1.12 Investigations on resonant acoustic waves in open pipes

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1 INTRODUCTION

The problem under consideration is as follows. In a straight cylindrical pipe a piston executes small harmonic oscillations at one end, whereas the other end is open to the atmosphere. One asks for the engendered oscillating pressure and velocity amplitudes in the pipe. As long as the frequency is far from resonant frequencies the solution is well known and can be obtained from linearized equations. However, near resonant frequencies this linear solution breaks down and any attempt to solve the problem must necessarily take recourse to nonlinear methods. This places this problem well in the context of this Symposium. Important classic work was done by Lord Rayleigh [3] who determined experimentally the magnitude of one nonlinear effect of importance, namely the shift of the classical linear resonant frequency brought about by the multidirectional acoustic radiation from the mouth of the pipe. When $\omega_0$ is the lowest resonant frequency, occurring when the wavelength of the standing waves in the pipe is just four times the pipe length, then Rayleigh found that by the mentioned effect (the directional distribution of radiation from the mouth of the pipe) this value has to be corrected by an amount $0.6 \, R/L$, when $R$ is the radius of the pipe and $L$ its length. In 1948 Levine and Schwinger [2] calculated this effect, confirming Rayleigh's result. However, the question remained of how the amplitude of pressure and velocity oscillations are determined. When viscosity and heat conduction can be neglected, the motion in the pipe is governed by the equations

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} + (u \pm a) \frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} \end{array} \right\} \left\{ \begin{array}{l} u \\ u \pm 2a \\ \gamma - 1 \end{array} \right\} = 0, \quad (1)$$

where $u$ and $a$ are the particle and sound speed respectively. Equations (1) imply that along the characteristics

$$\frac{dx}{dt} = u \pm a, \quad (2)$$

the Riemann invariants $u \pm 2a/(\gamma - 1)$ are constant.

2 THEORY OF VAN WIJNGAARDEN

In applying these equations to the present problem, van Wijngaarden [5] wrote $u$, $a$, $t$ and $x$ as functions of the characteristics $\alpha$ and $\beta$. Assuming a small deviation from acoustic waves we then have

$$u = u_0 + euv_1(\alpha, \beta) + e^2uv_2(\alpha, \beta) + \ldots$$

$$a = a_0 + eav_1(\alpha, \beta) + e^2av_2(\alpha, \beta) + \ldots$$

$$x = x_0 + exv_1(\alpha, \beta) + e^2xv_2(\alpha, \beta) + \ldots$$

$$t = t_0 + etv_1(\alpha, \beta) + e^2tv_2(\alpha, \beta) + \ldots$$

(3)

In (3) $e$ is a small parameter. In terms of $\alpha$ and $\beta$ (1) may be written as

$$\frac{\partial u}{\partial \alpha} = \frac{2}{\gamma - 1} \frac{\partial a}{\partial \alpha}, \quad (4)$$

$$\frac{\partial u}{\partial \beta} = \frac{2}{\gamma - 1} \frac{\partial a}{\partial \beta},$$

$$\frac{\partial x}{\partial \alpha} = (u + a) \frac{\partial t}{\partial \alpha},$$

$$\frac{\partial x}{\partial \beta} = (u - a) \frac{\partial t}{\partial \beta}.$$

The characteristics are labelled such that

$$\alpha = \beta = t \text{ at } x = 0. \quad (5)$$

The expansions (3) are introduced into (4) and (5), and terms of equal order in $e$ collected. For each of the functions $u_0, \alpha_0, \mu_1, \alpha_1$ etc., solutions can be obtained when sufficient boundary conditions are given.

One of these is that at the piston, $x = L + \delta \cos \omega t$ the velocity, $-\omega \delta \sin \omega t$, is given. Another is that the motion is periodic with period $T = 2\pi/\omega$. Here $\delta$ is the amplitude of the piston oscillation and $\omega$ the angular frequency. Since this is close to the lowest resonance frequency $\omega_0$, given by

$$\omega_0 = \frac{\pi}{2} \frac{a_0}{L}, \quad (6)$$

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we may write

$$\omega = \omega_0 + \epsilon \omega_1. \quad (7)$$

Two questions remain: what is the boundary condition at the open end \( x = 0 \), and what is the governing small parameter \( \epsilon \). Van Wijngaarden [2] assumed that at outflow, \((u)_{x=0} < 0\), the flow is jet-like with constant pressure \( p = p_\infty \), whereas at inflow the flow is as if a sink is located in the mouth of the pipe with the pressure varying according to momentum calculations based on this picture.

