

## THE VALUE FUNCTION OF A MODIFIED JACOBI FUNCTIONAL

E. van Groesen  
Department of Applied Mathematics  
University of Twente  
7500 AE Enschede, The Netherlands

ABSTRACT. Second order Hamiltonian systems with convex potentials are considered. To relate the value of the energy and the period for certain periodic motions, the value function of a modified Jacobi functional is investigated. A family of saddle points of this functional parameterized by the energy, provides periodic solutions for which the minimal period belongs to the generalized derivative of the value function.

### 1. INTRODUCTION

In this contribution we consider second order Hamiltonian systems

$$-\ddot{q} = V'(q), \quad q(t) \in \mathbf{R}^N, \quad (1.1)$$

with  $V$  the potential energy, a smooth function on  $\mathbf{R}^N$  and  $V'$  its gradient. For periodic solutions of (1.1) we want to relate the period  $T$  to the value  $E$  of the total energy:

$$\frac{1}{2} \dot{q}^2 + V(q) = E. \quad (1.2)$$

In Gordon [6] and Lewis [11] there are some partial results in this direction, presupposing the existence of a smooth manifold of such periodic solutions; the requirement of smoothness seems to be difficult to verify in general. In this paper we will characterize for each value of  $E$  a periodic solution as a critical point of a functional  $J_E$  which is a modification of the usual Jacobi functional. This modified functional is easier to deal with from a functional analytic point of view; in particular, we obtain a one parameter family of periodic solutions as explicit saddle points of  $J_E$  for each  $E$ . This saddle point is in fact a minimizer of  $J_E$  on a naturally constrained subset  $N_E$ . The corresponding (mini-max) value of  $J_E$  defines a value function (see Clarke [5] and his contribution to these proceedings for other examples and applications of value functions). We will show that, under suitable conditions,

the period of the solution belongs to the generalized derivative of this value function. This result can be used to further investigate the relation between  $E$  and  $T$  and to prove, for instance, a monotone dependence in specific cases. The relation between the period and the value function is not a consequence of standard results about value functions since, in our case, the critical value is a genuine saddle value or, expressed differently, both the functional  $J_E$  as well as the equality constraint in the definition of  $N_E$  depend on the parameter  $E$ .

For simplicity of exposition we will suppose from now on that  $V$  is an even function (see Remark 6 how the methods can be adapted to treat the general case). Then periodic solutions can be obtained by an appropriate continuation of the solution of a boundary value problem (cf. Rosenberg [14], Berger [3]). Specifically, on a normalized time interval, a solution  $x(t)$  of

$$\begin{aligned} -\ddot{x} &= T^2 V'(x) \\ x(0) &= \dot{x}(1) = 0 \end{aligned} \tag{1.3}$$

provides a so-called normal mode solution of period  $4T$  by continuation of the function  $q(t)$  defined for a quarter period  $T$  by

$$q(t) = x(t/T) . \tag{1.4}$$

(So, normal modes are brake orbits that go through the origin of configuration space.) The transformation (1.4) relates the energy  $E$  and the 1/4-period  $T(> 0)$  like

$$\frac{1}{2} \frac{1}{T^2} \dot{x}^2 + V(x) = E . \tag{1.5}$$

## 2. THE MODIFIED JACOBI FUNCTIONAL $J_E$

On the space of functions  $X := \{x \in H_1([0,1], \mathbb{R}^N) \mid x(0) = 0\}$  we consider the following functional  $J_E : X \rightarrow \bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$ :

$$J_E(x) = \begin{cases} 2 \left[ \int \frac{1}{2} \dot{x}^2 \cdot \{E - \int V(x)\} \right]^{1/2} & \text{if } \int V(x) < E \\ -\infty & \text{otherwise ;} \end{cases} \tag{2.1}$$

here  $\int$  denotes integration over the interval  $(0,1)$ . Note that  $J_E$  is a product of two functionals and that it differs in that respect essentially from the usual Jacobi functional

$$2 \int \sqrt{E - V(x)} \cdot \sqrt{\frac{1}{2} \dot{x}^2} .$$

We will call  $J_E$  a modified Jacobi functional since it results from the action functional on phase space in a way which is analogous to the way the usual Jacobi functional is obtained, the difference being that for the modification the energy constraint is prescribed in an integrated way instead of in a pointwise fashion. This integrated form of the energy constraint was introduced, and exploited in a different way, in the fundamental paper of Rabinowitz [13]. (See [7] for a historical account on the Euler-Maupertuis principle and details about the corresponding modification leading to (2.1).) The next result shows that  $J_E$  serves our purposes; the proof of it is straightforward.

