A BEHAVIORAL APPROACH TO H_2 OPTIMAL CONTROL

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Abstract

In this paper we consider a formulation of the H_2 optimal control problem in which the controller is not viewed as an operator that transforms measurements to controls. Instead, the controller is assumed to be a device that constrains a set of a-priori specified interconnection variables so as to achieve a controlled system whose manifest variables have a free component and satisfy an upperbound on the $\mathcal{L}_2$ norm of its impulse response. We formalize and solve this problem in a behavioral setting, that is, without reference to system representations and input-output structures. The equivalance between the H_2 control problem, the generalized H_2 control problem and the LQ optimal control problem are established.

1 Introduction

The theory of H_2 optimal control is undoubtedly among the most important tools for multi-variable control system analysis and design. In its most general formulation, the H_2 optimal control problem amounts to synthesizing a feedback controller for a given plant so as to minimize the H_2 norm of the transfer function of the controlled system which maps exogenous disturbances to a controlled output variable. When the exogenous disturbances are given a stochastic interpretation and assumed to be independent zero mean white noise sequences of unit variance then the H_2 optimal control problem is known as the Linear Quadratic Gaussian (LQG) problem, a problem which has been studied since 1960 and which has marked the beginning of modern control theory.

We refer to [4] and references therein for an extensive and recent treatment of the H_2 optimal control problem in its many variations. In this paper we propose a formulation of this problem which is motivated by recent achievements in the behavioral theory of dynamical systems. See [2,6–9]. In the behavioral setting, a controller is not viewed as an operator which transforms measurements to controls. Rather, the role of the controller is to constrain a distinguished set of interconnection variables of a plant, so as to achieve a more desirable behavior. Control system synthesis then amounts to designing an additional set of laws for a given plant, so as to achieve a specified control objective. The controlled system is the interconnection of the plant with the controller and is simply the restriction of the plant behavior to those trajectories which also satisfy the laws imposed by the controller. This setting has various advantages. Firstly, it is independent of input-output partitionings of the interconnection variables, i.e., there is no need to distinguish between measurement and control variables prior to the design of a control system. In particular, this means that the controller is not viewed as a signal processor that transforms measurements in controls. Secondly, the causality structure of the controlled system can be disregarded in the control synthesis problem. Thirdly, we believe that for many practical control problems this setting is more natural.

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2 Controlled systems

We consider discrete time systems with time set $T = \mathbb{Z}$ and finite dimensional real signal spaces $P = \mathbb{R}^{\mathtt{p}}$. Let $\ell(T, P)$ denote the set of all mappings $p : T \to P$. The systems we wish to control are given by subsets

$$\mathcal{P} \subseteq \ell(T, P)$$

which are linear, time-invariant and closed in the topology of pointwise convergence. It is well known [6,7] that these systems can be descried by

$$R(\sigma)p = 0$$

where $R(\xi) \in \mathbb{R}^{\cdot\times\mathtt{p}}[\xi]$ is a matrix with $\mathtt{p}$ columns of polynomials with real valued coefficients, and σ denotes the shift, defined for $p \in \ell(T, P)$ by $(\sigma p)(t) := p(t + 1)$. Hence, the systems we consider here are the solution sets of auto-regressive polynomial difference equations. This class of systems has been extensively studied [2, 6, 7] and will be denoted by $\mathbb{L}^{\mathtt{p}}$ or by $\mathbb{L}$ if the dimension of the signal space is clear from the context. In other words, $\mathbb{L}$ consists of those systems $\mathcal{P}$ which can be represented as $\mathcal{P} = \ker R(\sigma)$ for some matrix polynomial R.

Let $\mathcal{P} \in \mathbb{L}^{\mathtt{p}}$ be a dynamical system and suppose that its signal space P is the Cartesian product $P = W \times C$ with $W = \mathbb{R}^{\mathtt{w}}$ and $C = \mathbb{R}^{\mathtt{c}}$ both non-trivial. This means that trajectories $p \in \mathcal{P}$ are partitioned as $p = \mathrm{col}(w, c)$, where w has dimension $\mathtt{w}$ and c has dimension $\mathtt{c}$. We call w the *manifest variable*, c the *interconnection variable* and we refer to $\mathcal{P}$ as the *full plant behavior*. A *controller* (for $\mathcal{P}$) is a dynamical system $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$ which constrains the interconnection variables c of $\mathcal{P}$, (and thus also the manifest variables w). The *interconnection* of the plant $\mathcal{P}$ and the controller $\mathcal{C}$ is the system

$$\mathcal{K}_{\text{full}} := \{(w, c) \mid (w, c) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}$$

which we will refer to as the *full controlled system*. The idea is depicted in Figure 1.

