Sampling per mode simulation for switching diffusions

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Plan

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    - and each component may influence the evolution of the other component.
- Our motivation is to estimate the probability that the continuous component hits a critical set.
- We use a splitting technique adapted to the context of switching diffusions: the sampling per mode algorithm introduced by Krystul in [Krystul, 2006]

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- Let $A \subset \mathbb{R}^d$ be a closed critical region in which X_t could enter but with a very small probability.
- If T_A denotes the hitting time of A, we would like to estimate $\mathbb{P}(T_A \leq T)$ with T a deterministic or a stopping time.

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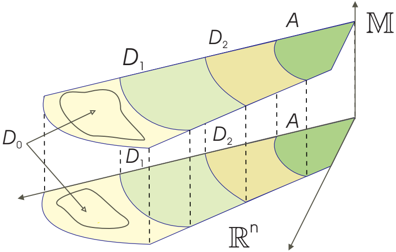

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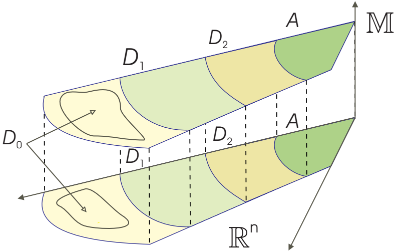

- With $B = A \times \mathbb{M}$ and $B_k = D_k \times \mathbb{M}$, we define for $k = 1, \cdots, n$

$$T_k = \inf\{t \geq 0 : Z_t \in B_k\} = \inf\{t \geq 0 : X_t \in D_k\},$$

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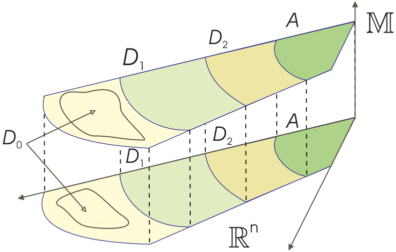

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- Then

$$\mathbb{P}(T_A \leq T) = \mathbb{P}(T_B \leq T) = \prod_{k=1}^{n} \mathbb{P}(T_k \leq T | T_{k-1} \leq T),$$

where conditional probabilities are not very small.

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  - to sample more particles from mode with higher probability,
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- Increasing the number of particles should improve the estimate but only at the cost of increased simulation time,
- Idea: keep constant the number of particles in each visited mode at each resampling step,

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Multilevel Feynman-Kac distributions

- To capture the behaviour of Z between each thresholds, we consider the random excursions $\mathcal{Z}_k$ of Z between T_{k-1} and $T_k \wedge T$

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- and we introduce the selection functions,

$$g_k(\mathcal{Z}_k) = 1_{\{Z_{T_k \wedge T} \in B_k\}}, \quad g_k^j(\mathcal{Z}_k) = 1_{\{Z_{T_k \wedge T} \in D_k \times \{j\}\}}, \quad j \in \mathbb{M},$$

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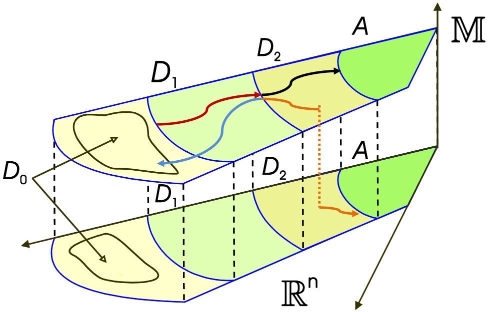

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- Clearly,

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$$\gamma_k(f) = \mathbb{E}\left[f(\mathcal{Z}_k) g_{k-1}(\mathcal{Z}_{k-1})\right] = \mathbb{E}\left[f((X_t, \theta_t),\ T_{k-1} \leq t \leq T_k \wedge T) 1_{\{T_{k-1} \leq T\}}\right]$$

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- and the corresponding normalized measures defined by

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- Then, we recover

$$\mathbb{P}(T_n \leq T) = \prod_{p=0}^{n} \eta_p(g_p),$$

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- In order to keep trace of the discrete mode, we construct for any $j \in \mathbb{M}$

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- We have the decompositions

$$\widehat{\eta}_k = \sum_{j \in \mathbb{M}} \omega_k^j \widehat{\eta}_k^j, \qquad \eta_{k+1} = \sum_{j \in \mathbb{M}} \omega_k^j \eta_{k+1}^j,$$

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- so, the following two separate selection/mutation transitions

$$\eta_k \xrightarrow{\text{selection}} \widehat{\eta}_k := \Psi_k(\eta_k) \xrightarrow{\text{mutation}} \eta_{k+1} = \widehat{\eta}_k\mathcal{M}_{k+1}.$$

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$$\eta_k^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_k^i} \xrightarrow{\text{selection}} \widehat{\eta}_k^N := \frac{1}{N}\sum_{i=1}^{N} \delta_{\widehat{\xi}_k^i} \xrightarrow{\text{mutation}} \eta_{k+1} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\xi_{k+1}^i}.$$