Proposition 1. Let  $x(\neq 0)$  be a critical point of  $J_E$  on  $X$ . Then  $x$  satisfies the boundary value problem (1.3) for a value of  $T$  given by

$$T = \frac{d}{dE} J_E(x), \quad \text{i.e.} \quad T^2 = \frac{\int \frac{1}{2} \dot{x}^2}{E - \int V(x)}. \quad (2.2)$$

The corresponding normal mode defined by (1.4) has  $T$  as its 1/4-period and has energy  $E$ .

### 3. MINIMIZING $J_E$ ON NATURALLY CONSTRAINED SETS $N_E$

It is obvious that  $J_E$  is neither bounded from below nor from above on  $X$ . For rather general potentials  $V$  it is possible to prove the existence of at least one critical point using the mountainpass lemma of Ambrosetti and Rabinowitz [2]. Here we shall require  $V$  to be strictly convex which will enable us to provide a very explicit characterization of a saddle point of  $J_E$ . It is no restriction to assume that  $V$  is normalized such that  $V(0) = 0$  and  $V(x) > 0$  for all  $x \in \mathbb{R}^N$ .

Let us first introduce the natural constraints for the functional  $J_E$  (the idea goes back to Nehari [12]; Berger and Schechter [4] have considered such constraints in some generality; for an application of the idea in the context of Hamiltonian systems see also Berger [3] and Ambrosetti and Mancini [1]). For  $E > 0$  let  $N_E$  be the subset

$$N_E := \{x \in X \mid E = \int V(x) + \frac{1}{2} V'(x) \cdot x\}. \quad (3.1)$$

Proposition 2. For each  $E > 0$  the set  $N_E$  is a natural constraint for the functional  $J_E$  in the following sense:

- (i) any critical point of  $J_E$  belongs to  $N_E$ , and
- (ii) critical points of  $J_E$  on  $N_E$  are also critical points of  $J_E$  on all of  $X$ .

Proof. (i) Multiply the equation (1.3) by  $x$  and integrate by parts and use (2.2). (ii) By convexity of  $V$ , the set  $N_E$  is a regular manifold and Lagrange's multiplier rule is applicable. A

critical point  $x$  of  $J_E$  on  $N_E$  thus satisfies

$$-x = T^2 V'(x) + \lambda \left( \frac{3}{2} V'(x) + \frac{1}{2} V''(x) \cdot x \right)$$

for  $T^2$  given by (2.2) and some  $\lambda \in \mathbb{R}$ . To get the result, show that  $\lambda = 0$  by multiplying this equation by  $x$  and integrating by parts, and using  $x \in N_E$  and the convexity of  $V$ .

(The vanishing of the multiplier  $\lambda$  that results from the restriction to  $N_E$  explains the name "natural constraint".)  $\square$

As a consequence of this proposition, we may just as well look for critical points of  $J_E$  on  $N_E$  instead of on all of  $X$ . But on  $N_E$ , the functional  $J_E$  is strictly positive and we may look for a minimizer. The next result can be established (see [9]).

Proposition 3. For any  $E > 0$  there exists at least one solution of the constrained minimization problem

$$\inf\{J_E(x) \mid x \in N_E\} . \quad (3.2)$$

Moreover, for such a solution, the corresponding normal mode has  $T$  given by (2.2) as its minimal  $1/4$ -period.

Remark 4. By convexity of  $V$ , on each ray in  $X$  through the origin the functional  $J_E$  takes its maximum value at a unique point. The set  $N_E$  is precisely the collection of all these points and is strictly starshaped with respect to the origin. Consequently, introducing polar coordinates in  $X$ , with  $S$  the unit sphere  $\{y \in X \mid \int \dot{y}^2 = 1\}$ , the constrained minimization problem (3.2) is an explicit characterization of the following saddle point formulation:

$$\inf\{J_E(x) \mid x \in N_E\} = \inf_{y \in S} \max_{\rho > 0} J_E(\rho y) . \quad (3.3)$$

#### 4. THE VALUE FUNCTION AND ITS RELATION TO THE PERIOD

We introduce the value function as the value of  $J_E$  at the critical point characterized by (3.2):

$$j(E) := \inf\{J_E(x) \mid x \in N_E\} . \quad (4.1)$$

Proposition 5. For  $E > 0$ , the value function  $j(E)$  is continuous and monotonically increasing. If the right and left-hand side derivatives are denoted by  $j_+^!(E)$  and  $j_-^!(E)$  respectively, then if  $\hat{x}$  is any solution of the constrained minimization problem (3.2), the minimal  $1/4$ -period  $\hat{T}$  satisfies