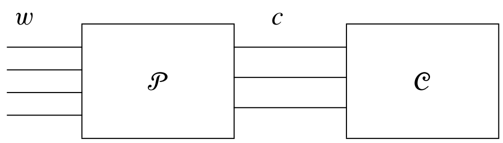

Figure 1: System interconnections

By eliminating the interconnection variable from the plant and the controlled system we obtain the following sets

$$\mathcal{P}_{\max} := \{w \mid \exists\, c \text{ such that } (w,c) \in \mathcal{P}\} \tag{1a}$$
$$\mathcal{K} := \{w \mid \exists\, c \text{ such that } (w,c) \in \mathcal{K}_{\text{full}}\} \tag{1b}$$
$$\mathcal{P}_{\min} := \{w \mid (w,0) \in \mathcal{P}\}. \tag{1c}$$

These are, respectively, the *maximal plant behavior*, the *controlled system behavior* and the *minimal plant behavior*. The maximal plant behavior has unconstrained interconnection variables and is therefore maximal in a set theoretic sense. The minimal plant behavior is maximally constrained in the sense that its interconnection variables are all set to zero. It is an easy exercise to show that for any plant $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ and controller $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$, the sets (1) all belong to $\mathbb{L}^{\mathtt{w}}$. We write $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$ to denote the interconnection of the plant $\mathcal{P}$ and the controller $\mathcal{C}$.

In this setting, control will mean to find $\mathcal{C}$, given $\mathcal{P}$, such that the controlled system $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$ consists of trajectories that are desirable or acceptable in a well defined sense. For this, the set $\mathcal{K}$ needs to be *implementable* and satisfy the *control objectives*. We will define these notions first.

Definition 2.1 (Implementability) A set $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ is *implementable* (for the plant $\mathcal{P}$) if there exists $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$ such that $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$.

That is, $\mathcal{K}$ is implementable if it is the interconnection of the plant $\mathcal{P}$ with a suitable controller $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$. To formalize desirable or acceptable behavior, we introduce the notion of a *control objective* as follows (see also [11]).

Definition 2.2 (Control objectives) A *control objective* is a quadruple $\mathcal{O} = (\mathcal{R}_{\min}, \mathcal{S}_{\min}, \mathcal{R}_{\max}, \mathcal{S}_{\max})$ of subsets of $\ell(T, W)$. A system $\mathcal{K}$ is said to *satisfy the control objective* $\mathcal{O}$ if

$$\mathcal{S}_{\min} \subseteq \mathcal{K} + \mathcal{R}_{\min} \tag{2a}$$
$$\mathcal{K} \cap \mathcal{R}_{\max} \subseteq \mathcal{S}_{\max}. \tag{2b}$$

The control objective $\mathcal{O}$ specifies desirable behavior in the sense that we find $\mathcal{K}$ acceptable only if it satisfies (2). Note that the inclusions (2) are dual. We will call (2a) a *minimal* and (2b) a *maximal requirement* for $\mathcal{K}$. The minimal requirement formalizes the idea that $\mathcal{K}$ should be "sufficiently rich" in that a suitable extension of this system contains at least a specified set of trajectories (such as disturbances, reference trajectories or norm-bounded signals). The maximal requirement specifies performance of the system.

The following result precisely characterizes the set of implementable systems. For a proof we refer to [1, 5].

Theorem 2.3 *$\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ is implementable if and only if*

$$\mathcal{P}_{\min} \subseteq \mathcal{K} \subseteq \mathcal{P}_{\max} \tag{3}$$

Hence, only the systems whose behaviors are wedged in between $\mathcal{P}_{\min}$ and $\mathcal{P}_{\max}$ are implementable. The main implication of Theorem 2.3 is that we can focus on the synthesis of controlled systems $\mathcal{K}$ rather than on the synthesis of controllers. Indeed, if $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ satisfies the inclusions (3), then an explicit algorithm for the synthesis of a controller $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$ such that $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$ is as follows.

1. Let $\mathcal{P} = \ker \begin{bmatrix} R_w(\sigma) & R_c(\sigma) \end{bmatrix}$ with $R = [R_w \;\; R_c] \in \mathbb{R}^{\cdot\times(\mathtt{w}+\mathtt{c})}[\xi]$ a polynomial kernel representation of the plant $\mathcal{P}$ and suppose that (3) holds.
2. Calculate a polynomial $U(\xi) \in \mathbb{R}^{\cdot\times\mathtt{c}}[\xi]$ such that $\mathcal{K} = \ker U(\sigma) R_w(\sigma)$. (Such U exists).
3. Set $C(\xi) = U(\xi) R_c(\xi)$ and define the controller
$$\mathcal{C} = \ker C(\sigma).$$

Then $\mathcal{C} \in \mathbb{L}^{\mathtt{c}}$ and it follows that $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$.

