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Relations Between Finite Zero Structure of The Plant and The Standard H_∞ Controller Order Reduction

Takao Watanabe[1] Anton A. Stoorvogel[23]

Abstract

This paper finds new facts about a relation between the finite unstable as well as stable invariant zero structure of the generalized plant and the order reduction of the H_∞ controller, and moreover characterizes a class of reduced-order H_∞ controllers on the basis of the invariant zeros.

1 Introduction

Over the last few decades, there has been a great interest in the issue of solving the H_∞ control problem, possibly due to a wide recognition that the H_∞ control is one of the most promising tools for robust control problems. Since then, solutions to the problem have been proposed in many researches. Thus, we have many ways to design an H_∞ controller. On the other hand, the H_∞ controller often needs to be obtained with a reduced order that is less than the order of the generalized plant, however it contains difficulties in designing a reduced-order H_∞ controller.

This paper focuses on obtaining information about the relation between the structure of a plant and the order of the controller that is to be designed for the plant. If we have *a priori* knowledge about the relation, we can predict the order of the H_∞ controller when constructing a generalized plant, and we can utilize it in designing a reduced-order H_∞ controller. For example, we know that an output feedback controller that is composed of a minimal-order observer and a state feedback can stabilize a closed-loop system with an order less than that of the plant. Based on this knowledge, methods for designing reduced-order H_∞ controllers have been obtained [1, 2]. Thus, one important thing in the reduced-order H_∞ controller design might be to know in advance facts about the relation between the structure of the generalized plant and the order reduction of the H_∞ controller.

[1]Dept. of Electrical Engineering
Tokyo Metropolitan University
1-1 Minami-osawa Hachioji, Tokyo 192-0397
Japan
takao@eei.metro-u.ac.jp
[2]Dept. of Information Technology and Systems, Delft University of Technology, P.O. Box 5031, 2600 GA Delft
[3]Dept. of Mathematics and Computing Science
Eindhoven University of Technology
P.O. Box 513, 5600 MB Eindhoven
The Netherlands
A.A.Stoorvogel@tue.nl

Notation: I_n denotes the identity matrix of dimension $n \times n$. We denote the set of real numbers by $\mathbb{R}$, the entire complex plane by $\mathbb{C}$, and subclasses of these sets as $\mathbb{R}^+ = \{s \in \mathbb{R} \mid s > 0\}$, $\mathbb{C}^0 = \{s \in \mathbb{C} \mid \mathbf{Re}(s) = 0\}$, $\mathbb{C}^+ = \{s \in \mathbb{C} \mid \mathbf{Re}(s) > 0\}$, and $\mathbb{C}^- = \{s \in \mathbb{C} \mid \mathbf{Re}(s) < 0\}$. The class of stable real rational transfer functions is denoted by $\mathcal{RH}_\infty$. For a matrix $A \in \mathbb{C}^{n\times m}$, we denote the set of its singular values by $\sigma(A)$, the transpose matrix by A^T, the conjugate transpose by A^*, the Moore-Penrose generalized inverse by $A^\dagger$ and an orthogonal compliment by $A^\perp$. In this paper, when the matrix A has full rank, the matrix $A^\perp$ is selected in such a way that $A^\perp$ is either inner or co-inner, satisfies $A^\perp A^\dagger = 0$ and either $(A \quad A^\perp)$ or $(A^T \quad (A^\perp)^T)$ is square and invertible. The transmission zeros of a system are defined as the zeros of its transfer matrix. The invariant zeros of a realization for a system are defined via the Rosenbrock system matrix.

2 Preliminaries

2.1 Reduced-order H_∞ control problem

Consider the following linear continuous-time system

$$\Sigma: \begin{array}{l} \dot{x} = Ax + B_1 w + B_2 u \\ z = C_1 x + D_{11} w + D_{12} u \\ y = C_2 x + D_{21} w + D_{22} u \end{array} \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^{m_1}$ is the disturbance input, $u \in \mathbb{R}^{m_2}$ is the control input, $z \in \mathbb{R}^{p_1}$ is the controlled output, $y \in \mathbb{R}^{p_2}$ is the measurement output. We make a standing assumption that the triple (A, B_2, C_2) is stabilizable and detectable. We denote the transfer function of the system Σ as

$$\Sigma(s) = \begin{pmatrix} \Sigma_{11}(s) & \Sigma_{12}(s) \\ \Sigma_{21}(s) & \Sigma_{22}(s) \end{pmatrix} \qquad (2)$$

where a subsystem $\Sigma_{ij}(s)$ is represented by $\Sigma_{ij}(s) = D_{ij} + C_j (sI - A)^{-1} B_i, i, j = 1, 2$.

