Local and Global Bifurcations of a Three Particle System

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1. Introduction

Systems of differential equations with symmetry will bifurcate differently compared to systems without symmetry. For instance at a Hopf bifurcation point, where a symmetric steady state looses its stability, more than one branch of periodic states may be produced. Each of these branches can be detected by restricting the state space suitably, and applying standard Hopf bifurcation results within the restricted state space.

We show how the same ideas can be applied to obtain global branches by variational methods. Again, symmetry considerations lead to the choice of suitable functions spaces. The level branches can be continued globally by minimizing suitable functionals. We give the details for a three particle system.

2. Local $D_3 \times S_1$ Equivarant Bifurcations

The general theory of symmetric Hopf bifurcation was developed in Golubitsky and Stewart [4]. A system of N identically coupled particles has the symmetry corresponding to discrete rotation and reflection. Phrased differently, the system of ODE is equivariant with respect to the dihedral group of 2N elements, D_N. The corresponding Hopf bifurcation results have been found independently by Golubitsky and Stewart [5] and Van Gils and Valkering [3].

Here we are interested in a system of 3 particles coupled by a nearest neighbour potential. We review some of the known results on periodic states.

The equations of motion are given by

$$(2.1) \qquad \ddot{y}_n = -\frac{\partial V}{\partial y_n} \qquad (\dot{\ } = \frac{d}{dt}, \quad y_n = y_{n(\mathrm{mod}3)}).$$

where V is a nearest neighbour potential determined by a C^∞-function W, $W(0) = W'(0) = 0$, $W''(0) > 0$

$$(2.2) \qquad V = \sum_1^3 W(y_n - y_{n-1}).$$

System (2.1) admits the trivial solution y_n = constant. After transforming (2.1) to a first order ODE, one finds that the linearization around this solution has a double eigenvalue at zero, with geometrical multiplicity one, corresponding to the motion of the mass centre, and a pair of semi-simple purely imaginary complex conjugate eigenvalues which are double, due to the symmetry.

First one needs coordinates in which the symmetry is described naturally. The group D_3 has a two-dimensional irreducible representation. Dividing out the motion of the mass centre (reducing to a four dimensional system) we can thereafter choose coordinates in $\mathbb{R}^4 \simeq \mathbb{C}^2$ such that the action of D_3 on $\mathbb{C}^2$ is generated by the rotation φ and

(2.3a) $\gamma \cdot (z_1, z_2) = \left(e^{i\frac{2\pi}{3}} z_1, e^{-i\frac{2\pi}{3}} z_2\right)$

(2.3b) $\kappa \cdot (z_1, z_2) = (z_2, z_1)$

and the vectorfield in the new coordinates commutes with this action. After a near identity transformation, we may also achieve that, up to any finite order in the Taylor expansion in $(z_1, \bar{z}_1, z_2, \bar{z}_2)$ the vector field commutes with the circle action given by the flow generated by the linear part of the vector field. Or, phrased differently, the appropriate bifurcation function obtained by a Liapunov-Schmidt reduction will commute with this circle action. The circle action is given by:

(2.3c) $\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2)$.

Given an element $u \in \mathbb{C}^2$, by Σ_u we denote those elements from the group $D_3 \times S^1$ which leave u invariant. Fix (Σ_u) is the set of all $z \in \mathbb{C}^2$ which are Σ_u-invariant.

The groep $D_3 \times S^1$ acting on $\mathbb{C}^2$ as described in (2.3a-c) has three non trivial isotropy subgroups, each of them has a two dimensional fixed point space. These are listed below.

	Isotropy subgroup	Fixed point space	Periodic state
a)	$\tilde{\mathbb{Z}}_3 = \{(\gamma, -\frac{2\pi}{3})\}$	$(z_1, 0)$	rotating wave.
b)	$\mathbb{Z}_2 = \{(id, 0), (\kappa, 0)\}$	(z_1, z_1)	standing wave.
c)	$\mathbb{Z}_2^c = \{(id, 0), (\kappa, \pi)\}$	$(z_1, -z_1)$	standing wave.

