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# NON-STATIONARY OSCILLATIONS OF SANDWICH PLATES UNDER LOCAL DYNAMIC LOADING

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**Abstract**

The paper addresses the elastic response of composite sandwich panels to local dynamic loading. The plane and axisymmetric formulations are considered; no overall bending is assumed. The governing equations are derived using the static Lamé equations for the core and the plate Kirchoff-Love dynamic theory for the faces. The closed form solutions for the non-stationary excitation are obtained using integral transformations technique. The solutions allow to predict the stress-strain state of the structure and are in good agreement with finite element analysis.

## INTRODUCTION

One of the inherent properties of sandwich structures is low transversal stiffness. As a result, these structures are susceptible to local damages due to interaction with attached structures or impact. Thus, it is of a practical importance to predict the elastic stress-strain response of sandwich structures subject to localized dynamic loading.

Besides experimental and finite element analysis, e.g. Refs. [1, 2], there are two approaches to analytical modeling of sandwich structure local behaviour. These approaches are based on different descriptions of a core deformation. The simplified approach assumes that the plate is resting on a continuous set of independent springs, the stiffness of which defines a foundation modulus; thus, the interface stress depends only on the deflection at the same point. This so called Winkler model is used for solutions of static problems in [2,3]. Dynamic analysis for the given modulus is performed in [4,5]. In many cases the Winkler model or the more advanced Winkler-Pasternak model [2] provides a satisfactory agreement with experimental results, but it is not universal for a general case of the sandwich constitution.

The more precise approach is based on an elastic continuum model that results in interconnectivity of all points of an interface. Applying the linear theory of elasticity, a static behaviour of a plate resting on an elastic layer was analyzed in [6, 7], stationary oscilations are considered in [4,10]. These results are generalizations of the well-known solutions for a semi-infinite medium that models a thick core [8,9]. In the present paper, the elastic continuum model is used for the analysis of non-stationary oscillations of a locally loaded sandwich panels of arbitrary thickness.

**Notations.** Subscript *"f"* belongs to the face, *"c"* – to the core, *"if"* – to the face-core interface, *"F"*, *"H"* and *"L"* – to the Fourier, Hankel and Laplase transforms, respectively, *"pl"* – to the plane state and *"ax"* – to the axisymmetrical state. Subscript *"0"* means that a function is examined at co-ordinate origin.

## ANALYTICAL MODELLING

A three-layered sandwich panel is studied under a point or line load. No overall bending or Hertzian indentation is considered. The panel consists of two thin, non-stretchable face sheets with thickness $h_f$ and relatively thick and light-weight core with thickness $h_c$. The densities of the faces and the core are $\rho_f$ and $\rho_c$, respectively.

The local bending of the top face is considered in the plane (plane stress and plane strain) and axisymmetric formulations. The face is modelled as transversely isotropic, infinite plate with bending stiffness $D_f$ bonded to the core layer (Fig. 1). No difference is assumed between displacements of the midplane and the interface and no influence of the shear stresses on bending. Under these assumptions, the thin plate Kirchoff-Love dynamic theory is used for the face bending under external excitation $P(t)$ and normal core reaction $\sigma_{if}$, see Fig. 1. Denoting the width of the beam as $b$ and the Dirac delta-function as $\delta$, the governing equation of the face deflection $w_f$ is

$$D_f \frac{\partial^4 w_f(x,t)}{\partial x^4} + \rho_f h_f \frac{\partial^2 w_f(x,t)}{\partial t^2} = \frac{\delta(x)}{2b} P(t) - \sigma_{if}(x,t), \tag{1}$$

for the plane formulation, where $0 \leq x \leq \infty$, and for the axisymmetric formulation is

$$D_f \Delta\Delta w_f(r,t) + \rho_f h_f \frac{\partial^2 w_f(r,t)}{\partial t^2} = \frac{\delta(r)}{2\pi r} P(t) - \sigma_{if}(r,t),\ \Delta = \frac{\partial^2}{\partial r^2} + \frac{\partial}{r\partial r}. \tag{2}$$