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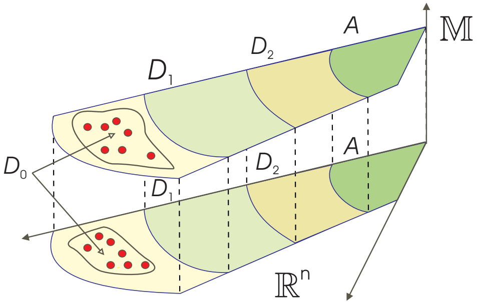
- Obviously, the total number of particles can change at each time some mode is not visited, or empty mode is visited afresh.
- Let $\widehat{N}_k$ and N_k denote the total numbers of particles $\widehat{\xi}_k$ and ξ_k, and $\omega_k^{j,N}$ the weights associated with the modes, we have the evolution scheme

$$(N_k, (\omega_{k-1}^{j,N})_{j \in J_{k-1}}, \xi_k) \to (\widehat{N}_k, (\omega_k^j)_{j \in J_k}, \widehat{\xi}_k) \to (N_{k+1}, (\omega_k^{j,N})_{j \in J_k}, \xi_{k+1})$$

where J_k denotes the set of non empty modes at step k

- In each mode j, we sample N^j particles $\xi_0^\kappa = \widehat{\xi}_0^\kappa = (0, (X_0^\kappa, j)) \sim \eta_0^j$.



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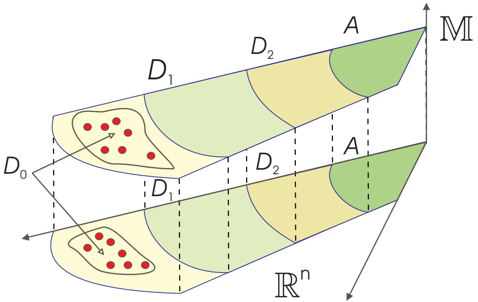

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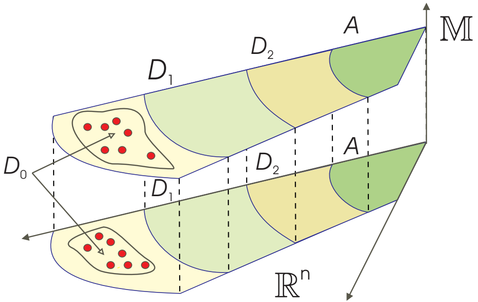

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- Here J_0^j is the set of the indices of the particles in the mode j.

Mutation $(\widehat{N}_k, \omega_k^N, \widehat{\xi}_k) \to (N_{k+1}, \omega_k^N, \xi_{k+1})$

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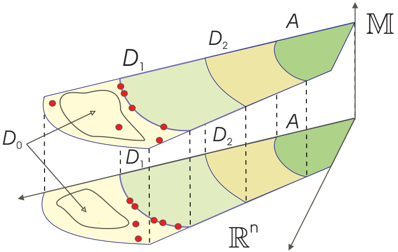

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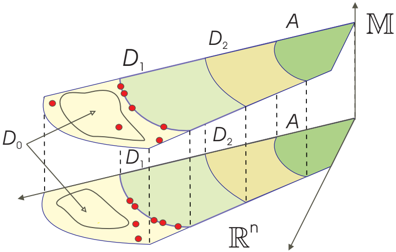

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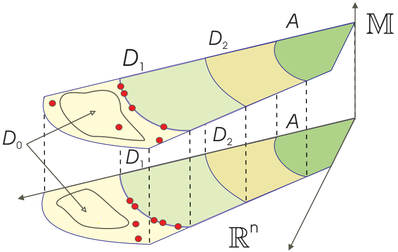

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- So $\eta_{k+1}^N = \sum_{j \in J_k} \omega_k^{j,N} \eta_{k+1}^{j,N}$, with $\eta_{k+1}^{j,N} = \frac{1}{N^j} \sum_{\kappa \in J_k^j} \delta_{\xi_{k+1}^\kappa}$, where J_k^j is the set of the labels of the particles in mode $j \in J_k$.