Given $\mathcal{P}_{\max}, \mathcal{P}_{\min}, \mathcal{R}_{\min}, \mathcal{R}_{\max}, \mathcal{S}_{\min}$ and $\mathcal{S}_{\max}$, the *control problem* is therefore to find $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$, if it exists, which is implementable (i.e., satisfies (3)) and which satisfies the control objective $\mathcal{O}$ (i.e., (2)). Any such $\mathcal{K}$ is said to be a *solution* of the control problem and in that case we will say that $\mathcal{K}$ *achieves $\mathcal{O}$ for the plant $\mathcal{P}$*. It is obvious that for a given $\mathcal{P}$ and given control objectives $\mathcal{O}$, solutions for the corresponding control problem may not exist. The following result provides a necessary condition for the existence of solutions.

Theorem 2.4 *$\mathcal{K}$ achieves the control objective $\mathcal{O}$ for the plant $\mathcal{P}$ only if*

$$\mathcal{P}_{\min} \cap \mathcal{R}_{\max} \subseteq \mathcal{S}_{\max},$$
$$\mathcal{P}_{\max} + \mathcal{R}_{\min} \supseteq \mathcal{S}_{\min}$$

Proof. If $\mathcal{K}$ achieves $\mathcal{O}$ for $\mathcal{P}$, then (3) and (2b) implies that $\mathcal{P}_{\min} \cap \mathcal{R}_{\max} \subseteq \mathcal{K} \cap \mathcal{R}_{\max} \subseteq \mathcal{S}_{\max}$. Similarly, (3) and (2a) imply $\mathcal{P}_{\max} + \mathcal{R}_{\min} \supseteq \mathcal{K} + \mathcal{R}_{\min} \supseteq \mathcal{S}_{\min}$ which yields the result. □

In words, a necessary condition for the existence of a controller that achieves acceptable performance, is that the minimal plant behavior $\mathcal{P}_{\min}$ must satisfy the maximal requirement, and the maximal plant behavior $\mathcal{P}_{\max}$ must satisfy the minimal requirement. These conditions are clearly not sufficient. The following result shows that solutions of the control problem are in general not unique.

Theorem 2.5 *If $\mathcal{K}_1$ and $\mathcal{K}_2$ achieve $\mathcal{O}$ for $\mathcal{P}$, then any $\mathcal{K}$ with $\mathcal{K}_1 \subseteq \mathcal{K} \subseteq \mathcal{K}_2$ also achieves $\mathcal{O}$ for $\mathcal{P}$.*

Proof. Immediate from (2) and(3). □

The observation of Theorem 2.5 is at the basis of the following result.

Theorem 2.6 *Let $\mathcal{P}, \mathcal{P}' \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ be two plants and suppose that $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ achieves $\mathcal{O}$ for $\mathcal{P}$. If $\mathcal{K} \subseteq \mathcal{P}'_{\max} \subseteq \mathcal{P}_{\max}$ then $\mathcal{K}$ also achieves $\mathcal{O}$ for $\mathcal{P}'$.*

Proof. Since $\mathcal{P}_{\min} \subseteq \mathcal{P}'_{\max} \subseteq \mathcal{P}_{\max}$, Theorem 2.3 implies that $\mathcal{P}'_{\max}$ is implementable for $\mathcal{P}$. Consequently, there exists $\mathcal{C}$ such that $\mathcal{P}'_{\max} = \mathcal{P} \sqcap \mathcal{C}$. But then $\mathcal{P}'_{\min} = \{w \mid (w, 0) \in \mathcal{P}'\} = \{w \mid (w, 0) \in \mathcal{P}, 0 \in \mathcal{C}\} = \mathcal{P}_{\min}$, so that $\mathcal{P}'_{\min} \subseteq \mathcal{K} \subseteq \mathcal{P}'_{\max}$. Conclude from Theorem 2.3 that $\mathcal{K}$ is implementable for $\mathcal{P}'$. The result then follows. □

3 Three control problems

In this section we will formalize and discuss three control problems As for the notation, let ℓ_2 be the set of square summable sequences on $\mathbb{Z}$ and denote by ℓ_2^+ the set of elements in ℓ_2 which vanish for $t < 0$. That is $w \in \ell_2^+$ if $w(t) = 0$ for $t < 0$ and the norm $\|w\|_2^2 := \sum_{t=0}^{\infty} |w(t)|^2$ is finite. Further, let ℓ_2^{f} and ℓ_2^{p} be the restriction of ℓ_2 to $\mathbb{Z}_+$ and $\mathbb{Z}_-$, respectively. (the superscripts stand for 'future' and 'past', respectively). Laplace or z-transformed signals $w \in \ell_2$ are denoted by $\hat{w}$. H_∞ and H_2 denote the usual (normed) spaces of complex vector-valued functions that are, resp., bounded and square integrable on the unit circle and which have bounded analytic continuation outside the unit circle.

