On Anisotropy, Objectivity and Invariancy in finite thermo–mechanical deformations

J. Huétink

University of Twente, dep. Engineering Technology, P.O. Box 217, 7500 AE Enschede, Netherlands

URL: www.tm.ctw.utwente.nl *e-mail: j.huetink@utwente.nl*

ABSTRACT: In elastic–plastic finite deformation problems constitutive relations are commonly formulated in terms the Cauchy stress as a function of the elastic finger tensor and an objective rate of the Cauchy stress as a function of the rate of deformation tensor. For isotropic materials models this is rather straight forward, but for anisotropic material models, including elastic anisotropy as well as plastic anisotropy, this may lead to confusing formulations. It will be shown that it is more convenient to define the constitutive relations in terms of invariant tensors referred to the deformed metric. An alternative decomposition of the deformation tensor is introduced that can easily be linked to the additive decomposition of the velocity gradient into a spin tensor and a rate of deformation tensor. Constraints for constitutive equations are formulated based on thermodynamics.

Key words: anisotropy, objectivity, invariance, material experiments, sheet metal, composites

1 DEFORMATION AND ORIENTATION

The deformation gradient $\mathbf{F}$ maps the initial configuration to the current configuration: $\mathrm{d}\mathbf{x} = \mathbf{F} \cdot \mathrm{d}\mathbf{X}$. Commonly $\mathbf{F}$ is (polar)decomposed in a rotation and a stretch tensor: $\mathbf{F} = \mathbf{Q} \cdot \mathbf{U}$ and $\mathbf{F} = \mathbf{V} \cdot \mathbf{Q}$.
However in the case of elastic-plastic deformation this is not convenient as it will generally not result into a subsequent decomposition of the stretch tensor into an elastic stretch tensor and a plastic stretch tensor which are both symmetric. We may alternatively split the deformation tensor in a tensor $\mathbf{G}$ which is $\mathbf{NOT}$ necessarily symmetric, and a (subsequent) rotation:

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{G} \tag{1}$$

The decomposition (1) is not unique. The tensor $\mathbf{G}$ may contain deformation and material (lattice) orientation. The tensor $\mathbf{G}$ is invariant under rigid body rotations as an additional rotation can be regarded as a subsequent multiplication of $\mathbf{Q}$ and $\mathbf{R}$, and has no influence on $\mathbf{G}$. With the decomposition (1) the velocity gradient $\mathbf{L}$ (= $\mathbf{v}\overleftarrow{\boldsymbol{\nabla}}$) can be written as:

$$\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}} + \mathbf{R} \cdot \dot{\mathbf{G}} \cdot \mathbf{G}^{-1} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{W} + \mathbf{D} \tag{2}$$

where $\mathbf{W}$ is the spin-tensor and $\mathbf{D}$ the rate of deformation tensor. Apart from the spin tensor we define the rate of rotation tensor according to:

$$\boldsymbol{\Omega} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}}$$

The rate of rotation tensor $\boldsymbol{\Omega}$ is skew symmetric but may differ from the spin-tensor $\mathbf{W}$ which becomes clear from (2). They only coincide if $\dot{\mathbf{G}} \cdot \mathbf{G}^{-1}$ is symmetric. Or the other way around; if we require that $\boldsymbol{\Omega} = \mathbf{W}$, then there exist a multiplicative decomposition $\mathbf{F} = \mathbf{R} \cdot \mathbf{G}$ such that $\dot{\mathbf{G}} \cdot \mathbf{G}^{-1}$ is symmetric and $\mathbf{R} \cdot \dot{\mathbf{G}} \cdot \mathbf{G}^{-1} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{D}$.

