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Optimal Adaptive Control for a Class of Stochastic Systems

Arunabha Bagchi
Dept. of Applied Mathematics
University of Twente
P.O. Box 217
7500 AE Enschede
The Netherlands

Han-Fu Chen
Institute of Systems Science
Chinese Academy of Sciences
Beijing 100080
People's Republic of China

Abstract

We study linear-quadratic adaptive tracking problems for a special class of stochastic systems expressed in the state-space form. This is a long-standing problem in the control of aircraft flying through atmospheric turbulence. Using an ELS-based algorithm and introducing dither in the control law we show that the resulting control achieves optimal cost in the limit, while simultaneously the unknown parameters converge to their true values.

1. Introduction

There is an enormous literature on stochastic adaptive control starting with the pioneering work of Åström and Wittenmark on self-tuning regulator [1]. Most of the research in this area, however, concentrated on ARMAX models [2]. Parallel to this was the somewhat unrelated development of the design of adaptive flight control systems, starting with the thesis of Illiff [3]. Most researchers in this area start with the dynamical model of aircraft in flight and, consequently, formulate their problem in the state-space form. Unfortunately, the literature on stochastic adaptive control for systems in state-space form is rather limited. Kumar [4] made a thorough analysis on the problem of controlling an unknown linear-Gaussian system with quadratic criterion, but he had to restrict himself to the case of complete observation of the system states. We study here the problem of controlling a linear system with incomplete and inaccurate observation of the system states so that a quadratic tracking criterion is minimized in the situation when the matrix multiplying the control term in the state equation is unknown. We do not assume that the observation noise is Gaussian, but do restrict ourselves to the situation with no state noise. This problem arises naturally in controlling the flight of an aircraft in atmospheric turbulence where the objective is to minimize the normal acceleration or gust response in the angle of attack [5]. This is done in order to improve passanger and pilot comfort. It corresponds to our problem when the so-called control derivatives of the aircraft are unknown.

2. Problem Formulation

Consider the following discrete-time dynamical system

$$x_{k+1} = Ax_k + Bu_k \tag{1}$$

$$y_k = Cx_k + w_k \tag{2}$$

where x_k and y_k, for fixed k, are $\Re^n-$ and $\Re^m$-valued state and observation vectors, respectively; u_k is an $\Re^p$-valued control vector, $\{w_k\}$ is a noise sequence to be specified below, the matrices A and C are known, but the matrix B is unknown. Our objective is to minimize the tracking criterion

$$J = \lim_{N\to\infty} \sup \frac{1}{N} J_N(u) \tag{3}$$

where

$$J_N(u) \triangleq \sum_{k=0}^{N-1} [(x_k - x_k^*)' Q_1 (x_k - x_k^*) + u_k' Q_2 u_k] \tag{4}$$

with $Q_1 \geq 0, Q_2 > 0$ and $\{x_k^*\}$ a prescribed sequence of desired path.

We follow the self-tuning approach which is based on the certainty equivalence principle. In the specific problem considered here, Kalman filter method yields readily a recursive estimator for B [6]. The estimator also has been proved in [6] to converge to the true value in the mean-square sense. When we close the control loop the analysis becomes more complicated. Asymptotic optimality of the resulting control law has not been established yet. One difficulty is that the estimator loses the interpretation of being the conditional espectation when the system operates in closed loop. We propose in the next section an ELS-based method to estimate B.

The rest of this section is devoted to determining the optimal control law when B is known. For this, we make the following assumptions:

- A1 $\alpha^{-1}(e^{i\lambda})+\alpha^{-1}(e^{-i\lambda})-1>0 \quad \forall\lambda\in[0,2\pi)$, where $\alpha(z)\triangleq\det(I-zA)\equiv 1+a_1z+\cdots+a_nz^n, z\in\mathcal{C}$. This is the *strict positive real* (SPR) condition. Note that this condition implies $\alpha(z)\neq 0 \ \forall|z|\leq 1$ (see [7, corollary 4.1]).
- A2 (A,C) is observable.
- A3 (A,B) is controllable and (A,D) is observable, where D is any matrix satisfying $D'D=Q_1$.
- A4 $\{w_k,\mathcal{F}_k\}$ is a martingale difference sequence (mds) with, for some $\beta\geq 2$,

$$\lim_{k\to\infty}\sup E\left[\| w_k\|^\beta \,|J_{k-1}\right]<\infty \text{ a.s.}$$

