which the projection $\Pi_{\mathcal{L}'|\mathcal{L}''}$ is contractive, we resort to some terminology from convex analysis [3].

Definition 4.1. *Let $f : \mathcal{X} \to \mathbb{R}$ be a convex function. A vector $w \in \mathcal{X}$ is said to be a* subgradient of f at $x \in \mathcal{X}$ *if*

$$f(z) \geq f(x) + \langle w, z - x \rangle \tag{4.6}$$

for all $z \in \mathcal{X}$. The set of all subgradients of f at $x \in \mathcal{X}$ is called the subdifferential of f at *x and denoted by $\nabla f(x)$, i.e.,*

$$\nabla f(x) := \{ w \in \mathcal{X} \mid f(z) \geq f(x) + \langle w, z - x \rangle \text{ for all } z \in \mathcal{X} \} .$$

Inequality (4.6) is usually referred to as the *subgradient inequality* and has the simple geometric interpretation that the graph of f lies on or above the affine function $g(z) := f(x) + \langle w, z - x \rangle$ which is the tangent hyperplane of f at x. We remark that the subdifferential of f at x is a closed convex set. If $\nabla f(x)$ is non-empty, f is said to be *subdifferentiable* at x.

The next proposition shows that subdifferentials of the mapping $f : x \mapsto \|x\|_p$ precisely parametrize the complements of $\mathcal{L}'$ for which $\Pi_{\mathcal{L}'|\mathcal{L}''}$ is contractive. The following lemma shall be used to prove the proposition.

Lemma 4.2. *Consider $f : x \mapsto \|x\|_p$. Then $w \in \nabla f(x)$ if and only if $\langle w, x \rangle = \|x\|_p$ and $\langle w, z \rangle \leq \|z\|_p$ for all $z \in \mathcal{X}$.*

Proof. (if). Obvious.

(only if). Substitute $z = \alpha x$ $(0 \leq \alpha \leq 1)$ into (4.6) then we have $\langle w, x \rangle \geq \|x\|_p$. Take $z = \alpha x$ where $\alpha > 1$ then we have $\langle w, x \rangle \leq \|x\|_p$. Thus we have $\langle w, x \rangle = \|x\|_p$ and $\langle w, z \rangle \leq \|z\|_p$ for all $z \in \mathcal{X}$. □

Proposition 4.2. *Let $x' \in \mathcal{X}$ be nonzero and $\mathcal{L}' = \operatorname{span}(x')$. Then the pair $(\mathcal{L}', \mathcal{L}'')$ is complementary and $\|\Pi_{\mathcal{L}'|\mathcal{L}''}\|_{p\text{-}ind} \leq 1$ if and only if $\mathcal{L}''$ is given by (4.5) with $0 \neq w \in \nabla \|x'\|_p$.*

Proof. (if). From Lemma 4.2 we have $\langle w, x' \rangle = \|x'\|_p \neq 0$ which yields that the pair $(\mathcal{L}', \mathcal{L}'')$ is complimentary. Now the subgradient inequality (4.6) yields

$$\|z\|_p \geq \|x'\|_p + \langle w, z - x' \rangle$$

For any given nonzero $\lambda \in \mathbb{R}$ we have

$$\begin{aligned} &\|\lambda z\|_p \geq \|\lambda x'\|_p + \frac{|\lambda|}{\lambda} \langle w, \lambda z - \lambda x' \rangle \\ \Longrightarrow\quad &\|\bar{z}\|_p \geq \|\lambda x'\|_p + \langle v, \bar{z} - \lambda x' \rangle \end{aligned} \tag{4.7}$$

where we set $\bar{z} = \lambda z$ and $v = |\lambda| w / \lambda$. Since z is arbitrary and $\lambda \neq 0$, (4.7) yields the subgradient inequality for all $\bar{z} \in \mathcal{X}$ with $\bar{z} \notin \mathcal{L}''$. By the fact that the pair $(\mathcal{L}', \mathcal{L}'')$ is complimentary any $\bar{z} \in \mathcal{X}$ has a unique decomposition $\bar{z} = \bar{z}' + \bar{z}''$ with $\bar{z}' = \Pi_{\mathcal{L}'|\mathcal{L}''} \bar{z}$ and $\bar{z}'' = \bar{z} - \bar{z}' \in \mathcal{L}''$. Since $\bar{z}' \in \mathcal{L}'$ it follows that there exists $\lambda \in \mathbb{R}$ such that $\bar{z}' = \lambda x'$. Now, from (4.7) we then have $\|\bar{z}\|_p \geq \|\bar{z}'\|_p$. Since $\bar{z}$ is arbitrary, it follows that $\|\Pi_{\mathcal{L}'|\mathcal{L}''}\|_{p\text{-ind}} \leq 1$.