If $\boldsymbol{\Omega} \neq \mathbf{W}$, then

$$\tfrac{1}{2}\mathbf{R} \cdot (\dot{\mathbf{G}} \cdot \mathbf{G}^{-1} + (\dot{\mathbf{G}} \cdot \mathbf{G}^{-1})^{\mathrm{T}}) \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{D}$$

In addition to the the velocity gradient (2) we define:

$$\mathbf{L}^{g} = \dot{\mathbf{G}} \cdot \mathbf{G}^{-1} \tag{3}$$

Note: We may require that $\mathbf{W} = \boldsymbol{\Omega}$. $\mathbf{R}$ is then path dependent and is to be solved from the evolution equation $\dot{\mathbf{R}} = \mathbf{W} \cdot \mathbf{R}$, and not from a polar decomposition.

2 STRAINS AND STRESSES

Commonly two strains are defined that are equal to the unit tensor in the case of rigid rotation, respectively the *right* and *left Cauchy–Green tensor*:

$$\mathbf{C} = \mathbf{F}^{\mathrm{T}} \cdot \mathbf{F} = \mathbf{U}^2 \text{ and } \mathbf{B} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{T}} = \mathbf{V}^2$$

For the *right Cauchy–Green tensor* we can also write:

$$\mathbf{C} = \mathbf{G}^{T} \cdot \mathbf{R}^{T} \cdot \mathbf{R} \cdot \mathbf{G} = \mathbf{G}^{T} \cdot \mathbf{G}$$

The Cauchy stress can be expressed in components referring to either the global base vectors $\mathbf{e}_i$ or local corotating base vectors $\mathbf{g}_i$:

$$\boldsymbol{\sigma} = \mathbf{e}_i \sigma_{ij} \mathbf{e}_j = \mathbf{g}_i \tau_{ij} \mathbf{g}_j$$

with $\mathbf{g}_i = \mathbf{R} \cdot \mathbf{e}_i$, and consequently:

$$\boldsymbol{\sigma} = \mathbf{R} \cdot \mathbf{e}_i \tau_{ij} \mathbf{e}_j \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R} \cdot \boldsymbol{\tau} \cdot \mathbf{R}^{\mathrm{T}} \tag{4}$$

with $\boldsymbol{\tau} = \mathbf{e}_i \tau_{ij} \mathbf{e}_j$. A similar stress tensor is introduced by Bergander et al [1]. The stress tensor $\boldsymbol{\tau}$ is invariant under superimposed rigid rotation.

2.1 Strain rates and Stress rates

The rate of deformation tensor is linked to the rate of the *right Cauchy–Green tensor* by:

$$\dot{\mathbf{C}} = 2\mathbf{F}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{F} = 2\mathbf{G}^{\mathrm{T}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{R} \cdot \mathbf{G} = 2\mathbf{G}^{\mathrm{T}} \cdot \mathbf{d} \cdot \mathbf{G}$$

with the invariant rate of deformation tensor:

$$\mathbf{d} = \tfrac{1}{2}(\mathbf{L}^{g} + \mathbf{L}^{g\mathrm{T}}) = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{D} \cdot \mathbf{R} \tag{5}$$

The rate of the Cauchy stress can with (4) be written as

$\dot{\boldsymbol{\sigma}} = \dot{\mathbf{R}} \cdot \boldsymbol{\tau} \cdot \mathbf{R}^{\mathrm{T}} + \mathbf{R} \cdot \dot{\boldsymbol{\tau}} \cdot \mathbf{R}^{\mathrm{T}} + \mathbf{R} \cdot \boldsymbol{\tau} \cdot \dot{\mathbf{R}}^{\mathrm{T}}$
or
$\dot{\boldsymbol{\sigma}} - \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot (\dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{R} \cdot \dot{\boldsymbol{\tau}} \cdot \mathbf{R}^{\mathrm{T}}$ (6)
The right hand side of (6) is objective. Consequently the left hand side of (6) is an objective rate, referred to as the *Green–Naghdi* or *Green–McInnis* rate [2]
$\overset{\diamond}{\boldsymbol{\sigma}} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega} \cdot \boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \boldsymbol{\Omega}^{\mathrm{T}} = \mathbf{R} \cdot \dot{\boldsymbol{\tau}} \cdot \mathbf{R}^{\mathrm{T}}$ (7)
The next step is to find expressions for the invariant stress $\boldsymbol{\tau}$ and stress rate $\dot{\boldsymbol{\tau}}$