In summary

$$\begin{align*}
x &= 0 & p &= p_\infty & \text{when } u < 0; \\
p &= p_\infty - \rho u^2 - 1.2R \frac{\partial p}{\partial t} & \text{when } u > 0.
\end{align*} \quad (8)$$

In (8), the term \( 1.2R(\partial/\partial t)(\rho u) \) stems from the 'end correction' suggested by Rayleigh. This picture of the flow also settles the question about \( \epsilon \). During inflow, energy is conserved along streamlines and equal to the energy of the gas in its rest state in the outside atmosphere. During outflow, however, the pressure is equal to the outside pressure but the gas has in addition a kinetic energy \( \frac{1}{2} \rho u^2 \) per unit mass. This energy is dissipated eventually in the air, surrounding the pipe, so that in a cycle the energy loss is proportional to \( \rho_0 u_0^2 \omega L \), where \( u_0 \) stands for the amplitude of the velocity oscillation. The work done by the piston during a cycle is proportional to \( \rho_0 u_0^2 \omega L \). Both expressions must, on the basis of the energy balance of the flow, be of equal magnitude, whence, using (6)

$$\frac{\tilde{u}}{40} \approx \left( \frac{\delta}{L} \right)^{1/2}. \quad (9)$$

The proper small parameter \( \epsilon \) therefore is

$$\epsilon = \left( \frac{\delta}{L} \right)^{1/2}. \quad (10)$$

Using the perturbation technique, outlined above, one obtains

$$u_1 = \frac{1}{2} \left\{ A(\alpha) + A(\beta) \right\},$$

$$\sigma_1 = \frac{\gamma - 1}{4} \left\{ A(\beta) - A(\alpha) \right\},$$

\( A(\alpha) \) being given, through considering the second order equations, by

$$\left( \frac{\partial}{\partial t} + \frac{\pi \omega_1}{\omega_0^2} \frac{\partial}{\partial a} \right) \frac{A|A|}{\partial a} + \frac{\omega_1}{\omega_0} = 2 \omega_1 L \cos \omega_1 t. \quad (11)$$

The coefficient \( \delta \) is related to Rayleigh's frequency correction by

$$\sigma \epsilon = 1.2 \frac{R}{L}. \quad (12)$$

From (11) it follows that resonance occurs at

$$\frac{\omega_1}{\omega_0} = -\frac{\sigma}{2},$$

which, upon using (7), gives Rayleigh's result.

Under these conditions the first order (in terms of \( \epsilon \)) pressure at the piston is

$$\frac{P - P_0}{P_0} = \frac{\gamma}{\gamma - 1} \sin \omega_1 t \frac{\sigma}{2} \text{ sgn}(\sin \omega_1 t). \quad (13)$$

3 THEORY OF SEYMOUR AND MORTELL

A different scheme, and another boundary condition, were introduced by Seymour and Mortell [4]. They took the Riemann invariants as dependent variables,

$$\begin{align*}
f(\beta) &= u + \frac{2a}{\gamma - 1}, \\
g(\alpha) &= u - \frac{2a}{\gamma - 1}.
\end{align*} \quad (14)$$

They expressed \( x \) and \( t \) as a series, as in (3)

$$\begin{align*}
x &= x_0 + x_1(\alpha, \beta) + x_2(\alpha, \beta) + \ldots, \\
t &= t_0 + t_1(\alpha, \beta) + t_2(\alpha, \beta) + \ldots,
\end{align*} \quad (15)$$

when \( x_1 \) and \( t_1 \) are of order \( f \), \( x_2 \) and \( t_2 \) of order \( f^2 \).

This has the advantage that no estimate of \( f \) needs to be made at the outset. Further, they used lagrangian variables, i.e. the fluid particles are labelled with the position they occupy in the state of rest prior to the motion of the piston. The boundary condition at the piston simply is: \( x = L, \)

$$u = -5 \omega \sin \omega t. \quad (16)$$

The crucial point in the work of Seymour and Mortell is the boundary condition at the other end of the pipe. They supposed that particles, initially in contact with the surrounding atmosphere, remain at this contact surface, given by \( x = 0 \) in lagrangian coordinates. It is of some importance to note that this is no longer true when fluid elements leave the pipe and are replaced by particles, originally in the outside atmosphere. Seymour and Mortell further assumed that the motion in the pipe is a superposition of two simple waves, travelling in opposite directions. Then, at \( x = 0 \), there is a coupling between \( p \) and \( u \), which in their theory is represented by a 'lumped' damping factor \( j \):

$$p = -ju, \quad j \ll 1. \quad (16)$$

Calculation of the time that it takes a wave to travel up and down the tube leads, using (16) and the condition that two traversals correspond with the given period of the motion, with the perturbation scheme outlined above, to an equation for \( f \) of the type

$$\frac{\epsilon^2}{2} \frac{df}{d\eta} + \left( \frac{\omega - \omega_0}{\omega_0} + k \right) \frac{df}{d\eta} + 4jf = 2(1 + j) \frac{\delta}{L} \cos \eta. \quad (17)$$
Here $f$ is the Riemann invariant, non-dimensionalized with $a$, on the 'right' characteristics, whereas $\eta$ is the non-dimensional time, $\eta = \omega_0 t$. The constant $b$ is of unit order, the constant $k$ represents the effect of interaction between the right-going and the left-going simple wave, given by

$$k = \int_\eta^{\eta+2\pi} f^2(s)ds.$$  \hspace{1cm} (18)

Different possibilities for the magnitude of $f$ arise from a detailed study of the solutions of (17). Some of the results of Seymour and Mortell are given here.