Since $\rho_c$ is usually extremely small, the core inertia is neglected and the core behaviour is described by static Lamé equations for isotropic elastic continuum. The Lamé equations are solved by means of Fourier or Hankel integral transformation technique. For the symmetric functions (e.g. deflection and normal interfacial stress), the

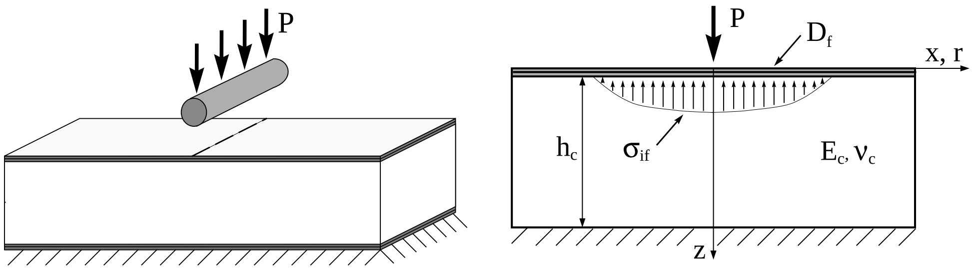


Figure 1: Impact set-up for plane formulation (left) and physical model (right).

cosine Fourier or zero-order Hankel transformation

$$w_f^F(\omega) = \int_0^\infty w_f(x)\cos(\omega x)\,\mathrm{d}x, \quad w_f^H(\omega) = \int_0^\infty w_f(r)\mathrm{J}_0(\omega r)r\,\mathrm{d}r$$

is used in the plane and axisymmetric formulations, respectively. The sine Fourier or first-order Hankel transformation is used for the other functions (rotation, etc.).

The transformations for the Lamé equations are performed under the boundary conditions of zero longitudinal and transverse displacements of the bottom face, as well as zero longitudinal displacement and the given deflection $w_f$ of the upper face. The transformed Lamé equations produce the following relation between the images of the face deflection and normal stress at the interface [7]

$$\sigma_{if}^{F(H)}(\omega, t) = E_1\omega F(\omega h_c) w_f^{F(H)}(\omega, t), \tag{3}$$

$$F(\omega h_c) = \frac{\cosh(\omega h_c)\sinh(\omega h_c) + \psi\omega h_c}{\sinh^2(\omega h_c) - (\psi\omega h_c)^2},$$

where the reduced parameters of the core stiffness $E_1$ and $\psi$ are introduced using Young's modulus, $E_c$, and Poisson's ratio, $\nu_c$, of the core as

$$E_1 = 2\psi E_c/(1+\nu_c)^2, \quad \psi = (1+\nu_c)/(3-\nu_c)$$

for the plane stress state and as

$$E_1 = 2\psi E_c(1-\nu_c)/(1+\nu_c), \quad \psi = 1/(3-4\nu_c)$$

both for the plane strain and axisymmetric states.

The equation of motion of the impactor at the contact with the face is

$$m\ddot{w}_0(t) = -P(t), \tag{4}$$

where $m$ is the mass of the impactor, $w_0(t)$ is the face deflection under the impactor, $P(t)$ is the contact force. The effect of the impactor rebound is not considered. Thus, the impactor is assumed "to stick" to the face sheet once the contact had onset.

## RESPONSE TO PRESCRIBED LOADING

The double Fourier-Laplace transformation of Eq. (1) and the double Hankel-Laplace transformation of Eq. (2) produce

$$D_f \omega^4 w_f^{F(H)L}(\omega, p) + \sigma_{if}^{F(H)}(\omega, p) + \rho_f h_f p^2 w_f^{F(H)L}(\omega, p) = P^L(p)/2\varphi, \qquad (5)$$

where $\varphi = b$ or $\varphi = \pi$ for plane or axisymmetric formulations, respectively. Taking into account Eq. (3), the image of the face deflection is

$$w_f^{F(H)L}(\omega, p) = \frac{1}{2\varphi} \frac{P^L(p)}{D_f \omega^4 + E_1 \omega F(\omega h_c) + p^2 \rho_f h_f}, \qquad (6)$$