It is a trivial, but nevertheless important, observation that the vectorfield leaves a fixed point space invariant. As the above spaces are two dimensional, we can apply in each of them Liapunov's Center Theorem to (2.1).

This yields three branches of periodic solutions with the symmetry of the corresponding isotropy subgroup.

a) For the rotating wave the invariance under $\tilde{\mathbb{Z}}_3$ means that

$$y_i(t) = y_{i-1}(t-T/3), \text{ where T is the period.}$$

b) For the standing wave invariant under $\mathbb{Z}_2$ we find that:

$$y_1 \equiv 0; \qquad y_2(t) = -y_3(t).$$

c) For the standing wave invariant under $\mathbb{Z}_2^c$ we find that

$$y_1(t) = -y_1(t+\tfrac{T}{2}), \qquad y_2(t) = -y_3(t+\tfrac{T}{2}).$$

All orbits that bifurcate from the equilibrium state are related to these three by the group action of D_3 and/or a time shift. For instance, applying a reflection to a) yields a wave that rotates the other way around

a') $\qquad y_i(t) = y_{i-1}(t+\tfrac{T}{3}).$

Clearly the solutions b) and c) are invariant for reflection about the first particle (apart from time shift in case c). Applying other reflections or rotations one finds solutions that are invariant for reflection with respect to one of the other particles.

Solutions satisfying a)–c) can be given a standard form as follows. a) yields immediately that the wave can be written with one scalar function ξ and a constant a as

(2.4a) $\qquad y_i = a + \xi(t+i\tfrac{T}{3}), \quad \overline{\xi} = 0.$

The overbar denotes time averaging. In case b) one obtains similarly

(2.4b) $\qquad y_1 = 0, \quad y_2 = a+\xi, \quad y_3 = -a-\xi, \qquad \overline{\xi} = 0.$

To find a simple form in case c) introduce the symbol L for the reflection about the first particle

$$(Ly)_1 = -y_1, \quad (Ly)_2 = -y_3, \quad (Ly)_3 = -y_2.$$

Then c) reads $(Ly)(t) = y(t+\tfrac{T}{2})$. Shifting time over $\tfrac{T}{2}$ one finds, using the periodicity of y, $y(t) = (Ly)(t+\tfrac{T}{2})$. Adding these two equalities one obtains

$$(Ly+y)(t) = (Ly+y)(t+\tfrac{T}{2}).$$

This implies that Ly+y is constant, and especially $y_2 - y_3$ is constant. Characterisation c) above yields $\overline{y}_1 = -\overline{y}_1$, $\overline{y}_2 = -\overline{y}_3$ so that finally

(2.4c) $\qquad y_1 = \eta, \quad y_2 = \xi+a, \quad y_3 = \xi-a, \qquad \overline{\xi} = \overline{\eta} = 0.$

Expressions (2.4a–c) are obtained on the basis of a)–c) only. We now derive further restrictions from the fact that the equations (2) are invariant for time reversal and from the uniqueness of the solutions a)–c).

Consider a solution y(t) of a). Then y(-t) is in the family a'), which is found from a) by applying L. Consequently there is a t_0 so that

$$(2.5) \qquad (Ly)(t+t_0) = y(-t) \qquad .$$

Inserting (2.4a) one obtains $a+\xi(t) = -a-\xi(-t+t_0)$ from which it follows that the time origin can be chosen so that ξ is odd

$$(2.6a) \qquad \xi(t) = -\xi(-t), \qquad a = 0.$$

For a remark about the condition a = 0 see below. Time reversed solutions in the cases b) and c) must yield a solution in each family respectively. Consequently in case b) we have $a+\xi(t) = a+\xi(-t+t_0)$ so that we may choose

$$(2.6b) \qquad \xi(t) = \xi(-t).$$

For the last case c) one finds similarly

$$(2.6c^1) \qquad \xi(t) = \xi(-t), \qquad \eta(t) = \eta(-t).$$

Since it also holds that $\eta(t) = -\eta(t+\frac{T}{2})$ and the same for ξ, both functions are odd about $\frac{T}{4}$.