First, when $|f| = 0, (\delta/L)^{1/2} = 0(\epsilon)$, then $f = 0(\epsilon)$ and the third order terms can be neglected.

This reduces (17) to

$$\left(\frac{\omega - \omega_0}{\omega_0}\right) \frac{df}{d\eta} + 4f = 2\epsilon^2 \cos \eta.$$ \hspace{1cm} (19)

When comparing this equation with the result (11) of van Wijngaarden's theory, two differences may be observed. In (11) a term

$$\frac{A}{A_0}$$

occurs instead of the term $4f$ in (19), indicating the effect of the different boundary conditions. Both theories lead here to a signal with amplitude of order $(\delta/L)^{1/2}$. Resonance, however, occurs in (19) when $\omega = \omega_0$, because a term comparable with $\sigma L/A_0$, in (11), is absent in equation (19).

The third order terms in (17) cooperate when $j = 0(\epsilon^4/\delta)$. Then (17) has continuous solutions with $f = 0(\epsilon^2/\delta)$, or $f = 0(\delta/L)^{1/2}$. Resonance takes place at a frequency lower than $\omega_0$, the difference being given by

$$\frac{\omega - \omega_0}{\omega_0} = -k^2 = 0(\epsilon^4/\delta).$$

Depending on the value of $j$, (17) admits therefore continuous solutions of order $(\delta/L)^{1/2}$ and of order $(\delta/L)^{1/2}$.

**4 EXPERIMENTS**

In 1968 van Wijngaarden [5] reported pressure measurements at the piston of order $(\delta/L)^{1/2}$. Since, as shown above, pressure amplitudes of this order may result from quite different boundary conditions and quite different models for the flow behaviour in the pipe, it seemed worthwhile to the present authors to build a suitable experimental apparatus permitting measurement of both velocity and pressure at and near the mouth of the pipe.

The dimensions of importance are:

- Length $L$ of the pipe (steel) = 2.756 m;
- Radius $R$ of the pipe = 55.10$^{-3}$ m; $(R/L = 0.02)$
- Amplitude of piston (aluminium) $\delta = 2.7 \times 10^{-3}$ m;
- Velocity $a_0 = 345.0$ m/s;
- $\omega_0 = 196.5$ s$^{-1}$.

Velocities were measured with hot-wire anemometers, pressures were recorded with piezo-electric pressure transducers. The experimentally obtained values of the pressure at the piston reproduced the findings of van Wijngaarden [5], but we learned a lot more from inspection of the measured pressure and velocity profile at and near $x = 0$.

In the figures the important experimental results are summarized. Fig.1 gives the pressure amplitude at the piston as a function of $\omega_1/\omega_0 = 4^{-1}(\omega_1 - \omega_0)$. The pressure is of order $(\delta/L)^{1/2}$, resonance occurs near the value, following from Rayleigh's formula. The shift is of order $\epsilon$, anyway. Fig.2 gives the phase difference between piston path and pressure at the piston.

In Fig.3 pressure and velocity at the centre of the open end are given. The velocity is of order $\epsilon$, the pressures of order $\epsilon^2$. However, this figure clearly shows that neither the boundary condition (8), assumed by van Wijngaarden [5], nor the boundary condition (16) assumed by Seymour and Mortell [4], applies. Inspection of Fig.3 shows that at $x = 0$, $r = 0$, the pressure behaves roughly as $8\eta d\eta$, with additional peaks as the velocity changes sign.

In Fig.4a–4g the axial velocity profile at the open end is schematically drawn for a whole cycle. At $x = 80$ mm the velocity is almost uniform. Between $x = 0$ and $x \approx 80$ the velocity is inhomogeneous. The flow seems to be such that in the first phase of inflow a zone of recirculation is built up near the open end. The extent of this vorticity region is of order of 100 mm. Because this is small with respect to the wavelength (of order L), we may assume the flow to be incompressible in that region.