Let w be partitioned as

$$w = \begin{pmatrix} d \\ z \end{pmatrix} \tag{4}$$

where d has dimension $\mathtt{d} > 0$, z has dimension $\mathtt{z} > 0$ and $\mathtt{w} = \mathtt{d} + \mathtt{z}$. The H_2 *control objective* is then defined by the quadruple $\mathcal{O}_{H_2} = (\mathcal{R}_{\min}, \mathcal{S}_{\min}, \mathcal{R}_{\max}, \mathcal{S}_{\max})$ where

$$\mathcal{R}_{\min} := \ker \begin{pmatrix} I_d & 0 \end{pmatrix} \tag{5a}$$

$$\mathcal{S}_{\min} := \ell_2^+, \tag{5b}$$

$$\mathcal{R}_{\max} := \{w \in \ell_2^+ \mid \sigma d = 0,\ \|d\|_2 \leq 1\} \tag{5c}$$

$$\mathcal{S}_{\max} := \{w \in \ell_2^+ \mid \|z\|_2 < 1\} \tag{5d}$$

This yields the following formulation of the H_2 control problem.

Definition 3.1 (H_2 control problem) The H_2 control problem amounts to finding a system $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ that achieves the objective $\mathcal{O}_{H_2}$ for a given plant $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$.

The control objective $\mathcal{O}_{H_2}$ can be interpreted as follows. With the specifications $\mathcal{O}_{H_2}$, the minimal requirement (2a) for the controlled system $\mathcal{K}$ is equivalent to the condition that d is a *free* ℓ_2^+ variable. This means that for every $d \in \ell_2^+$ there should exist at least one $z \in \ell_2^+$ such that $\mathrm{col}(d, z)$ is compatible with the controlled system in the sense that $\mathrm{col}(d, z) \in \mathcal{K}$. Stated otherwise, the controller which is supposed to achieve $\mathcal{O}_{H_2}$ is not allowed to constrain the variable d of the plant. To interpret the maximal requirement (2b), note that $w \in \mathcal{R}_{\max}$ if and only if $w = \mathrm{col}(d, z)$ where d satisfies $d(t) = 0$ for all nonzero $t \in \mathbb{Z}$, while $d(0)$ may be an arbitrary element in the unit ball of $\mathbb{R}^{\mathtt{d}}$. Any such d is therefore an impulse, $\mathcal{R}_{\max}$ is a set of *impulses* and it is for this reason that we refer to $\mathcal{K} \cap \mathcal{R}_{\max}$ as the *impulsive behavior* of $\mathcal{K}$. The maximal requirement therefore states that z satisfies the norm constraint $\|z\|_2 < 1$ in the impulsive behavior of the controlled system.

A second control problem is the following. Let ℓ_∞^+ denote the set of all signals w on $\mathbb{Z}$ which vanish for $t < 0$ and which have bounded amplitude

$$\|w\|_\infty := \sup_{t \geq 0} \sqrt{w^\top(t) w(t)}.$$

The *generalized H_2 control objective* is defined by the quadruple $\mathcal{O}_{GH_2} = (\mathcal{R}_{\min}, \mathcal{S}_{\min}, \mathcal{R}_{\max}, \mathcal{S}_{\max})$ where

$$\mathcal{R}_{\min} := \ker \begin{pmatrix} I_d & 0 \end{pmatrix} \tag{6a}$$

$$\mathcal{S}_{\min} := \ell_2^+, \tag{6b}$$

$$\mathcal{R}_{\max} := \{w \in \ell_2^+ \mid \|d\|_2 \leq 1\} \tag{6c}$$

$$\mathcal{S}_{\max} := \{w \in \ell_2^+ \mid \|z\|_\infty < 1\}. \tag{6d}$$

Hence, the minimal requirement in the control objective $\mathcal{O}_{GH_2}$ is equal to the minimal requirement in $\mathcal{O}_{H_2}$, whereas the maximal requirement in $\mathcal{O}_{GH_2}$ amounts to saying that in the controlled system all signals z, that are compatible with a d in the unit ball of ℓ_2^+, should have amplitude at most one.

Definition 3.2 (Generalized H_2 control problem) The generalized H_2 control problem amounts to finding a system $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ that achieves the objective $\mathcal{O}_{GH_2}$ for a given plant $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$.

Note the difference between the three control problems formulated in Definition 3.1 and Definition 3.2. We refer to [4] and [3] for a detailed treatment of, resp., the H_2 and generalized H_2 control problem in the usual input-output setting.