The final solutions are determined by the double inverse transforms. Among possible functions $P(t)$, the fundamental force-time dependence is an impulse function $P(t) = I\delta(t)$ ($P^L(p) = I$). In this case, the originals of the face deflection and the interfacial stress under the force are

$$w_0^{pl}(\tau) = \frac{I}{\pi b E_1 t_n} \int_0^\infty \frac{\mathrm{d}\bar{\omega}}{\eta(\bar{\omega}, \tau)}, \quad \sigma_0^{pl}(\tau) = \frac{I}{\pi b t_n x_n} \int_0^\infty \frac{\bar{\omega} F(\bar{\omega}/\varepsilon)\mathrm{d}\bar{\omega}}{\eta(\bar{\omega}, \tau)} \qquad (7)$$

in the plane formulation and

$$w_0^{ax}(\tau) = \frac{I}{2\pi E_1 t_n x_n} \int_0^\infty \frac{\bar{\omega}\mathrm{d}\bar{\omega}}{\eta(\bar{\omega}, \tau)}, \quad \sigma_0^{ax}(\tau) = \frac{I}{2\pi t_n x_n^2} \int_0^\infty \frac{\bar{\omega}^2 F(\bar{\omega}/\varepsilon)\mathrm{d}\bar{\omega}}{\eta(\bar{\omega}, \tau)} \qquad (8)$$

in the axisymmetric formulation. Here, the dimensionless variables $\bar{\omega} = \omega x_n$ and $\bar{\omega}/\varepsilon = \omega h_c$ are introduced using the characteristic length $x_n = \sqrt[3]{D_f/E_1}$ and non-dimensional parameter $\varepsilon = x_n/h_c$. The parameter $\varepsilon$ characterizes the relationship between the bending stiffness of the face sheet and the core layer; $\varepsilon$ is small for the majority of real sandwich structures. The function $\eta(\bar{\omega}, \tau)$ and dimensionless time $\tau$ are introduced using the characteristic time $t_n = \sqrt{\rho_f h_f x_n/E_1}$ as

$$\eta(\bar{\omega}, \tau) = \frac{\sqrt{\bar{\omega}^4 + \bar{\omega} F(\bar{\omega}/\varepsilon)}}{\sin\left(\tau\sqrt{\bar{\omega}^4 + \bar{\omega} F(\bar{\omega}/\varepsilon)}\right)}, \qquad \tau = \frac{t}{t_n}.$$

The structural response to the impulse $I$=1 N$\cdot\,\mathrm{sec}$ is illustrated in Fig. 2. The response to an arbitrary function $P(t)$ can be derived using the Duhamel's integral.

## RESPONSE TO SMALL MASS IMPACT

Denoting initial impact velocity as $v$, the Laplace transformation of Eq. (4) produces

$$m p^2 w_0^L(p) = -P^L(p) + mv, \qquad (9)$$

Using dimensionless Laplace transformation variable $\bar{p} = p\ t_n$, which corresponds to the dimensionless time $\tau$, the combined solution of Eqs. (6) and (9) produces the following images of the face deflection, interfacial stress and contact force

$$w_0^L(\bar{p}) = \frac{vt_n}{\bar{p}^2 + \gamma^2/f_j},\ \sigma_0^L(\bar{p}) = \frac{1}{x_n}\frac{g_j E_1 v t_n}{\bar{p}^2 f_j + \gamma^2},\ P^L(\bar{p}) = \frac{1}{t_n}\frac{mv\gamma^2}{\bar{p}^2 f_j + \gamma^2}, \tag{10}$$

for the plane ($j$=1) and axisymmetric ($j$=2) formulations, where

$$\gamma^2 = \begin{cases} \rho_f h_f x_n b/m & \text{– in the plane formulation;} \\ 2\rho_f h_f x_n^2/m & \text{– in the axisymmetric formulation.} \end{cases}$$

The non-dimensional parameter $\gamma$ relates the face mass in the deformed zone to the impactor mass $m$ and characterizes the type of the impact. If this parameter is small, the influence of the face inertia on the structure response is negligible and the response is close to the quasi-static case. For large values of $\gamma$ the quasi-static approach is invalid.