We assume that a controller is represented by

$$\Sigma_K: \begin{array}{l} \dot{\eta} = A_k \eta + B_k y \\ u = C_k \eta + D_k y \end{array}, \ \eta \in \mathbb{R}^{n_k}, \qquad (3)$$

with the minimal realization. The H_∞ suboptimal control problem is to find a controller that stabilizes the resulting closed loop system composed of Σ and Σ_K, and that makes the H_∞-norm of the closed loop system strictly less than an *a priori* given bound γ, if one exists. Here we call n_k the order of the H_∞ controller. The reduced-order H_∞ control problem that we consider in this paper includes clarifying

1. *a priori* knowledge about the relation between the H_∞ controller-order reduction and the structure of the plant, and obtaining
2. a way how we obtain a reduced-order H_∞ controller practically by using the knowledge.

In our recent paper, we have characterized a class of reduced order H_∞ controllers by using the structure of invariant zeros of Σ_{21}.

Theorem 1
Suppose that the H_∞ control problem for Σ is solvable. Then there exists an H_∞ controller of order

$$n_{k_c} \leq n - \left(\operatorname{rank}\begin{pmatrix} C_2 & D_{21}\end{pmatrix} - \operatorname{rank}\left(\Sigma_{21}(\alpha)\right)\right) \qquad (4)$$

where $\alpha > 0$ is an invariant zero of the system Σ_{21}.

2.2 The goal of this paper

In the above result there are some issues to be solved.

(i) The result uses only one zero on the positive real axis. Hence, if there exist plural zeros on the positive real axis, it is impossible to use those zeros in order to reduce the order of the controller further.

(ii) In the case where the system Σ_{21} has invariant zeros in the open right half plane, it is unknown whether we can obtain a reduced-order H_∞ controller by using a zero of complex number.

(iii) In the case where the system Σ_{21} has invariant zeros in the open left half plane, it is unknown whether we can obtain a reduced-order H_∞ controller by using them.

Thus, the goal of this paper is to investigate relations between the finite invariant zeros and the order reduction of the H_∞ controller that is derived with the two-ARE approach in the standard H_∞ problem.

3 AREs and H_∞ controllers

3.1 Assumptions

In this section we recall the standard solution that is derived by solving two algebraic Riccati equations [3–5]. Again, we consider the H_∞ control problem for the generalized plant Σ in (1). From the context of our issues, we make an assumption that

A1 Σ_{21} has invariant zeros in $\mathbb{C}^+$ as well as in $\mathbb{C}^-$.

In order to analyze the structure of the H_∞ controller, avoiding the complexity of the analysis, we make the following assumptions:

A2 $D_{12}^T D_{12} > 0$

A3 $D_{21} D_{21}^T > 0$

A4 $\operatorname{rank}\begin{pmatrix} sI - A & B_2 \\ C_1 & D_{12}\end{pmatrix} = n + m_2, \forall s \in \mathbb{C}^0$

A5 $\operatorname{rank}\begin{pmatrix} sI - A & B_1 \\ C_2 & D_{21}\end{pmatrix} = n + p_2, \forall s \in \mathbb{C}^0$

A6 $D_{11} = 0, \quad D_{22} = 0.$

The assumptions **A2** and **A3** exclude singular cases where D_{ij} $(i, j \in 1, 2, i \neq j)$ loses full rank or inequalities $D_{12}D_{12}^T > 0$ or/and $D_{21}^T D_{21} > 0$ hold. In [6], we studied the singular case where either $D_{12}D_{12}^T > 0$ or $D_{21}^T D_{21} > 0$ is not satisfied. The assumptions **A4** and **A5** are from the reason that we employ the ARE approach. The assumption **A6** is for the sake of simplicity of the analysis. This has no loss of generality in our result, and can be excluded by using techniques in [7].