As a third control problem, we consider the Linear Quadratic (LQ) control problem which has been discussed and solved in the behavioral setting in [12]. The partitioning (4) of the manifest variable w will not be relevant for this problem. Let $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ be a given plant, $\mathtt{p} = \mathtt{w} + \mathtt{c}$ and let $p \in \ell(T, \mathbb{R}^{\mathtt{p}})$ be an arbitrary trajectory. The restrictions $p^- := p|_{\mathbb{Z}_-}$ and $p^+ := p|_{\mathbb{Z}_+}$ will be referred to as the *past* and the *future* of p, respectively. The set of *continuations* of p is defined as

$$\mathcal{P}(p) := \{\bar{p} \in \mathcal{P} \mid \bar{p}^- = p^-\}.$$

In words, the set of continuations of p consists of all trajectories whose past coincides with the past of p and whose futures are compatible with the laws of the plant. Note that $\mathcal{P}(p)$ is non-empty if and only if $p^- \in \mathcal{P}^-$ and note that $\mathcal{P}(p)$ only depends on the past of p. Define the control objective $\mathcal{O}_{LQ}$ by the quadruple:

$$\mathcal{R}_{\min} := \mathcal{P}_{\max}(0), \tag{7a}$$

$$\mathcal{S}_{\min} := \mathcal{P}_{\max}, \tag{7b}$$

$$\mathcal{R}_{\max} := \cup_{w^- \in \ell_2^{\mathrm{p}},\ \|w^-\|_2 \le 1} \mathcal{P}_{\max}(w) \tag{7c}$$

$$\mathcal{S}_{\max} := \{w \in \ell_2 \mid \|w^+\|_2 < 1\}. \tag{7d}$$

With these specifications, the minimal requirement (2a) of the LQ control objective is equivalent of saying that the state dimension of the controlled system should be no smaller than the state dimension of the plant. That is, the controller is not allowed to reduce the dimension of the state variable of the plant. Details of this claim can be found in [12]. The maximal requirement states that in the controlled system all futures that are compatible with a past in the unit ball of ℓ_2^{p} should have ℓ_2^{f}-norm at most one.

Definition 3.3 (LQ control problem) The LQ control problem amounts to finding a system $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ that achieves the objective $\mathcal{O}_{LQ}$ for a given plant $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$.

4 Main results

The main results in this section claim that the three control problems of Section 3 are, under suitable conditions, equivalent in a well-defined sense.

First, we will characterize the control objectives $\mathcal{O}_{H_2}$ and $\mathcal{O}_{GH_2}$. For this, consider the ℓ_2 behavior of a dynamical system $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$ which is defined as $\mathcal{K} \cap \ell_2^+$. Infer from [10] that $\mathcal{K} \cap \ell_2^+$ can be written as either the kernel or image of a rational, stable operator in H_∞, i.e., there exist rational $\Theta, \Psi \in H_\infty$ such that

$$\begin{aligned}\mathcal{K} \cap \ell_2^+ &= \{w \in \ell_2^+ \mid \Theta\hat{w} = 0\}\\ &= \{w \in \ell_2^+ \mid \hat{w} \in \Psi H_2\}\end{aligned}$$

where $\hat{w}$ denotes the z-transform of w.

Theorem 4.1 *Let $\mathcal{K} \in \mathbb{L}^{\mathtt{w}}$. Then the following statements are equivalent.*

1. *$\mathcal{K}$ satisfies the control objective $\mathcal{O}_{H_2}$.*
2. *$\mathcal{K}$ satisfies the control objective $\mathcal{O}_{GH_2}$.*
3. *For every full row rank representation*

$$\begin{pmatrix}\Theta_d & \Theta_z\end{pmatrix}\begin{pmatrix}\hat{d}\\ \hat{z}\end{pmatrix} = 0$$

of $\mathcal{K} \cap \ell_2^+$, Θ_z is square and invertible, $T := \Theta_z^{-1}\Theta_d$ belongs to H_∞ and

$$\|T\|_{H_2}^2 := \frac{1}{2\pi}\lambda_{\max}\left(\int_0^{2\pi} T^*(e^{i\omega})T(e^{i\omega})d\omega\right) < 1. \tag{8}$$

4. *For every full column rank representation*

$$\hat{w} = \begin{pmatrix}\hat{d}\\ \hat{z}\end{pmatrix} \in \begin{pmatrix}\Psi_d\\ \Psi_z\end{pmatrix} H_2$$

of $\mathcal{K} \cap \ell_2^+$, Ψ_d is square and invertible, $T := \Psi_z\Psi_d^{-1}$ belongs to H_∞ and satisfies (8).

Proof. (1 ⇒ 3). Suppose $\mathcal{K}$ satisfies $\mathcal{O}_{H_2}$. Then Θ_z is injective. Indeed, $\Theta_z z = 0$, $z \neq 0$ implies that $\mathrm{col}(0, \alpha z) \in \mathcal{K} \cap \mathcal{R}_{\max}$, and it follows that $\|\alpha z\| < 1$. Since α is arbitrary, this leads to a contradiction, proving that Θ_z is injective. By (2a), for all $d \in \ell_2^+$ there exists $z \in \ell_2^+$ such that $\Theta_d d + \Theta_z z = 0$. Hence, $\Theta_d H_2 \subseteq \Theta_z H_2$. As Θ has full rank, and Θ_z is injective, Θ_z must be square and invertible. Injectivity of Θ_z moreover implies that z is uniquely determined by d. Consequently, there exists a map $T : H_2 \to H_2$ such that $(d, z) \in \mathcal{K} \cap \ell_2^+$ if and only if $z = Td$. Obviously, $T = -\Theta_z^{-1}\Theta_d$ and T belongs to H_∞. Now use the isometry between ℓ_2^+ and H_2 (Parseval) to infer that (8) is equivalent to (2b).