(only if). By Lemma 4.1, there exists $w \in \mathcal{X}$, with $\langle w, x' \rangle \neq 0$ such that $\mathcal{L}''$ is given by (4.5). Since $\langle w, x' \rangle$ is nonzero, we may as well assume that

$$\|x'\|_p = \langle w, x' \rangle$$

By complimentary of $(\mathcal{L}', \mathcal{L}'')$ we can decompose uniquely any $z \in \mathcal{X}$ in terms of $z = z' + z''$ where $z' = \lambda x'$ for $\lambda \in \mathbb{R}$ and $z'' \in \mathcal{L}$. Since $\|\Pi_{\mathcal{L}'|\mathcal{L}''}\|_{p\text{-ind}} \leq 1$ we have $\|z\|_p \geq \|z'\|_p$. Then

$$\|z\|_p \geq \|\lambda x'\|_p \geq \lambda \|x'\|_p = \lambda \langle w, x' \rangle = \langle w, \lambda x' \rangle = \langle w, z \rangle$$

From this point we have obtained $\langle w, x' \rangle = \|x'\|_p$ and $\langle w, z \rangle \leq \|z\|_p$ for all $z \in \mathcal{X}$. Consequently, by Lemma 4.2, $w \in \nabla \|x'\|_p$ as desired. □

The following theorem is an immediate consequence of Theorem 4.1 and Proposition 4.2.

Theorem 4.2. *Let $M : \mathcal{X} \to \mathcal{Y}$ have rank n. For every p there exists $M^* : \mathcal{X} \to \mathcal{Y}$ with $\operatorname{rank} M^* < n$ such that*

$$\|M - M^*\|_{p\text{-}ind} = \min_{\operatorname{rank} M' \leq n-1} \|M - M'\|_{p\text{-}ind} = \sigma_n^{(p)}.$$

Moreover, any M^* given by $M^*|_{\mathcal{L}'} = 0$ with $\mathcal{L}' = \operatorname{span}(x') \in \mathbb{L}_n$ and $M^*|_{\mathcal{L}''} = M|_{\mathcal{L}''}$ with $\mathcal{L}''$ given by (4.5) *with $w \in \nabla \|x'\|_p$ is an optimal approximant of rank $< n$.*

At this stage it is unclear whether for arbitrary p, the p-induced singular values $\sigma^{(p)}(M)$ precisely characterize the minimal achievable approximation errors in that

$$\min_{\operatorname{rank}(M') \leq k} \|M - M'\|_{p\text{-ind}} = \sigma_{k+1}^{(p)}$$

holds for all k. This question is currently under investigation.

5 Optimal system identification

Consider the optimal system identification formulated in Section 3. Let $\tilde{w}(t)$, $t = 0, \ldots N$ be a real valued observed time series of dimension q, i.e., $\tilde{w}(t) \in \mathbb{R}^q$, and suppose that we wish to find an optimal autoregressive model

$$\sum_{i=0}^{n} x_i w(t+i) = 0$$

where the model coefficients x_i are row vectors of dimension q, and n is the model order. Let $x = (x_0, \ldots, x_n)^\top \in \mathbb{R}^q(n+1)$ denote the model coefficient vector and set

$$M = \begin{pmatrix} \tilde{w}^\top(0) & \cdots & \tilde{w}^\top(n) \\ \vdots & & \vdots \\ \tilde{w}^\top(N-n) & \cdots & \tilde{w}^\top(N) \end{pmatrix}.$$

It is immediate that the misfit

$$\mu(x,\tilde{w}) = \frac{\|Mx\|_p}{\|x\|_p}.$$

Consequently, $\mathcal{L} \subseteq \mathbb{R}^{q(n+1)}$ satisfies $\mu(\cdot,\tilde{w})|_{\mathcal{L}} \le \varepsilon$ if and only if $\|M|_{\mathcal{L}}\|_{p\text{-ind}} \le \varepsilon$. Hence, by definition, all subsets $\mathcal{L} \subseteq \mathbb{R}^{q(n+1)}$ with this property are characterized by $\mathcal{L} \in \mathbb{L}_j^{(p)}$, $j \ge k$ where k is such that

$$\sigma_{k-1}^{(p)}(M) > \varepsilon \ge \sigma_k^{(p)}(M). \tag{5.9}$$

This proves the following result:

Theorem 5.1. *If k satisfies* (5.9)*, then all $x \in \mathcal{L}$ with $\mathcal{L} \in \mathbb{L}_j^{(p)}$, $j \ge k$, solve the identification problem in that the misfit*

$$\mu(x,\tilde{w}) \le \varepsilon.$$

The identification problem has no solution if no such k exists. Furthermore, every $x^* \in \mathcal{L}$ with $\mathcal{L} \in \mathbb{L}_{q(n+1)}^{(p)}$ defines an optimal *model of (minimal) misfit $\mu(x^*,\tilde{w}) = \sigma_{n+1}^{(p)}$.*

Note that this result provides a complete solution to the system identification problem for any p.

6 Conclusions

In this paper we introduced the notion of induced p-norm singular values and showed their relevance for a number of problems. In particular, we addressed the optimal rank approximation problem and derived sufficient conditions for the existence of optimal approximants which minimize the induced p-norm of the error.

References

[1] C.A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.*, Academic Press, London, 1975.

[2] G.H. Golub and C.F. van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, 1989.

[3] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[4] G.W. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, London, 1990.