$$j_+^!(E) < \hat{T} < j_-^!(E) . \quad (4.2)$$

Proof. The monotonicity is an immediate consequence of the monotonicity of the function  $E \rightarrow J_E(x)$  for each fixed  $x$ , and of the explicit mini-max characterization (3.3). To prove the first inequality of (4.2), the second one is analogous, let for  $e > 0$  a number  $\rho = \rho(e) \in \mathbf{R}$  be defined such that  $\rho(e) \cdot \hat{x} \in N_{E+e}$ . Then  $\rho$  is uniquely determined and  $\rho(0) = 1$  and  $\rho'(0) =$   
 $[\int V'(\hat{x}) \cdot \hat{x} + \frac{1}{2} V''(\hat{x}) \cdot \hat{x} \cdot \hat{x}]^{-1}$  is finite. Since  $j(E + e) < J_{E+e}(\rho\hat{x})$  by definition of  $j$ , the result follows from the fact that the first variation  $\delta J_E(\hat{x}; \cdot)$  vanishes and from (2.2):

$$\begin{aligned} j'_+(E) &< \lim_{e \rightarrow 0} \frac{1}{e} [J_{E+e}(\rho\hat{x}) - J_E(\hat{x})] \\ &= \frac{d}{dE} J_E(\hat{x}) + \delta J_E(\hat{x}; \hat{x}) \cdot \rho'(0) = \hat{T} . \quad \square \end{aligned}$$

In [10] this value function is exploited to find relations between  $T$  and  $E$ . For instance, for a specific class of subquadratic potentials it is shown that  $j$  is a convex function and hence, because of (4.2), is differentiable. Moreover, all solutions of (3.2) then have the same minimal 1/4-period  $T = j'(E)$ , and  $T$  runs from  $j'_T(0)$  to  $\infty$  if  $E$  runs from 0 to  $\infty$ ; at a given  $E$ , all solutions of (3.2) are also minimizers on  $X$  of the usual Lagrange functional for solutions of prescribed period  $T = j'(E)$ :

$$\frac{1}{T} \int \frac{1}{2} \dot{x}^2 - T \int V(x) .$$

See [10] for further results and for superquadratic potentials.

Remark 6. In case  $V$  is convex but not necessarily even, one can look for brake-orbits by considering  $J_E$  on all of  $H_1([0,1], \mathbf{R}^N)$ . Then  $\dot{x}(0) = \dot{x}(1) = 0$  result as natural boundary conditions and  $T$  is half of the period. Then analogous results can be obtained if one uses as natural constraints the sets

$$M_E := \{x \mid \int V'(x) = 0, \quad E = \int V(x) + \frac{1}{2} V'(x) \cdot x\}$$

(of codimension  $N + 1$ , in this case). See also Berger [3] for the additional constraint, and [9] for the existence result in this specific case.

#### REFERENCES

1. Ambrosetti, A. and G. Mancini, 'Solutions of minimal period for a class of convex Hamiltonian systems', Math. Analen 255 (1981), 405-421.

2. Ambrosetti, A. and P. H. Rabinowitz, 'Dual variational methods in critical point theory and applications', J. Funct. Anal. 14 (1973), 349-381.
3. Berger, M. S., 'Periodic solutions of second order dynamical systems and isoperimetric variational problems', Amer. J. Math. 93 (1971), 1-10.
4. Berger, M. S. and M. Schechter, 'On the solvability of semilinear gradient operator equations', Adv. in Math. 25 (1977), 97-132.
5. Clarke, F. H., Optimization and nonsmooth analysis, Wiley Interscience, New York, 1983.
6. Gordon, W. B., 'On the relation between period and energy in periodic dynamical systems', J. Math. Mech. 19 (1969), 111-114.
7. van Groesen, E. W. C., Hamiltonian flow on an energy surface: 240 years after the Euler-Manpertuis principle, in R. Martini (ed.), L. N. Physics, vol. 239, Springer, 1985.
8. van Groesen, E. W. C., 'On small period, large amplitude normal modes of natural Hamiltonian systems', Nonlin. Anal. T.M.A. 10 (1986), 41-53.
9. van Groesen, E. W. C., 'Analytical mini-max methods of Hamiltonian brake orbits of prescribed energy', J. Math. Anal. Appl., in press.
10. van Groesen, E., 'Duality between period and energy of certain periodic Hamiltonian motions', J. London Math. Soc., in press.
11. Lewis, D. C., 'Families of periodic solutions of systems having relatively invariant line integrals', Proc. Amer. Math. Soc. 6 (1955), 181-185.
12. Nehari, Z., 'On a class of nonlinear second order differential equations', Trans. Amer. Math. Soc. 95 (1960), 101-123.
13. Rabinowitz, P. H., 'Periodic solutions of Hamiltonian systems', Comm. Pure Appl. Math. 31 (1978), 157-184.
14. Rosenberg, R. M., 'Normal modes of non-linear dual-mode systems', J. Appl. Math. 27 (1960), 263-268.