(3 ⇒ 1). For all $d \in \ell_2^+$, $z := Td$ satisfies $\Theta_d d + \Theta_z z = 0$, implying that $\mathrm{col}(d, z) \in \mathcal{K}$. Hence (2a) holds. That (8) implies (2b) has already be proven.

(1 ⇔ 2). We will take a rather 'indirect' route via state space representations. Using the equivalence (1 ⇔ 3), we infer that (1) is equivalent of saying that $\mathcal{K}$ admits a representation of the form $x(t+1) = Ax(t) + Bd(t)$, $z(t) = Cx(t) + Dd(t)$ for which there exists $Y = Y^\top$ such that the inequalities

$$\begin{cases} Y > 0\\ Y > BB^\top + AYA^\top\\ DD^\top + CYC^\top < I\end{cases} \tag{9}$$

hold. Indeed, d should be a free variable and therefore qualifies as input of a suitable state representation of the operator T, whereas for impulsive d with $d_0 = d(0)$ and $\|d\|_2 \le 1$ we infer that

$$\|z\|_2^2 < d_0^\top d_0 \le 1$$

if and only if there exists $X = X^\top$ which satisfies the inequalities

$$\begin{cases} X > 0\\ X > C^\top C + A^\top XA\\ D^\top D + B^\top XB < I\end{cases} \tag{10}$$

Using duality, we infer that there exists X satisfying (10) if and only if there exists Y satisfying (9). With $x(0) = 0$, the existence of such a Y is equivalent to saying that there exists $\varepsilon > 0$ such that for all $d \in \ell_2^+$ and all $t \ge 0$

$$\begin{cases} x(t)^\top Y^{-1}x(t) \le \sum_{k=0}^t d^\top(t)d(t)\\ z^\top(t)z(t) \le (1-\varepsilon)x(t)^\top Y^{-1}x(t)\end{cases}$$

Hence,

$$z^\top(t)z(t) \le (1-\varepsilon)\sum_{k=0}^t d^\top(t)d(t) \le (1-\varepsilon)\|d\|_2^2$$

Taking the supremum over $t \geq 0$ yields $\|z\|_\infty \leq (1-\varepsilon)\|d\|_2^2$, which yields statement (2). Since this argument can be conversed, (2) also implies (1).

The equivalence ($1 \Leftrightarrow 4$) is the dual of ($1 \Leftrightarrow 3$) and proven in a similar way. □

An immediate consequence of Theorem 4.1 is that the H_2 and generalized H_2 control problem are equivalent. Precisely,

Theorem 4.2 *Let $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ be a given plant. Then the H_2 control problem is solvable if and only if the generalized H_2 control problem is solvable. Moreover, $\mathcal{K}$ achieves $\mathcal{O}_{H_2}$ for $\mathcal{P}$ if and only if $\mathcal{K}$ achieves $\mathcal{O}_{GH_2}$ for $\mathcal{P}$.*

Next, we will achieve the equivalence of either of these problems with the LQ control problem as formalized in Definition 3.3, for the so called full information case.

The full information case

The *full information case* concerns plants $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ with the property that the manifest variables w are observable from the interconnection variables c. Precisely, if $(w', c), (w'', c) \in \mathcal{P}$ implies that $w' = w''$ then w is said to be observable from c. The following result characterizes this property.

Theorem 4.3 *Let $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$. The following statements are equivalent.*

1. *in $\mathcal{P}$, w is observable from c.*
2. *$\mathcal{P}_{\min} = 0$.*
3. *there exists $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{c}}[\xi]$ such that $(w, c) \in \mathcal{P}$ implies $w = M(\sigma)c$.*

The full information case is therefore characterized by the property that all relevant knowledge of the manifest variables can be deduced from knowledge of the interconnection variables.

Consider the H_2 control problem as defined in Definition 3.1. Let $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ be a given plant and let its manifest variables $w = \operatorname{col}(d, z)$ be partitioned as in (4). Let π_z denote the projection of w onto the component z, i.e., $\pi_z(w) = z$. By Theorem 2.4, a necessary condition for solvability of the H_2 problem is that $\mathcal{S}_{\min} \subseteq \mathcal{P}_{\max} + \mathcal{R}_{\min}$. As noted before, this is equivalent of saying that d is a free variable in $\mathcal{P}_{\max}$. We assume this to be the case and make the following additional assumptions:

A1 $\mathcal{P}$ is controllable.