![Fig.1 Pressure amplitude at piston.](image-url)
Since vorticity is convected with the local speed it follows that a typical length scale for the vorticity region is
\[ 1 = \frac{\beta}{\omega} = \frac{(G/a_0)A}{L} \approx 150 \text{ mm}, \]
of the same order of magnitude as that observed in the experiments. In Fig. 5 the direction of the velocity is shown in the region \( 0 < x < 50 \text{ mm} \) at \( t = \frac{1}{2} T \). This shows the occurrence of vorticity very clearly. At the middle of a cycle, the velocity at \( x \approx 100 \text{ mm} \) is zero, but at the open end there is inflow at \( r = 0 \), outflow however at the wall, \( r = R \). Between \( t = 0 \) and \( t = \frac{1}{4} T \) the inflow velocity is higher near the centre than at the pipe wall. Between \( t = \frac{1}{4} T \) outflow gradually occurs in the centre too, but the velocity of outflow is significantly larger at the wall. Our interpretation is that vorticity is shed from the pipe wall in the period \( \frac{1}{4} T < t < \frac{1}{2} T \), corresponding with the high velocity gradient near the wall. After that, in the period \( \frac{1}{4} T < t < T \), the velocity distribution at the open end is practically homogeneous. This was assumed in van Wijngaarden [5] for the whole half of the cycle in which outflow takes place. The shedding of vorticity from the open end is the result of the Kutta condition at the open end. To verify the validity of this condition, which has been subject of discussion in recent work on aerodynamic sound (see Crighton [11]) we measured also the direction and magnitude of the velocity in the open end section just outside the pipe. For this purpose the anemometer was positioned subsequently normal to the axis of the pipe and aligned to it. In the first position the received signal is proportional to the square of the axial velocity, in the second position to the square of the radial velocity. The results are shown in Fig. 6a and Fig. 6b (The scale on these figures is not linear. In fact they represent the anemometer readings.) The time \( t = 0 \) corresponds with...
$t = 0$ in Fig. 3, and marks the moment in which the piston is at the position $x = L - \delta$. Fig. 6a shows that the axial velocity just outside the pipe is very low during outflow, much higher during inflow. Fig. 6b, with the same time origin as in Fig. 6a, records the absolute value of the radial velocity at about the same station. The radial velocity is low during outflow, with the exception of a peak corresponding with the shedding of vorticity.

Another support for this interpretation is found in Fig. 7, where $|u|$ is recorded at various stations, at $r = 0$, in front of the pipe. Also here $t = 0$ corresponds with $t = 0$ in Fig. 3. Fig. 7 shows that at $x = -55$ the inflow is hardly noticeable, whereas the axial velocity is appreciable during outflow out of the pipe.

Finally we consider the pressure distribution. Fig. 8 shows the pressure in the open end at various values of $r$. The time $t = 0$ is the same as in the other figures, except for Figs 4 and 5. Fig. 8 shows that a significant pressure difference exists during inflow between the region near the pipe wall and the region near the centre. During the final part of the outflow period the pressure is nearly uniform over the cross-section. In Fig. 9 the pressure is given at $x = 110$. It follows that there is no appreciable change of the pressure with $r$ here, indicating that at $x = 110$, we are beyond the vorticity-containing region.

Fig. 6 (a) $|u| at x = 0, r = 68 \text{ mm.}$ (b) $|u| at x = 0$ and $r = 63 \text{ mm.}$

Fig. 7 $|u|$ at various axial distances from $x = 0$ and $r = 0$.

Fig. 8 Pressure variation in time at various values of $r$, $x = 0$

Fig. 9 Pressure variation in time at various values of $r$, $x = 110 \text{ mm.}$
5 DISCUSSION

The experiments reported above indicate that none of the boundary conditions at the open end, and used in existing theories, describes the conditions at the open end adequately.

While an improved theoretical treatment needs further time, we conclude this report with a tentative estimate for the relation between \( \varepsilon, \delta, R \) and \( L \), under the conditions met in our experiments. The length \( L \) of the vorticity-containing region is of order \( \varepsilon L \). Assuming the velocity induced by vorticity to be of order of magnitude \( \varepsilon \), an estimate for the circulation is

\[ \Gamma \sim \varepsilon \omega_0 l = \varepsilon^2 \alpha_0 L. \]

The vorticity, shed from the pipe in one cycle, then has also this order of magnitude. The associated energy, leaving the pipe, is \( \rho_0 \varepsilon^2 R^2 \), or \( \rho_0 \varepsilon^4 \alpha_0^2 L^2 R \). Equating this to the work done by the piston, of order \( \rho_0 \varepsilon^2 \delta^2 R^2 \), gives

\[ \varepsilon \sim \left( \frac{\delta R}{L^2} \right)^{1/3}. \]

In our experiments, where \( R/L \) is of order \( (\delta/L)^{1/2} \), this leads to

\[ \varepsilon \sim \left( \frac{\delta}{L} \right)^{1/2}. \]

Of course, this reasoning is rather speculative, and what is needed is the solution of the theoretical problem in which the one-dimensional compressible flow in the main part of the pipe is matched to the incompressible three-dimensional flow near the mouth. This problem is the subject of our research now.

6 REFERENCES