3.2 Parametrization of the H_∞ controllers

We recall the parametrization of H_∞ controllers represented by stabilizing solutions of AREs under the above assumptions:

$$\begin{aligned} X\left(A - B_2 D_{12}^\dagger C_1\right) + \left(A - B_2 D_{12}^\dagger C_1\right)^T X \\ + X\left[\gamma^{-2} B_1 B_1^T - B_2 D_{12}^\dagger \left(B_2 D_{12}^\dagger\right)^T\right] X \\ + C_1^T D_{12}^\perp \left(D_{12}^\perp\right)^T C_1 = 0 \end{aligned} \qquad (5)$$

$$\begin{aligned} Y\left(A - B_1 D_{21}^\dagger C_2\right)^T + \left(A - B_1 D_{21}^\dagger C_2\right) Y \\ + Y\left[\gamma^{-2} C_1^T C_1 - \left(D_{21}^\dagger C_2\right)^T D_{21}^\dagger C_2\right] Y \\ + B_1 \left(D_{21}^\perp\right)^T D_{21}^\perp B_1^T = 0. \end{aligned} \qquad (6)$$

Lemma 1
For the generalized plant Σ, there exists an H_∞ controller if and only if two AREs in (5) and (6) have positive semidefinite stabilizing solutions, and these satisfy

$$\rho(XY) < \gamma^2. \qquad (7)$$

If there exists an H_∞ controller, the class of all H_∞ controllers is given by

$$\Sigma_K \quad \begin{aligned} \dot{\eta} &= A_Y \eta - \hat{B}_2 u + H_\infty y \\ u &= -C_K(s)\eta + \Pi(s) y \end{aligned} \qquad (8)$$

where parameters are given by

$$\begin{cases} A_Y = A + \gamma^{-2} Y C_1^T C_1 + H_\infty C_2 \\ H_\infty = -B_1 D_{21}^\dagger - Y \left(D_{21}^\dagger C_2 \right)^T D_{21}^\dagger \\ \hat{B}_2 = B_2 + \gamma^{-2} Y C_1^T D_{12} \\ C_K(s) = \left(F_\infty - E_{12}^{-\frac{1}{2}} N(s) E_{21}^{-\frac{1}{2}} \hat{C}_2 \right) Z \\ \Pi(s) = E_{12}^{-\frac{1}{2}} N(s) E_{21}^{-\frac{1}{2}} \\ F_\infty = -D_{12}^\dagger C_1 - D_{12}^\dagger \left(B_2 D_{12}^\dagger \right)^T X \\ \hat{C}_2 = \gamma^{-2} D_{21} B_1^T X + C_2, \quad Z = \left(I - \gamma^{-2} Y X \right)^{-1} \\ E_{12} = D_{12}^T D_{12}, \quad E_{21} = D_{21} D_{21}^T \end{cases}, \tag{9}$$

and $N(s) \in \mathcal{RH}_\infty$ is a free parameter.

3.3 Characterization of invariant zeros of $\Sigma_{21}(s)$

We can characterize invariant zeros of Σ_{21} by applying a state transformation $x = T\bar{x}, |T| \neq 0$.

Lemma 2

Suppose that assumptions **A3** *and* **A5** *are satisfied. Then there exists a suitable basis such that the matrices $A - B_1 D_{21}^\dagger C_2$ and $B_1 \left(D_{21}^\perp \right)^T$ have the following block decompositions with A_- stable and A_+ antistable*

$$A - B_1 D_{21}^\dagger C_2 = \begin{pmatrix} A_- & 0 & 0 \\ 0 & A_+ & 0 \\ \tilde{A}_a & \tilde{A}_b & \tilde{A} \end{pmatrix} \tag{10}$$

$$B_1 \left(D_{21}^\perp \right)^T = \begin{pmatrix} 0 \\ 0 \\ \tilde{B} \end{pmatrix} \tag{11}$$

where $\left(\tilde{A}, \tilde{B} \right)$ is controllable.

In the argument below, it is assumed that we have already made the state transformation for Σ, and we have a generalized plant that satisfies (10). In this case, it is clear that Lemma 1 still holds.

Lemma 3

The positive semidefinite stabilizing solution of ARE (6) can be represented by

$$Y = \begin{pmatrix} 0 & 0 \\ 0 & Y_r \end{pmatrix}, \quad Y_r > 0. \tag{12}$$

Here, we decompose the solution of ARE (5) in accordance with Y in (12) as

$$X = \begin{pmatrix} * & * \\ * & X_r \end{pmatrix}, \quad X_r \in \mathbb{R}^{r \times r}, \tag{13}$$

and define a matrix $Z_r = \left(I - \gamma^{-2} Y_r X_r \right)^{-1} \in \mathbb{R}^{r \times r}$.