A2 $\mathcal{P}_{\min} = 0$.

A3 For all $z \in \ell_2$ with $z^- \in \pi_z \mathcal{P}_{\max}^-$ there exists $d \in \ell_2^+$ such that $z \in \pi_z \mathcal{P}_{\max}$.

For a definition of controllability in the behavioral setting we refer to [6,9]. Assumption A2 implies that we consider the full information case only. Assumption A3 can be interpreted as a property of *instantaneous controllability* of the state of the plant through the variable d. Indeed, A3 can be restated as $\pi_z[\mathcal{P}_{\max}(0)]^+ = \ell(\mathbb{Z}_+, Z)$, that is, every trajectory $z : \mathbb{Z}_+ \to Z$ can be obtained from the equilibrium response of $\mathcal{P}_{\max}$.

Define the *noise-free behavior* of the plant $\mathcal{P}$ as the set

$$\mathcal{Z} := \{(z, c) \in \ell(T, \mathbb{R}^{\mathtt{z}+\mathtt{c}}) \mid (0, z, c) \in \mathcal{P}\}.$$

It is immediate that $\mathcal{Z} \in \mathbb{L}^{\mathtt{z}+\mathtt{c}}$, and since $\mathcal{P}$ is controllable, also $\mathcal{Z}$ is controllable. We will consider the Linear Quadratic (LQ) control problem for the plant $\mathcal{Z}$. That is, let $\mathcal{Z}_{\max}$ and $\mathcal{Z}_{\min}$ denote the maximal and minimal behavior of $\mathcal{Z}$, respectively. Observe that $\mathcal{P}_{\min} = 0$ implies that $\mathcal{Z}_{\min} = 0$ and controllability of $\mathcal{Z}$ implies controllability of $\mathcal{Z}_{\max}$. The LQ problem for the plant $\mathcal{Z}$ is now specified by $\mathcal{O}_{LQ}$ and has formalized in Definition 3.3

Theorem 4.4 *Let $\mathcal{P} \in \mathbb{L}^{\mathtt{w}+\mathtt{c}}$ be a given plant and suppose that Assumptions A1, A2 and A3 hold. Let $\mathcal{Z}$ denote the noise free behavior of $\mathcal{P}$. Then there exists a non-singular matrix $R \in \mathbb{R}^{d\times d}$ such that for*

$$\mathcal{P}' := \{(z, d', c) \mid d' = Rd \text{ and } (z, d, c) \in \mathcal{P}\}$$

the following statements are equivalent

1. *The H_2 control problem is solvable for $\mathcal{P}'$.*
2. *The generalized H_2 control problem is solvable for $\mathcal{P}'$.*
3. *The LQ problem is solvable for $\mathcal{Z}$.*

Moreover, any controller $\mathcal{C}$ that solves one of these problems, will solve either of the other.

This means that under the given assumptions the three problems of Section 3 are equivalent up to a rescaling of the variable d.

5 A design example

The merits of the behavioral framework have been criticized more than once with the question whether something new has been contributed to the existing tools of analysis and synthesis in control engineering. The following example shows that a traditional input-output setting of the H_2 optimal control problem may lead to performance levels which can be out-performed when the setting is replaced by a behavioral one.

We consider the example of an active suspension of a transport vehicle as described in [11]. The plant is modeled by the equations

$$\begin{cases} m_2\ddot{q}_2 + b_2(\dot{q}_2 - \dot{q}_1) + k_2(q_2 - q_1) - F = 0 \\ m_1\ddot{q}_1 + b_2(\dot{q}_1 - \dot{q}_2) + k_2(q_1 - q_2) + k_1(q_1 - q_0) + F = 0 \end{cases}$$

where m_1, m_2 are masses of the axle mass and the chassis mass, resp., F is a force acting on m_2, q_i, $i = 1, 2$ denote the displacement of the mass m_i, and q_0 represents the vertical deviations in the road profile with respect to some equilibrium position and some velocity of the vehicle. The numbers k_1, k_2 and b_2 denote spring and damping constants. This defines a linear, time-invariant (continuous time) plant $\mathcal{P}'$ with manifest variables $w' := \mathrm{col}(q_0, \ddot{q}_2, q_2 - q_1, q_1 - q_0, F)$ and with interconnection variables $c := \mathrm{col}(\ddot{q}_2, q_2 - q_1, F)$.