By A3, there is a unique solution to the following algebraic Riccati equation in the class of positive definite matrices:

$$S=A'SA-A'SB(Q_2+B'SB)^{-1}B'SA+Q_1 \quad (5)$$

and the matrix

$$F\triangleq A-B(Q_2+B'SB)^{-1}B'SA \quad (6)$$

is asymptotically stable. Define

$$L\triangleq -(Q_2+B'SB)^{-1}B'SA \quad (7)$$

$$b_k\triangleq -\sum_{j=0}^{\infty}(F^j)'Q_1x^*_{k+j}=F'b_{k+1}-Q_1x^*_k \quad (8)$$

$$d_k\triangleq -(Q_2+B'SB)^{-1}B'b_{k+1} \quad (9)$$

Direct calculations show (see [8] for details) that

$$J_N(u)=x_0'Sx_0-x_N'Sx_N+2b_0'x_0-2b_N'x_N$$
$$+\sum_{k=0}^{N-1}\left(x_k^{*\prime}Q_1x_k^*-d_k'B'b_{k+1}\right)$$
$$+\sum_{k=0}^{N-1}(u_k-Lx_k-d_k)'(Q_2+B'SB)$$
$$(u_k-Lx_k-d_k) \quad (10)$$

Let $\mathcal{U}$ be the class of admissible controls which will be specified below. The point to note at this moment is that, whatever class $\mathcal{U}$ we choose for admissible controls,

$$\inf_{u\in\mathcal{U}}\cdot\lim_{N\to\infty}\sup\frac{1}{N}J_N(u)\geq\lim_{N\to\infty}\sup\frac{1}{N}\sum_{k=0}^{N-1}\left[x_k^{*\prime}Q_1x_k^*-d_k'B'b_{k+1}\right] \quad (11)$$

Thus, if in the class of desired control laws we are able to choose a control for which $\lim_{N\to\infty}\sup\frac{1}{N}J_N(u)$ equals the right hand side of (11), then this will automatically yield the optimal control desired.

3. Recursive Estimator for B

Let

$$C\text{ adj }(I-zA)\equiv C+C_1z+\cdots+C_{n-1}z^{n-1} \quad (12)$$

where "adj" stands for the adjoint of a matrix and C_i are $m\times n$ matrices, $i=1,\cdots,n-1$. Set

$$\theta'=\left[CB\vdots C_1B\vdots\cdots\vdots C_{n-1}B\right] \quad (13)$$

and

$$\phi_k'=\left[u_k'\vdots u_{k-1}'\vdots\cdots\vdots u_{k-n+1}'\right] \quad (14)$$

where CB and C_iB are $m\times p$ matrices.

Let us define

$$\Sigma\triangleq\begin{bmatrix}C\\C_1\\ \vdots\\ C_{n-1}\end{bmatrix}\text{ and }\Pi=\begin{bmatrix}CB\\C_1B\\ \vdots\\ C_{n-1}B\end{bmatrix} \quad (15)$$

Lemma 3.1 *With the notations above,*

$$B = (\Sigma'\Sigma)^{-1}\Sigma'\Pi$$

Proof All we have to show is that Σ is of full row-rank so that $\Sigma'\Sigma$ is invertible. Suppose that

$$\Sigma x = 0, \quad x \epsilon \Re^n.$$

Then

$$C(I - zA)^{-1}x = 0.$$

Then for $|z|$ sufficiently small,

$$C(I + zA + z^2A^2 + \cdots)x \equiv 0.$$

$$\Longleftrightarrow Cx = 0, \;\; CAx = 0, \cdots, CA^n x = 0.$$

Since (A, C) is observable, we must have $x = 0$, establishing that Σ is of full row rank. □