4 Main results

The H_∞ controller presented by (8) contains the free parameter $N(s) \in \mathcal{RH}_\infty$. If we put the parameter zero, we obtain the so-called central controller whose order is less than that of Σ. On the other hand, by utilizing the freedom in the parameter, we can obtain a controller that might have an order less than that of the central controller. As for the free parameter selection, using this technique Li and Chang [8] and Yung [9] have considered the reduced-order H_∞ controller design. However, the relation between the invariant zeros of the plant and the controller reduction is not clarified in those papers. Moreover, in [10,11], to avoid the stable pole-zero cancellation between the plant and the H_∞ controller the SISO H_∞ controller order reduction is investigated based on the invariant zeros on the left half plane. In this section, by using the zeros of Σ_{21} we derive a condition under which a reduced order H_∞ controller exists, and show a way to find a free parameter that reduces the order of the H_∞ controller while preserving the stability and the closed loop H_∞-norm less than γ.

Theorem 2

Suppose that the H_∞ control problem for the generalized plant (1) under the assumptions **A1** *to* **A6** *is solvable, and the generalized plant satisfies (10) where $A_+ \in \mathbb{R}^{r_+ \times r_+}$ is the matrix associated with the unstable invariant zeros of Σ_{21}. Also, suppose that $m_2 \leq p_2$, where $m_2 = \dim(u)$ and $p_2 = \dim(y)$.*

Let $\tilde{V}_m \in \mathbb{R}^{r_+ \times m}$ be an arbitrary matrix that satisfies rank $\tilde{V}_m = m$ *and*

$$A_+^T \tilde{V}_m = \tilde{V}_m J, \quad J \in \mathbb{R}^{m \times m}. \tag{14}$$

where $m \leq r_+$, then the following properties hold.

(i) If the following rank condition

$$\text{rank} \left(\hat{C}_2 Z Y V \right) = m \tag{15}$$

is satisfied, there exists a static solution $N \in \mathbb{R}^{m_2 \times p_2}$ to the equation $C_K(s) Y V = 0$, where the matrix V is defined as

$$V = \begin{pmatrix} 0_{r_- \times m} \\ \tilde{V}_m \\ 0_{(n-r) \times m} \end{pmatrix} \in \mathbb{R}^{n \times m} \tag{16}$$

and $r = r_- + r_+$, $A_- \in \mathbb{R}^{r_- \times r_-}$, $A_+ \in \mathbb{R}^{r_+ \times r_+}$.

(ii) If the solution satisfies the condition

$$\bar{\sigma}(N) < \gamma, \tag{17}$$

the solution reduces the order of the H_∞ controller in (8) to $n - m$, while preserving the closed loop *performance γ. Then the reduced order H_∞ controller is represented by*

$$\Sigma_{K_r} \quad \begin{array}{l} \dot{\eta}_r = (A_r + B_r C_r)\,\eta_r + (H_r - B_r \Pi_r)\,y \\ u = -C_r \eta_r + \Pi_r y \end{array} \tag{18}$$

where the state variable of the controller is

$$\eta_r = \left(V^T Y\right)^{\perp} \eta \in \mathbb{R}^{n-m}, \tag{19}$$

and parameters are

$$\begin{cases} A_r = \left(V^T Y\right)^{\perp} A_Y \,(YV)^{\perp} \\ B_r = \left(V^T Y\right)^{\perp} \hat{B}_2 \\ H_r = \left(V^T Y\right)^{\perp} H_\infty \\ C_r = F_\infty \left(I - \left(\hat{C}_2 ZYV\right)^{\dagger} \hat{C}_2\right) Z\,(YV)^{\perp} \\ \Pi_r = F_\infty \left(\hat{C}_2 ZYV\right)^{\dagger}. \end{cases}$$

Remark 1
The reduced-order controller represented in (18) gives the upper bound of its order. Hence, the controller obtained in this theorem has an order less than $n - m$, where n is the order of the plant and m is the sum of the algebraic multiplicities of eigenvalues of J.

Remark 2
One major difference between this reduced-order H_∞ controller and the central controller is that the reduced-order H_∞ controller is proper.