A realistic set of model parameters* is given by $m_1 = 1.5 \times 10^3$, $m_2 = 1 \times 10^4$, $k_1 = 5 \times 10^6$, $k_2 = 5 \times 10^5$ and $b_2 = 5 \times 10^4$ which we will use in the present example. We extend the dynamics of the plant so as to incorporate suitable weighting parameters for the manifest variables reflecting road, passenger comfort and physical system characterisics. Specifically, the manifest variable w' of $\mathcal{P}'$ is replaced by w where w' and w are related by

$$w = \begin{pmatrix} .01(s+1) & 0 & 0 & 0 & 0 \\ 0 & \frac{s+10}{.01s^2+s+50} & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} w'$$

Hence the plant $\mathcal{P}$ consists of all trajectories (w, c) which satisfy these equations. Note that this plant is linear, time-invariant with signal spaces $W = \mathbb{R}^5$ and $C = \mathbb{R}^3$.

In the standard input-output setting of the H_2 optimal control problem, the interconnection variables c need to be partitioned in a measurement and a control variable. From a physical point of view, a possible choice is $c = \mathrm{col}(u, y)$ where $u = \ddot{q}_2$ is the control and $y = \mathrm{col}(q_2 - q_1, F)$ is the measurement variable.

Let $w = \mathrm{col}(d, z)$ where $d = w_1$ reflects the (weighted) road profile and $z = \mathrm{col}(w_2, \ldots, w_5)$ the (weighted) to-be-controlled variables.

With these specifications, the transfer function of the plant which maps $\mathrm{col}(d, u)$ to $\mathrm{col}(z, y)$ is proper. However, there does *not* exist a proper controller such that the H_2 norm of the closed-loop transfer function is finite. In other words, the standard H_2 optimal control problem has no solution in this input-output setting.

On the other hand, in the behavioral formulation of this H_2 control problem, (properly adapted to the continuous time case), solutions *do* exist in the sense that there exist implementable controlled systems $\mathcal{K}$ in which d is a free variable, and the to-be-controlled output z in the impulsive behavior of $\mathcal{K}$ satisfies the norm constraint $\|z\|_2 < 1$.

The main point of this example is that a classical input-output formulation of the H_2 control problem does not lead to solutions, whereas a formulation of this control problem in a behavioral setting does lead to a solution. As for this example, we remark that the controlled behavior $\mathcal{K}$ admits a feedback implementation in the sense that there exists a controller $\mathcal{C}$, a partitioning $c = \mathrm{col}(u, y)$ of the interconnection variables and a proper rational transfer function C such that

$$\mathcal{C} = \{c \mid c = \mathrm{col}(u, y),\ u = Cy\}$$

such that $\mathcal{K} = \mathcal{P} \sqcap \mathcal{C}$.

*Courtesy of B. de Jager, Department of Mechanical Engineering, Eindhoven University of Technology.

References

[1] J.W. Polderman and I. Mareels, *A Behavioral Approach to Adaptive Control*, in: The Mathematics of Systems and Control: From Intelligent Control to Behavioral Systems, (Eds. J.W. Polderman and H.L. Trentelman), University of Groningen, The Netherlands, 1999.

[2] J.W. Polderman and J.C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Vol. 26 of 'Texts in Applied Mathematics', Springer, New York, NY, U.S.A., 1997.

[3] M.A. Rotea, *The Generalized H_2 Control Problem*, Automatica, Vol. 29:2, (1993), pp. 373-385.

[4] A. Saberi, P. Sannuti and B.M. Chen, *H_2 Optimal Control*, Prentice Hall International, Series in Systems and Control Engineering, 1995.

[5] H.L. Trentelman, *A Truly Behavioral Approach to the H_∞ Control Problem*, in: The Mathematics of Systems and Control: From Intelligent Control to Behavioral Systems, (Eds. J.W. Polderman and H.L. Trentelman), University of Groningen, The Netherlands, 1999.

[6] J.C. Willems, *Models for Dynamics*, Dynamics Reported, Vol. 2, (1989), pp. 171-269.

[7] J.C. Willems, *Paradigms and Puzzles in the Theory of Dynamical Systems*, IEEE Transactions on Automatic Control, Vol. 36, (1991), pp. 259-294.

[8] J.C. Willems, *Feedback in a Behavioral Setting*, in: Systems, Models and Feedback: Theory and Applications, (Eds. A. Isidori and T.J. Tarn), Birkhäuser, Boston, (1992), pp. 179–191.

[9] J.C. Willems, *On Interconnections, Control and Feedback*, IEEE Transactions on Automatic Control, Vol. 42, (1997), pp. 326-339.

[10] S. Weiland and A.A. Stoorvogel, *Rational Representations of Behaviors: Interconnectability and Stabilizability*, Mathematics of Control, Signals, and Systems, Vol. 10:2, (1997), pp. 125-164.

[11] S. Weiland, A.A. Stoorvogel and B. de Jager, *A Behavioral Approach to the H_∞ Optimal Control Problem*, Systems and Control Letters, Vol. 32, (1997), pp. 323-334.

[12] S. Weiland and A.A. Stoorvogel, *A Behavioral Approach to the Linear Quadratic Optimal Control Problem*, Proceedings of the 14th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Perpignan, France, June 19-23, session IS05A, (2000).