2.2 Elastic, Plastic and Thermal deformation

The deformation tensor $\mathbf{F}$ is assumed to be decomposed in a rotational part $\mathbf{R}$, a reversible part $\mathbf{F}_r$ and an irreversible part $\mathbf{F}_p$. The reversible part can be decomposed in an elastic part and a thermal (expansion) part.
$\mathbf{F} = \mathbf{R} \cdot \mathbf{G} = \mathbf{R} \cdot \mathbf{F}_r \cdot \mathbf{F}_p = \mathbf{R} \cdot \mathbf{F}_e \cdot \mathbf{F}_T \cdot \mathbf{F}_p$ (8)
The (invariant) velocity gradient $\mathbf{L}^{\mathbf{g}}$ can be split into an elastic, a thermal, and a plastic part:
$\mathbf{L}^{\mathbf{g}} = \dot{\mathbf{G}} \cdot \mathbf{G}^{-1} \quad = \mathbf{L}_e + \mathbf{L}_T + \mathbf{L}_p$ (9)
with
$\mathbf{L}_e = \dot{\mathbf{F}}_e \cdot {\mathbf{F}_e}^{-1}$
$\mathbf{L}_T = \mathbf{F}_e \cdot \dot{\mathbf{F}}_T \cdot {\mathbf{F}_T}^{-1} \cdot {\mathbf{F}_e}^{-1} = (\mathbf{F}_e \cdot {}^4\mathbf{I} \cdot {\mathbf{F}_e}^{-1^{\mathrm{T}}}) : (\dot{\mathbf{F}}_T \cdot {\mathbf{F}_T}^{-1})$
$\mathbf{L}_p = (\mathbf{F}_e \cdot \mathbf{F}_T \cdot {}^4\mathbf{I} \cdot ({\mathbf{F}_T}^{-1} \cdot {\mathbf{F}_e}^{-1})^{\mathrm{T}}) : (\dot{\mathbf{F}}_p \cdot {\mathbf{F}_p}^{-1})$
These are obtained by using the property that for arbitrary second order tensors
$\mathbf{A} \cdot \mathbf{B} \cdot \mathbf{C} = (\mathbf{A} \cdot {}^4\mathbf{I} \cdot \mathbf{C}^{\mathrm{T}}) : \mathbf{B}$ (10)
where ${}^4\mathbf{I} = \delta_{ik}\delta_{jl}\mathbf{e}_i\mathbf{e}_j\mathbf{e}_k\mathbf{e}_l$, the fourth order unit tensor. Substitution of (9) into (2) leads to: $\mathbf{L} = \dot{\mathbf{F}} \cdot \mathbf{F}^{-1} = \mathbf{W} + \mathbf{D} = \dot{\mathbf{R}} \cdot \mathbf{R}^{\mathrm{T}} + \mathbf{R} \cdot (\mathbf{L}_e + \mathbf{L}_T + \mathbf{L}_p) \cdot \mathbf{R}^{\mathrm{T}}$

2.3 Stress and free energy

Commonly the Helmholz free energy ψ is regarded as a function of $\mathbf{C}_e = \mathbf{F}_e^{\mathrm{T}} \cdot \mathbf{F}_e$ and the temperature T:
$\psi = \psi(\mathbf{C}_e, T)$
From the second law of thermodynamics it follows in a similar way as in [3] and [2] that:
$\boldsymbol{\tau} = 2\rho\mathbf{F}_e \cdot \dfrac{\partial\psi}{\partial\mathbf{C}_e} \cdot \mathbf{F}_e^{\mathrm{T}}$ (11)
In polymer models it is commonly assumed that the free energy ψ is a function of the left elastic Cauchy–Green tensor (or Finger tensor $\mathbf{B}_e$) , i.e. the Leonov model. This is not sufficient for invariance of ψ, as $\mathbf{B}_e$ is objective and hence changes under rigid body motions. For including anisotropy, ψ should then also be a function of $\mathbf{R}$. Note: Besseling [3] argued that proper state variables should be invariant and hence objective tensors cannot be regarded as proper state variables. Nevertheless if we assume that $\psi = \psi(\mathbf{B}_e)$, then it can be shown that:
$\boldsymbol{\sigma} = 2\rho\dfrac{\partial\psi}{\partial\mathbf{B}_e} \cdot \mathbf{B}_e$ (12)
However, it will not be possible to describe anisotropic material if (12) is assumed. This will be illustrated by the following example.