We are now in a position to propose a recursive algorithm for estimating B. We first propose the following scheme to estimate θ recursively:

$$\begin{aligned}\theta_{k+1} &= \theta_k + \gamma_k P_k \phi_k [\alpha(z) y_{k+1} \\ &- \theta_k' \phi_k - (\alpha(z) - 1)\hat{w}_{k+1}]\end{aligned} \quad (16)$$

$$P_{k+1} = P_k - \gamma_k P_k \phi_k \phi_k' P_k \quad (17)$$

$$\gamma_k = (1 + \phi_k' P_k \phi_k)^{-1} \quad (18)$$

$$\hat{w}_{k+1} = \alpha(z) y_{k+1} - \theta_{k+1}' \phi_k - (\alpha(z) - 1)\hat{w}_{k+1} \quad (19)$$

where z now stands for the unit delay operator and $\theta_0, P_0, \hat{w}_0$ are chosen arbitrarily.

Let us write $\hat{\theta}_k$ in the block matrix form

$$\theta_k' = \left[\theta_{k1} \vdots \cdots \vdots \theta_{kn}\right]$$

where θ_{ki} are $m \times p$ matrices, $i = 1, \cdots, n$, and set

$$\Pi_k = \begin{bmatrix} \theta_{k1} \\ \vdots \\ \theta_{kn} \end{bmatrix} \quad (20)$$

We propose the following recursive estimator for B:

$$B_k = (\Sigma'\Sigma)^{-1}\Sigma'\Pi_k \quad (21)$$

Theorem 3.2 *Assume that conditions A1, A2 and A4 hold. Then for any $\mathcal{F}_k$-measurable control u_k,* $\| B - B_{k+1} \|^2 =$

$$O\left(\frac{\log \lambda_{\max}(k)(\log\log \lambda_{\max}(k))^{\Delta(\beta-2)}}{\lambda_{\min}(k)}\right) \quad (22)$$

where $\lambda_{\max}(k)$ and $\lambda_{\min}(k)$ denote the maximum and minimum eigenvalues of P_{k+1}^{-1}, respectively, β is as defined in A4 and

$$\Delta(\beta-2) = \begin{cases} 0 & \text{if } \beta > 2 \\ c > 1, & \text{otherwise arbitrary, if } \beta = 2. \end{cases}$$

Proof We omit the proof. See [9] for details. □

4. Consistent Estimator for B

The previous theorem shows that the estimation error of B_k depends upon the behavior of P_k^{-1}. In general, we do not know whether B_k converges to the true B or not. To ensure strong consistency of the estimator of unknown parameters and achieve optimality of the control law at the same time is a very difficult problem. Direct certainty-equivalence based adaptive contol law can not achieve this goal in the linear-quadratic problem [10]. In stochastic adaptive control literature the idea of diminishing dither to the control law has been introduced for this purpose [11], [12] which will be used here.

Let $\{v_k\}$ be a sequence of $\Re^p$-valued random vectors which is independent of $\{w_k\}$, with $Ev_k = 0$, $Ev_k v_k' = I$ and $\| v_k \| \leq$ constant a.s. Define

$$u_k^d \triangleq \frac{v_k}{k^{\epsilon/2}}, \quad \epsilon \in \left[0, \frac{1}{2n}\right). \quad (23)$$

Without loss of generality, we may assume that $\mathcal{F}_k \equiv \sigma\{w_i, v_i, \; 0 \leq i \leq k\}$. Set

$$\mathcal{F}_k' \equiv \sigma\{w_i, 0 \leq i \leq k, \quad v_j, \;\; 0 \leq j \leq k-1\}.$$

Let u_k^s be any $\mathcal{F}_k'$-measurable control law at time k, obtained possibly by some certainty equivalence principle. We apply the diminishingly excited version u_k of u_k^s to the system:

$$u_k = u_k^s + u_k^d \quad (24)$$

Theorem 4.1 *If A1, A2, A4 hold, and if*

$$\sum_{i=1}^{k} \| u_i^s \|^2 = O\left(k^{1+\delta}\right), \;\; \delta \in \left[0, \frac{1-2\epsilon n}{1+2n}\right), \;\; a.s. \quad (25)$$