SELECTION/RESAMPLING $(N_{k+1}, \omega_k^N, \xi_{k+1}) \rightarrow (\widehat{N}_{k+1}, \omega_{k+1}^N, \widehat{\xi}_{k+1})$

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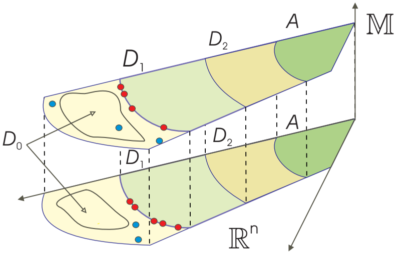

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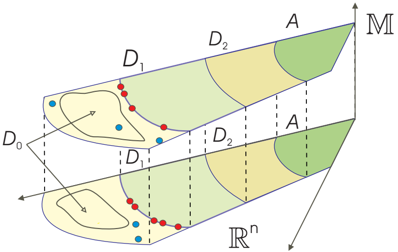

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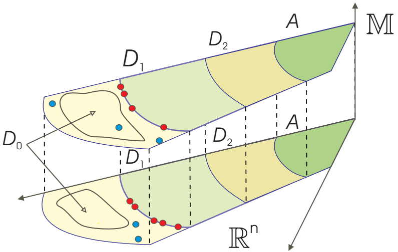

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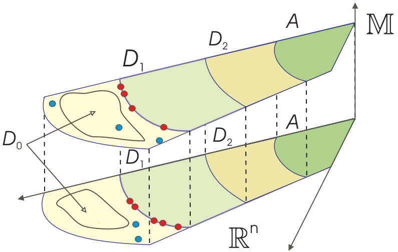

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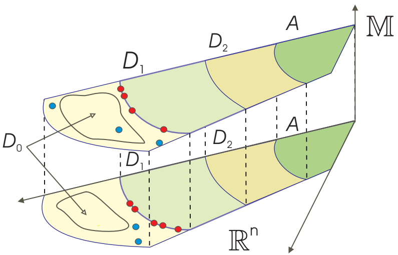

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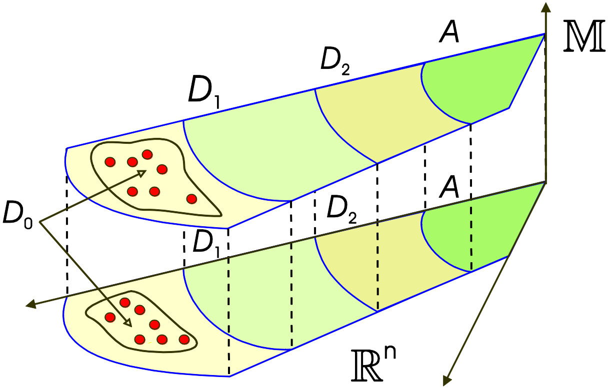
Sampling per Mode algorithm: Recapitulation



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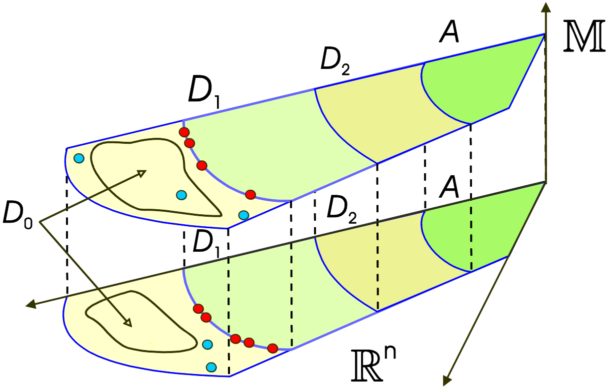

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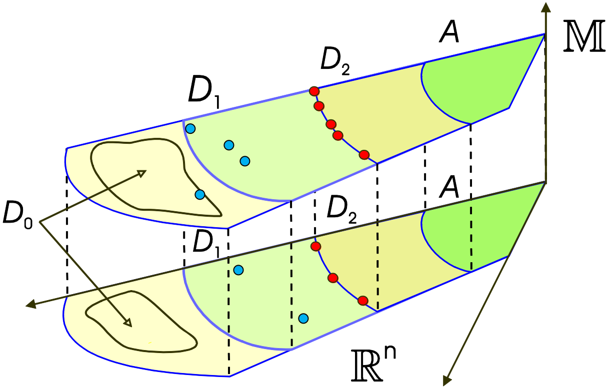

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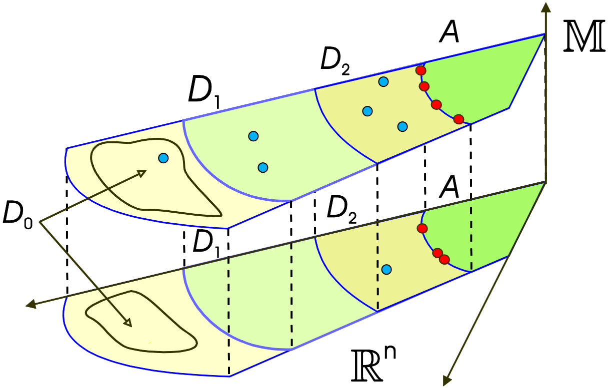

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$$\Omega_q = \sum_{j \in \mathbb{M}}(\omega_{q-1}^j)^2\rho_j^{-1} = 1 + \chi^2(\omega_{q-1}, \rho),$$

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- This algorithm is implemented in a software developed by National Aerospace Laboratory (NLR) and used to evaluate the safety characteristics of an arbitrary (new) operational Air Traffic Management concept [Blom, 2009].

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