REMARK. Our system has one more symmetry, viz a continuous shift of the coordinates $y_i \to y_i + x$. This implies that to any solution a constant can be added. For the above analysis this has two consequences. The requirement (2.5) is no longer correct: due to the fact that in (2.5) an arbitrary constant (d,d,d) has to be added. Consequently, the condition a = 0 in (2.6a) has to be dropped. In case c) we may require as extra condition that $y_1 + y_2 + y_3 = 0$, so that

$$(2.6c^2) \qquad \xi = -\frac{\eta}{2} .$$

3. Global Variational Methods

To extend the local bifurcation results of the foregoing section to global branches of finite amplitude solutions we will use the following variational formulation. On the set of 2π-periodic vector functions x in the first Sobolev space $H = H^1_{per}([0,2\pi],\mathbb{R}^3)$ we consider the functionals

$$(3.1) \qquad K(x) = \frac{1}{2}\int \dot{x}^2 d\tau \quad \text{and} \quad U(x) = \int V(x)d\tau$$

(integration over $[0,2\pi]$) and look for critical points of U on level sets of K. For given $R \in \mathbb{R}_+$ we describe this critical point problem by

$$(3.2) \qquad \text{crit}(U(x)\,|\,K(x) = R,\ x \in H).$$

Such critical points satisfy (under mild conditions) the equations

$$(3.3) \qquad -\ddot{x} = \lambda\frac{\partial V}{\partial x}$$

for some multiplier $\lambda \in \mathbb{R}$, and periodic boundary conditions.

Provided that λ is positive, a simple time-scaling shows that a solution of (3.3) corresponds to a periodic solution of (2.1) with period $\sqrt{\lambda}$. Therefore, periodic solutions of (2.1) will be obtained by proving the existence of critical points of (3.2) in a direct way.

This procedure has been used in this form for the first time by Berger [1,2] to obtain a whole family of periodic solutions, parametrized with the parameter R.

Instead of (3.2) variational principles can be formulated where the period or the total energy are prescribed and treated as parameters. Although closely related to (3.2) the critical points of the functionals in these formulations are generally more difficult to characterize than for (3.2) (see van Groesen [6,7,8,9]).

Returning to (3.2), as is often the case for such variational principles, non-trivial periodic solutions of (2.1) are genuine saddle-points for (3.2). To prove existence, topological methods may occasionally be used. Probably also for the case under consideration (see Rabinowitz e.a. [10] and the references therein for such topological methods which require certain index theories connected to the symmetry of the problem).

Here we prefer to explore a different way (following Berger [1,2], Van Groesen [6,7,8,9] and Valkering [11,12]), which will provide a very explicit characterization of the three different wave motions indicated in the foregoing section.

The general idea is to find a subset X of H such that, first of all, the Euler-Lagrange equation (3.3) is not altered by the restriction to X; for obvious reasons such a set X is called a natural constraint or a naturally constrained set. Moreover, we will be able to find such sets X for which U can in fact be maximized on the level sets of K in X. Hence, a periodic solution of (2.1) is then characterized by a solution of the maximization problem

$$(3.4) \qquad \sup\{U(x)\,|\,K(x) = R,\ x \in X\},$$

which solution corresponds to a saddle point of (3.2). (Note that the minimization problem for (3.2) always provides a trivial solution x = constant.)

The construction of a naturally constrained set X is by no means straightforward and no general theory is available yet. For the case under consideration we will exploit the fact that we know the symmetries of the system in the construction of the sets X. A consequence of the symmetries is that for each of the three different wave forms described in section 2, the set X of 3-vector functions depend in fact only on a scalar function (cf. (2.6a-c^2)). Explicitly:

$$(3.5)\qquad \begin{aligned} X_a &= \{x = (\xi(t),\xi(t-\tfrac{2\pi}{3}),\xi(t-\tfrac{4\pi}{3}))\,|\,\xi(t) = -\xi(-t),\ \xi \in \hat{H}\}, \\ X_b &= \{x = (0,\xi(t)+a,-\xi(t)-a)\,|\,a \in \mathbb{R},\ \xi(t) = \xi(-t),\ \xi \in \hat{H}\}, \\ X_c &= \{x = (-\tfrac{2}{3}\xi(t),\tfrac{1}{3}\xi(t)+a,\tfrac{1}{3}\xi(t)-a)\,|\,a \in \mathbb{R},\ \xi(t) = \xi(-t),\ \xi \in \hat{H}\}, \end{aligned}$$

where $\hat{H} = \{\xi \in H^1_{per}([-\pi,\pi],\mathbb{R})\,|\,\int_{-\pi}^{\pi} \xi = 0\}$.