Then

$$\| B - B_k \|^2 = O\left(\frac{\log k(\log\log k)^{\Delta(\beta-2)}}{k^\alpha}\right),$$
$$\alpha \in \left(\tfrac{1}{2}(1+\delta), 1 - n(\epsilon+\delta)\right] \quad (26)$$

Proof The proof is technical and is omitted. See [9] for details. □

5. Optimal Adaptive Control

Let us now go back to the adaptive control problem posed in section 2. It is clear from (10) that, if $\{x_k\}$ was completely observed and $\{B_k\}$ was known, the optimal control would be given by

$$u_k = Lx_k + d_k. \quad (27)$$

We use the ELS-based estimator B_k for B and define the certainty-equivalence control u^0 by

$$u_k^0 = L_k\hat{x}_k + \hat{d}_k \quad (28)$$

where

$$\hat{x}_{k+1} = A\hat{x}_k + B_k u_k^0 \quad (29)$$

$\hat{x}_0$ arbitrary

$$L_k = -(Q_2 + B_k' S_k B_k)^{-1} B_k' S_k A \quad (30)$$

$$S_k = A'S_{k-1}A - A'S_{k-1}B_k (Q_2 + B_k' S_{k-1} B_k)^{-1} B_k' S_{k-1} A + Q_1 \quad (31)$$

$S_0 \geq 0$, otherwise arbitrary

$$\hat{d}_k = -(Q_2 + B_k' S_k B_k)^{-1} B_k \hat{b}_{k+1} \quad (32)$$

$$\hat{b}_k = -\sum_{j=0}^{k} F_{k-1}^j Q_1 x_{k+j}^* \quad (33)$$

$$F_k = A - B_k L_k \quad (34)$$

We now define stopping times $\{\sigma_k\}$ and $\{\tau_k\}$ as follows: Set $\tau_1 = 1$. Let

$$\sigma_k = \sup\{t > \tau_k \left| \sum_{i=\tau_k}^{j-1} \| u_i^0 \|^2 \leq (j-1)^{1+\delta} + \| u_{\tau_k}^0 \|^2, \ \forall j \epsilon (\tau_k, t]\right.\} \quad (35)$$

$$\tau_{k+1} = \inf\left\{ t > \sigma_k \left| \sum_{i=\tau_k}^{\sigma_k - 1} \| u_i^0 \|^2 \leq \frac{t^{1+\delta}}{2^k} \wedge \sum_{i=\sigma_k}^{t} \| u_i^0 \|^2 \leq \frac{t^{1+\delta}}{2^k} \right.\right\} \quad (36)$$

The desired control law u_k^s is defined by

$$u_k^s = \begin{cases} u_k^0, & \text{if } k\epsilon[\tau_\ell, \sigma_\ell) \text{ for some } \ell \\ 0, & \text{otherwise} \end{cases} \quad (37)$$

and finally, the adaptive control law we are after is given by

$$u_k^* = u_k^s + u_k^d \quad (38)$$

Let $\mathcal{U}$ denote the class of admissible controls defined by

$$\mathcal{U} = \{u | u_k \text{ is } \mathcal{F}_k - \text{measurable such that the resulting state satisfies } \| x_k \|^2 = o(k) as.\}$$

Theorem 5.1 *Assume that A1 - A4 hold. Then* $\{u_k^*\}\epsilon\mathcal{U}$,

$$\lim_{N\to\infty} \sup \frac{1}{N} J_N(u^*) = \lim_{N\to\infty} \sup \frac{1}{N} \sum_{k=0}^{N-1} \left[x_k^{*'} Q x_k^* - d_k' B' b_{k+1} \right] \quad (39)$$

and

$$\| B - B_k \|^2 = O\left(\frac{\log k(\log\log k)^{\Delta(\beta-2)}}{k^\alpha}\right),$$
$$\alpha \in \left(\tfrac{1}{2}, 1 - n\epsilon\right] \quad (40)$$

Proof We first show consistency of B_k. If $\tau_\ell < \infty$ and $\sigma_\ell = \infty$ for some ℓ, then $\sum_{i=1}^{k} \| u_i^s \|^2 = O\left(k^{1+\delta}\right)$ and the strong consistency of B_k follows from Theorem 4.1. If $\sigma_\ell < \infty$ and $\tau_{\ell+1} = \infty$ for some ℓ, then $u_i^s = 0 \ \ \forall i \geq \ell$ and, again by Theorem 4.1, B_k is strongly consistent.