In Eq. (10), the following auxiliary functions are introduced

$$f_j(\bar{p}) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\omega}^{j-1}\mathrm{d}\bar{\omega}}{\bar{\omega}^4 + \bar{\omega}F(\bar{\omega}/\varepsilon) + \bar{p}^2},\ \ g_j(\bar{p}) = \frac{1}{\pi}\int_0^\infty \frac{\bar{\omega}^{j}F(\bar{\omega}/\varepsilon)\mathrm{d}\bar{\omega}}{\bar{\omega}^4 + \bar{\omega}F(\bar{\omega}/\varepsilon) + \bar{p}^2}. \tag{11}$$

The direct analytical inversion of Eq. (10) is impossible. However, an efficient technique can be used for an approximate inversion with arbitrary small error. This technique is based on expansion of the functions $f_j(\bar{p})$ and $g_j(\bar{p})$ from Eq. (11) in asymptotic power series of $1/\bar{p}$ at large $\bar{p}$ (i.e. at small $\tau$). The most simple results concern the case when $\varepsilon \to 0$. In this case, the functions (11) can be evaluated as

$$g_j(\bar{p}) = f_{j+1}(\bar{p}) = \frac{1}{\bar{p}^{(3-j)/2}}\sum_{i=1}^{\infty}\frac{(-1)^{i-1}a_{j+1,i-1}}{\bar{p}^{3(i-1)/2}}, \tag{12}$$

where

$$a_{j+1,i-1} = \frac{\Gamma\big((i+j)/4\big)\Gamma\big((3i-j)/4\big)}{4\pi\Gamma(i+j)},$$

and $\Gamma$ is the gamma-function. The transition of Eq. (10) to the originals is also performed using the series. For instance, substituting Eq. (12) into the image of the contact force given by Eq. (10) produces

$$P^L(\bar{p}) = \frac{mv\gamma^2}{a_{1,0}\bar{p}^{1/2} + \sum_{i=1}^{\infty}(-1)^i\frac{a_{1,i}}{\bar{p}^{(3i-1)/2}} + \gamma^2}, \tag{13}$$

from which follows

$$P^L(\bar{p}) = mv\gamma^2\sum_{i=1}^{\infty} c_i(\gamma)\bar{p}^{-i/2} \quad \to \quad P(\tau) = \frac{mv\gamma^2}{\tau t_n}\sum_{i=1}^{\infty} c_i(\gamma)\frac{\tau^{i/2}}{\Gamma(i/2)}, \tag{14}$$

where $c_i(\gamma)$ are coefficients of expansion of the fraction from Eq. (13) in the Maclaurin series of $\bar{p}$. The face deflection $w_0$ and interfacial stress $\sigma_0$ are derived by expansion in series or as the following convolutions

$$w_0(\tau) = \frac{t_n^2}{m\gamma^2}\int_0^\tau P(\tau-\tau_1)\phi_1(\tau_1)\mathrm{d}\tau_1, \sigma_0(\tau) = \frac{E_1 t_n^2}{m\gamma^2 x_n}\int_0^\tau P(\tau-\tau_1)\phi_2(\tau_1)\mathrm{d}\tau_1, \quad (15)$$

where the functions $\phi_j(\tau)$ are originals of the functions $f_j(\bar{p})$ from Eq. (12);

$$\phi_j(\tau) = \tau^{(2-j)/2}\sum_{i=1}^{\infty}\tau^{3i/2}\frac{(-1)^i a_{j,i}}{\Gamma\big((3i+4-j)/2\big)}, \quad j = 1, 2.$$

The formula (14) for the contact force has irregularity of the type $1/\sqrt{\tau}$ for the plane formulation at $\tau \to 0$, while the deflection and stress are finite. The solutions for the axisymmetric formulation can be obtained from Eqs. (14) and (15) by replacements $\phi_1 \to \phi_2$, $\phi_2 \to \phi_3$. In this case, all the variables are finite. The structural response to the small-mass impact is illustrated in Figs. 3 and 4.