Theorem 2 implies a sufficient condition for the reduction of the controller order based on the transmission zeros of Σ_{21}. In the preceding, we will investigate the relation between the zero mode and the conditions in (15) and (17). Knowing the relation, we can clarify that how the zeros relate to those conditions and the reduced-order H_∞ controller. First, we show a result that is based on an unstable invariant zero on the positive real axis.

Theorem 3
Suppose that the H_∞ control problem for the generalized plant (1) under the assumptions **A1** *to* **A6** *is solvable. Furthermore, suppose that Σ_{21} has an invariant zero on the positive real axis with the geometric multiplicity m. Then, we can obtain a reduced-order H_∞ controller by the formula in (18), while preserving the closed loop performance γ.*

Remark 3
This result coincides with the result in Theorem 1. However, by this result we can obtain a reduced-order H_∞ controller without using the bilinear transformation. Suppose that Σ_{21} has an invariant zero $\alpha > 0$ on the positive real axis with the geometric multiplicity m. Then, in the equation (14), we can choose a matrix $\tilde{V}_m$ in such a way that the matrix J is represented by

$$J = \alpha I_m. \tag{20}$$

Using a full column rank matrix V in (16) with this $\tilde{V}_m$, we can obtain a reduced-order H_∞ controller by the formula in (18).

Next, we show a result that is based on plural unstable invariant zeros on the positive real axis.

Theorem 4
Suppose that the H_∞ control problem for the generalized plant (1) under the assumptions **A1** *to* **A6** *is solvable. Furthermore, suppose that Σ_{21} has invariant zeros on the positive real axis, and we choose the matrix $\tilde{V}_m$ in (14) such that the matrix J is given by*

$$J = \mathrm{blockdiag}\,(\alpha_1 I_{m_1}, \alpha_2 I_{m_2}, \cdots, \alpha_l I_{m_l}), \quad \alpha_i \le \alpha_{i+1}, \forall i \in [1, \cdots, l-1] \subset \mathbb{Z} \tag{21}$$

where $\alpha_i > 0 (i = 1, 2, \cdots, l)$ are the invariant zeros of Σ_{21} and $m = \sum_{i=1}^{l} m_i$. Then if the following LMI:

$$2\alpha_1 F + \sum_{k=1}^{l-1} \epsilon_{1,k+1} \tilde{F}_{k+1} > 0 \tag{22}$$

holds, we can obtain a reduced-order H_∞ controller by the formula in (18), while preserving the closed loop performance γ. Here, parameters are given by

$$\begin{cases} F := \begin{pmatrix} \tilde{V}_m^T \\ 0 \end{pmatrix} Y_r Z_r \begin{pmatrix} \tilde{V}_m \\ 0 \end{pmatrix} > 0 \\ \epsilon_{1,k} := \alpha_k - \alpha_1 \ge 0 \\ \tilde{F}_k := \tilde{I}_k F + F \tilde{I}_k \\ \tilde{I}_k := \mathrm{blockdiag}\left(O_{\sum_{i=1}^{k-1} m_i}, I_{m_k}, O_{m - \sum_{i=1}^{k} m_i}\right) \end{cases},$$

where O_i is a zero matrix of size $i \times i$, and O_0 means empty.

Remark 4
In the LMI condition, if the parameter $\epsilon_{1,k+1}$ is small enough, the inequality is always satisfied. This means that if the invariant zeros are located close each other, the sufficient condition for the existence of a reduced-order H_∞ controller comes to be satisfied. In this sense, Theorem 4 includes Theorem 3.

Finally, we show the most general result that is based on unstable invariant zeros in $\mathbb{C}^+$.

Theorem 5
Suppose that the H_∞ control problem for the generalized plant (1) under the assumptions **A1** *to* **A6** *is solvable. Furthermore, suppose that we choose the matrix*