Consider now that $\tau_\ell < \infty, \sigma_\ell < \infty$ for all ℓ. Then

$$\sup_{\tau_\ell \leq k < \tau_{\ell+1}} \frac{1}{k^{1+\delta}} \sum_{i=1}^{k} \| u_i^s \|^2 = \sup_{\tau_\ell \leq k < \sigma_\ell} \frac{1}{k^{1+\delta}} \sum_{i=1}^{k} \| u_i^0 \|^2$$
$$\leq 3 + \frac{\| u_{\tau_\ell}^0 \|^2}{\tau_\ell^{1+\delta}} \leq 3 + \frac{1}{2^{\ell-1}} < 4.$$

Therefore, $\sum_{i=1}^{k} \| u_i^s \|^2 = O\left(k^{1+\delta}\right)$ and Theorem 4.1 again leads to the strong consistency of B_k.

By Theorem 3.4 and Remark 3.3 of 7,

$$S_k \to S, k \to \infty$$

from which it also follows that

$$F_k \to F, \ \| b_k - \hat{b}_k \| \to 0, \ \| d_k - \hat{d}_k \| \to 0, k \to \infty$$

Since F is a stable matrix and $F_k \to F$, there exists a $\rho \epsilon (0,1)$ and a $C > 0$ such that

$$\| F_k F_{k-1} \cdots F_1 \| \leq C\rho^k \quad \forall k \qquad (41)$$

We can now show (see [9]) that there exists an ℓ_0 such that

$$\tau_{\ell_0} < \infty \text{ and } \sigma_{\ell_0} = \infty \qquad (42)$$

This implies that

$$u_k^* = u_k^0 + u_k^d$$

and $\hat{x}_{k+1} = F_k \hat{x}_k + B_k \hat{d}_k, \ \forall k \geq \tau_{\ell_0}$.

By (41) and the boundedness of $B_k \hat{d}_k$ it is clear that $\{\hat{x}_k\}$ is also bounded and so is $\{u_k^0\}$, so that

$$\sum_{i=1}^{k} \| u_i^s \|^2 = O(k).$$

This and Theorem 4.1 implies (40).

To prove optimality, notice that

$$x_{k+1} - \hat{x}_{k+1} = A(x_k - \hat{x}_k) + (B - B_k)u_k^*$$

By the boundedness of $\{u_k^*\}$, the stability of A and the fact that $B - B_k \to 0$ we find that

$$x_k - \hat{x}_k \longrightarrow 0, k \to \infty \qquad (43)$$

This, along with the boundedness of $\{\hat{x}_k\}$, implies that $\{x_k\}$ is also bounded. Therefore, $\{u_k^*\} \in \mathcal{U}$. Using (43) and the facts that $L_k - L \to 0$ and $\| d_k - \hat{d}_k \| \to 0$ we conclude that

$$\frac{1}{N} \sum_{k=0}^{N-1} (u_k^* - Lx_k - d_k)'(Q_2 + B'SB)$$

$$(u_k^* - Lx_k - d_k) \longrightarrow 0, k \to \infty$$

Combining this with (10) establishes (39). □

6. Conclusion

We solved a class of stochastic adaptive control problems in the state space form which arise in controlling aircraft flying in gusty conditions. The important, although difficult, extensions which should be further looked into involve the state noise case and/or when the parameters A and C also contain unknown elements. The approach presented in this paper does not directly go over to this most general situation, but may possibly be used there in combination with some other techniques.

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