## FINITE ELEMENT ANALYSIS

The numerical modelling of an elastic response of a sandwich structure to small mass impact was performed using the Finite Element (FE) code *LS-DYNA*®. The sandwich structure was modeled as a thin plate attached to a thick layer as shown in Fig. 1. The mechanical properties of the analyzed structure are indicated in Table 1.

Table 1: Mechanical Properties of the Sandwich Constituents

| | Thickness, mm | Young's modulus, MPa | Poisson's ratio | density, kg/m$^3$ | yield stress, MPa | yield strain, % |
|---|---|---|---|---|---|---|
| **core** | 50 | 85 | 0.42 | 56 | 0.9 | 1.8 |
| **face** | 2.4 | 16500 | 0.25 | 1500 | – | – |

The face sheet was meshed using 4-node shell elements; 250 elements were used. The FE mesh was condensed towards the contact area between the impactor and the face sheet; the condensation factor was 2. The core was meshed using 8-node volume elements. Fifteen elements were used through the thickness of the core. All degrees of freedom were constrained at the lower boundary of the core layer.

The impactor ($m$=0.01 kg, $v$=1 m/sec) was modeled as a rigid body meshed using 8-node volume elements. All degrees of freedom for the impactor were constrained, except for translation in a direction normal to the plate. The contact area between the impactor and the face sheet was computed automatically by the FE code.

## RESULTS AND DISCUSSION

The analytical calculations were performed for the same sandwich configuration as in the FE analysis, see Table 1. Some calculations relate to the case of $\varepsilon \to 0$. Solutions for the impulse loading were obtained using Eqs. (7) and (8); Eqs. (14) and (15) were used for small mass/short time impact.

### Case of impulse excitation

The structural response to the impulse $I$=1 N$\cdot$ sec is illustrated in Fig. 2. The features of Eqs. (7) and (8) are that the stress in the plane formulation and the deflection in the axisymmetric formulation are finite, see Fig. 2, and the stress in the axisymmetric formulation is singular at $\tau = 0$. The solutions presented in Fig. 2 also demonstrate the strong effect of the core thickness on the oscillation frequencies. Thus, the amplitude and frequency increase with decreasing of the core thickness, while almost no oscillations are observed for the infinite core thickness.

### Small mass/short time impact

Figures 3 and 4 demonstrate the structural response to the small mass/short time impact. In general, the quasi-static solutions ($\rho_c$=0) produce conservative estimation for the face deflection, interfacial stress and contact force, especially in the plane formulation. This fact demonstrates important role of the face inertia.

The maximums of analytical dynamic solutions and FE analysis are found in a good agreement. The dynamic solution demonstrates fast dissipation of the impact energy; in the FE analysis, the oscillations are damped even faster due to the impactor rebound. In general, the FE analysis confirms the singular property of Eq. (14) for the plane formulation where the contact force has irregularity $1/\sqrt{\tau}$ at $\tau \to 0$.

The series in Eq. (14) converge only for small and moderate values of dimensionless time $\tau$. Such intervals of $\tau$ include maximums of the variables only for large values of $\gamma$. Thus, the long-duration impacts for small $\gamma$ (i.e. for large impact mass) should be described by numerical inversion of the transformations (10); although the quasi-static approach can be sufficient in this case.

## CONCLUSIONS

The presented analytical solutions deal with the elastic response of sandwich beams and panels to local forced excitation or impact by a rigid body. The study concerns the plane and axisymmetric formulations. The main results can be outlined as

- The closed-form solutions were obtained for the case of forced excitation. Explicit formulas were derived for an impulse loading. The analytical results were verified with FE analysis showing good agreement in general;
- Problem of non-stationary oscillations excited by impact was solved. It was shown that the quasi-static solutions are insufficient for the case of a small mass impact, and the face inertia can not be neglected;
- The presented solutions can be further used for prediction of failure onset in foam-cored sandwich structures subject to local loads.