Simple calculations show that the expresssions for K and U on the sets X_i, $i \in \{a,b,c\}$ reduce to

$$(3.6)\qquad \begin{aligned} K_a &= \tfrac{3}{2}\int \dot{\xi}^2 &, \quad U_a(\xi) &= 3\int_{-\pi}^{\pi} [W(\xi(t)-\xi(t-\tfrac{2\pi}{3}))], \\ K_b &= \int \dot{\xi}^2 &, \quad U_b(\xi,a) &= \int_{-\pi}^{\pi} [2W(\xi(t)+a)+W(-2\xi(t)-2a)], \qquad \text{and} \\ K_c &= \tfrac{1}{3}\int \dot{\xi}^2 &, \quad U_c(\xi,a) &= \int_{-\pi}^{\pi} [W(\xi(t)+a)+W(-2a)+ W(-\xi(t)+a] . \end{aligned}$$

So far we have exploited the symmetries of the problem and reduced the problem (3.2) to a scalar problem.

In a completely standard way, functional analytic methods show that in case a) the maximization problem has a solution for each $R > 0$:

$$\sup\{U_a(\xi)\,|\,K_a(\xi) = R,\ \xi \in \hat{H}\}.$$

The maximization problem for cases b) and c), however, is not well posed (take e.g. ξ fixed, $a \to \infty$). In fact, the behaviour of U with respect to the constant a and with respect to the constrained function ξ, leads one to look for a saddle point of the following explicit form:

$$(3.7)\qquad \sup_{\xi}\ \inf_{a\in\mathbb{R}} \{U_i(\xi,a)\,|\,K_i(\xi) = R,\ \xi \in \hat{H}\}, \qquad i \in \{b,c\}.$$

Under mild conditions on W this minimax problem has indeed a solution. For instance, if W is a convex function, the minimization over $a \in \mathbb{R}$ (for given $\xi \in \hat{H}$) has a unique solution. If we write, for simplicity, $U_i(\xi,a) = \int_{-\pi}^{\pi} \hat{W}_i(\xi,a)$ this optimal value of a is determined by the equation $\int_{-\pi}^{\pi} \frac{\partial \hat{W}_i}{\partial a}(\xi,a) = 0$. This equation, considered as an additional constraint on ξ, can be shown to be a natural constraint (see the papers of Berger and Van Groesen).

Exploiting this fact, the minimax problem (3.7) reduces again to a maximization problem:

$$\sup_{\xi} \left\{ \int_{-\pi}^{\pi} \hat{W}_i(\xi,a) \,\middle|\, K_i(\xi) = R, \int_{-\pi}^{\pi} \frac{\partial \hat{W}_i}{\partial a}(\xi,a) = 0, \ \xi \in \hat{H} \right\}, \qquad i \in \{b,c\}.$$

Elementary methods suffice to prove the existence of a solution of this problem.

In this way we have proved the existence of global branches of 3 different kind of waves by characterizing each element on a branch as a solution of an explicit minimization problem. Besides being able to prove existence, such an extremal characterization gives some additional information. In fact, from the extremal property it follows in a straightforward way that the solutions obtained have in fact 2π as minimal period (in the rescaled time), and not $\frac{2\pi}{k}$ for some $k \in \mathbb{N}$, $k \geq 2$. Moreover, the extremal formulation makes one to study the solution analytically for limiting values of R.

In the case $R \downarrow 0$ one obtains the local bifurcation results of section 2 and if R approaches its maximal value one studies the hard core interaction (Valkering [1982]). Another application is the construction of efficient numerical methods of intermediate values of R.

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