$\tilde{V}_m$ *in (14) in such a way that the matrix* J *is given by*

$$J = \text{blockdiag}\left(\hat{J}_1, \cdots, \hat{J}_l\right) \quad (23)$$

$$\hat{J}_i = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}, \quad i = 1, 2, \cdots, l, \quad (24)$$

$$\alpha_i \le \alpha_{i+1}, \forall i \in [1, \cdots, l-1] \subset \mathbb{Z}$$

where $\lambda_i := \alpha_i + j\beta_i (i = 1, 2, \cdots, l)$ *are invariant zeros of* Σ_{21}*, and* $m = 2l$*. Then, if the following LMI:*

$$2\alpha_1 F + \beta_1 \hat{F} + \sum_{k=1}^{l-1} \epsilon_{1,k+1} \tilde{F}_{k+1} + \sum_{k=1}^{l-1} \mu_{1,k+1} \bar{F}_{k+1} > 0 \quad (25)$$

holds, we can obtain a reduced-order H_∞ *controller by the formula in (18), while preserving the closed loop performance* γ*. Here, given a nonsingular matrix* $\hat{V}_m \in \mathbb{C}^{m \times m}$ *that satisfies*

$$J\hat{V}_m = \hat{V}_m \Lambda, \quad (26)$$

where Λ *is given by*

$$\Lambda = \text{blockdiag}\left(\begin{pmatrix} \lambda_1 & 0 \\ 0 & \bar{\lambda}_1 \end{pmatrix}, \cdots, \begin{pmatrix} \lambda_l & 0 \\ 0 & \bar{\lambda}_l \end{pmatrix} \right),$$

and the parameters appeared in the LMI (25) are defined by

$$\begin{cases} F := \hat{V}_m^* \begin{pmatrix} \tilde{V}_m^T \\ 0 \end{pmatrix} Y_r Z_r \begin{pmatrix} \tilde{V}_m \\ 0 \end{pmatrix} \hat{V}_m > 0 \\ \epsilon_{1,k} := \alpha_k - \alpha_1 \ge 0 \\ \tilde{F}_k := \tilde{I}_k F + F \tilde{I}_k \\ \tilde{I}_k := \text{blockdiag}\left(O_{2(k-1)}, I_2, O_{n-2k}\right) \\ \hat{F} := \Theta_m^* F + F \Theta_m \\ \mu_{1,k} := \beta_k - \beta_1 \\ \bar{F}_k := \tilde{\Theta}_k^* F + F \tilde{\Theta}_k \\ \tilde{\Theta}_k := \text{blockdiag}\left(O_{2(k-1)}, \Theta, O_{m-2k}\right) \\ \Theta := \begin{pmatrix} j & 0 \\ 0 & -j \end{pmatrix} \\ \Theta_m := \text{blockdiag}\left(\Theta, \cdots, \Theta\right) \in \mathbb{C}^{m \times m} \end{cases},$$

and O_0 *means empty.*

Remark 5
In the LMI condition (25), if the parameters $\epsilon_{1,k+1}$, β_1 *and* $\mu_{1,k+1}$ *are small enough the LMI is always satisfied. This means that if the invariant zeros are located close each other and moreover those locate near to the real axis, the sufficient condition for the existence of a reduced-order* H_∞ *controller intends to be satisfied. In this sense, Theorem 5 includes both Theorems 3 and 4.*

5 Extension to the stable zero case

In the previous sections, we have considered the H_∞ control problem under the assumption that Σ_{21} has invariant zeros in $\mathbb{C}^+$. Then, we have discussed the reduced-order H_∞ controller design based on the unstable invariant zeros. On the other hand, in this section we shall discuss the reduced-order H_∞ controller design based on invariant zeros located in $\mathbb{C}^-$ as well.

Consider a simple example. Suppose that a generalized plant is given by (2) where $\Sigma_{11}(s)$ and $\Sigma_{12}(s)$ are

$$\Sigma_{11}(s) = \frac{s+z}{(s+1)(s+p)}, \; \Sigma_{21}(s) = \frac{s+z}{s+p},$$

where $z \ne p, z > 0$. Since $\Sigma_{21}(s)$ has no zeros in $\mathbb{C}^+$ we cannot use the techniques from the previous section to reduce the order of the controller. However, by noticing the fact that, *if there exists an* H_∞ *controller for the following generalized plant:*

$$\hat{\Sigma}(s) = \begin{pmatrix} \Sigma_{11}(s)\frac{s-z}{s+z} & \Sigma_{12}(s) \\ \Sigma_{21}(s)\frac{s-z}{s+z} & \Sigma_{22}(s) \end{pmatrix},$$

the controller is also an H_∞ *controller for the generalized plant in (2)*, we know that a reduced-order H_∞ controller for the original problem can be obtained by solving the H_∞ control problem for $\hat{\Sigma}$. Moreover since the plant $\hat{\Sigma}$ satisfies the assumption **A1**, we can design a reduced-order H_∞ controller by using the method we have presented in the previous section. Thus, we have a conjecture that *we can obtain a reduced-order* H_∞ *controller for a generalized plant in which* Σ_{21} *and* $\begin{pmatrix} \Sigma_{11} \\ \Sigma_{21} \end{pmatrix}$ have common stable transmission zeros . A theorem that will be presented below supports our conjecture.