## ACKNOWLEDGEMENTS

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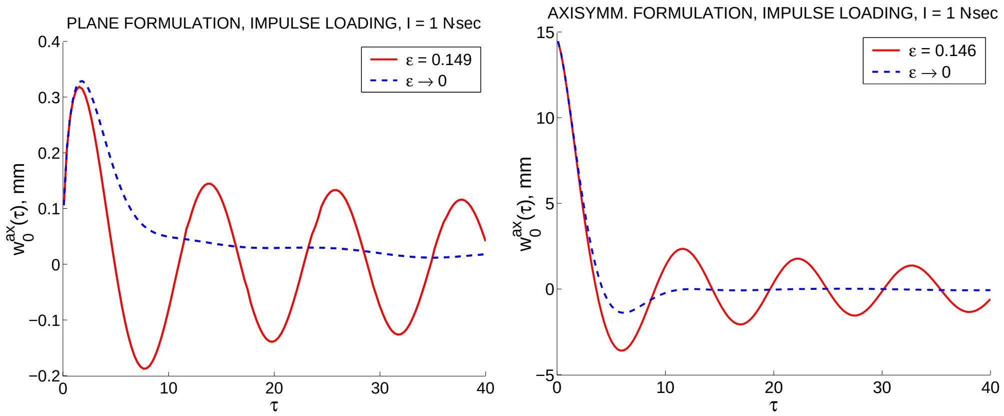


Figure 2: Maximum face deflection vs. dimensionless time

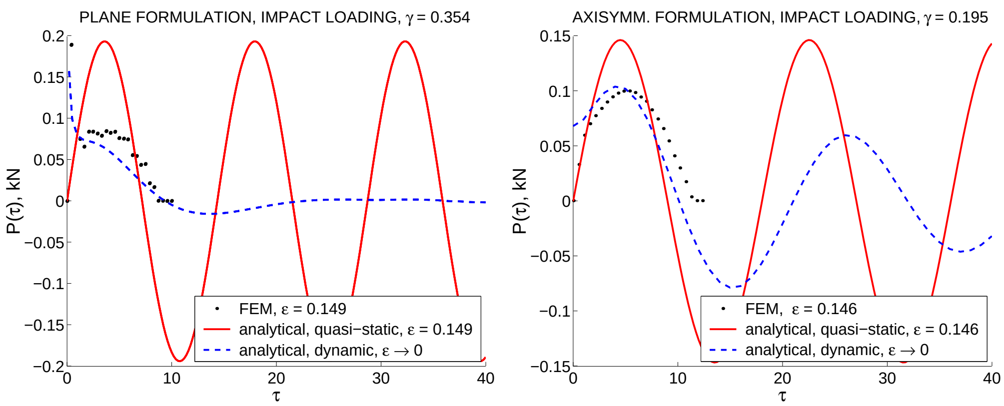


Figure 3: Contact force vs. dimensionless time.

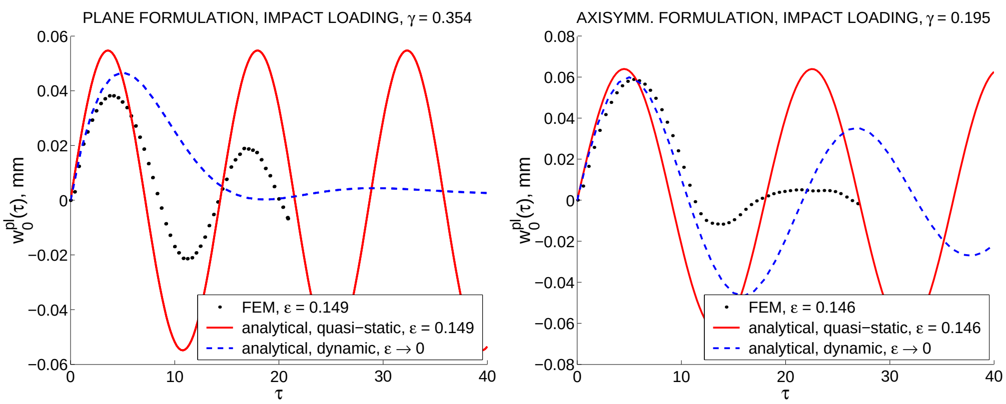


Figure 4: Maximum face deflection vs. dimensionless time.