To state the theorem, we introduce a preliminary definition of the inner function and its properties.

Definition 1
Consider a system $\Theta = (A, B, C, D)$ *where* $\sigma(A) \subset \mathbb{C}^-$*. The system* Θ *is called inner if the transfer matrix* $\Theta(s)$ *satisfies* $\Theta(-s)^T \Theta(s) = I$*.*

Lemma 4
Let Θ *be an inner system, and* Q *be an LTI system with appropriate input-output dimensions that allow the multiplication* $Q\Theta$*. Then the following statements are equivalent.*

(i) The system $Q\Theta$ *is stable and* $\|Q\Theta\|_\infty < 1$*.*

(ii) The system Q *is stable and* $\|Q\|_\infty < 1$*.*

Now, we are in a position to state the main part of this section.

Lemma 5
Suppose that the systems $\begin{pmatrix} \Sigma_{11} \\ \Sigma_{21} \end{pmatrix}$ *and* Σ_{21} *have common stable transmission zeros and, additionally,* $D_{11} = 0$ *and* Σ_{21} *is square. Then, there exists a matrix* U *such that* $UU^T = I$ *and*

$$\left(A - B_1 D_{21}^\dagger C_2\right) U^T = U^T A_- \quad (27)$$

$$C_1 U^T = 0. \quad (28)$$

Let us define a system as

$$\Theta = \left(A_-, P^{-1}U \left(D_{21}^{\dagger} C_2 \right)^T, -D_{21}^{\dagger} C_2 U^T, I \right) \quad (29)$$

where $P > 0$ is a solution of the following Lyapunov equation:

$$PA_- + A_-^T P + U \left(D_{21}^{\dagger} C_2 \right)^T D_{21}^{\dagger} C_2 U^T = 0.$$

Here, we can easily verify that the system Θ is inner. Using this inner system, we can obtain the following result.

Theorem 6
Suppose that the systems $\begin{pmatrix} \Sigma_{11} \\ \Sigma_{21} \end{pmatrix}$ and Σ_{21} have common stable transmission zeros with $D_{11} = 0$ and Σ_{21} is square. Then, we have an inner system Θ given by (29) and given a controller Σ_K, the following statements are equivalent.

(i) Σ_K solves the H_∞ control problem for Σ.

(ii) Σ_K solves the H_∞ control problem for

$$\hat{\Sigma} = \begin{pmatrix} \Sigma_{11}\Theta & \Sigma_{12} \\ \Sigma_{21}\Theta & \Sigma_{22} \end{pmatrix}, \quad (30)$$

where $\hat{\Sigma}$ satisfies the assumption **A1**, *and the McMillan degree of $\hat{\Sigma}$ is equal to that of Σ.*

Remark 6
Suppose that there exists an H_∞ controller for the system $\hat{\Sigma}$. Then, we can obtain a reduced-order H_∞ controller for the original system Σ where Σ_{21} has no zero in $\mathbb{C}^-$ but $\mathbb{C}^+$ by applying the way we have presented in the previous section to $\hat{\Sigma}$.

6 Conclusions

In this paper, we have clarified a structure of the controller order reduction in the standard H_∞ problem. It was shown that the mechanism of the controller reduction is related to the unstable zeros of an off-diagonal subsystem of the generalized plant. Moreover the mechanism is investigated in detail by using the ARE-based H_∞ controller that is represented by a free parameter. Furthermore, we have extended the result to cases where the subsystem has plural stable zeros. Using this new fact we proposed a method for the designing of a reduced-order H_∞ controller that has order less than the order of the generalized plant. The feature of the presented design is that we can obtain a reduced-order H_∞ controller by solving a couple of AREs provided that solutions of the AREs satisfy an LMI condition. Thus this design effectively computes a reduced-order H_∞ controller on the basis of the structural information of the generalized plant. Further topic of this study is to incorporate the design philosophy into the LMI approach.

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