2.3.a Example: Layers of Uniaxial fibres

We consider a layer of linear elastic uniaxial fibres in a composite. In this layer there exist a uniaxial tensile stress in the direction of the fibres $\sigma = E\varepsilon$. The free energy (per unit mass) is then equal to the elastic stored energy: $\psi = \frac{1}{2\rho^o}E\varepsilon^2$ where ρ^o is the mass density in a stress free state. The next step is to express the uniaxial strain ε in either $\mathbf{F_e}$, $\mathbf{C_e}$ or $\mathbf{B_e}$. A particle of the fibre is in the undeformed state given by a vector $\mathbf{a}^o$ and length ℓ^o, and in the deformed state by a vector $\mathbf{a}$ and length ℓ. The relation between the current and initial state satisfies: $\mathbf{a} = \mathbf{F} \cdot \mathbf{a}^o$ or with the decomposition (8): $\mathbf{a} = \mathbf{R} \cdot \mathbf{F_e} \cdot \mathbf{a}^o$. We define the scalar strain as:
$\varepsilon = \frac{1}{2}\frac{(\ell^2 - \ell^{o2})}{\ell^{o2}} = \frac{1}{2}\mathbf{a}^o\mathbf{a}^o : (\mathbf{C_e} - \mathbf{1})/\ell^{o2}$
and (with $\mathbf{B_e} = \mathbf{R} \cdot \mathbf{C_e} \cdot \mathbf{R}^{\mathrm{T}}$),
$\varepsilon = \frac{1}{2}\mathbf{aa} : (\mathbf{1} - {\mathbf{B_e}}^{-1})/\ell^{o2}$ (13)
Consequently the free energy becomes:
$\psi = \dfrac{E}{8\rho^o\ell^{o4}}(\mathbf{C_e} - \mathbf{1}) : \mathbf{a}^o\mathbf{a}^o\mathbf{a}^o\mathbf{a}^o : (\mathbf{C_e} - \mathbf{1})$ (14)
and with (11) the invariant stress $\boldsymbol{\tau}$ is then:
$\boldsymbol{\tau} = \frac{\rho E\ell^2}{2\rho^o\ell^{o6}}\mathbf{a}^o\mathbf{a}^o\mathbf{a}^o\mathbf{a}^o : (\mathbf{C_e} - \mathbf{1})$
The Cauchy stress tensor is then:
$\boldsymbol{\sigma} = \frac{\rho E}{2\rho^o\ell^{o4}}\mathbf{aaa}^o\mathbf{a}^o : (\mathbf{C_e} - \mathbf{1})$
or $\boldsymbol{\sigma} = \frac{\rho E}{2\rho^o\ell^{o4}}\mathbf{aaaa} : (\mathbf{1} - {\mathbf{B_e}}^{-1})$
If the free energy could be written as a function of $\mathbf{B_e}$ then the Cauchy stress could also be obtained by applying (12) directly. However, substituting (13) into the elastic stored energy yields:
$\psi = \frac{1}{8\rho^o}E(\mathbf{aa} : (\mathbf{1} - {\mathbf{B_e}}^{-1})/\ell^{o2})^2$
As the dyad $\mathbf{aa}$ is not constant and cannot be expressed as a function of $\mathbf{B_e}$ it is obvious that in the anisotropic case the free energy cannot be expressed as a function of $\mathbf{B_e}$ only, but is also dependent on $\mathbf{R}$.
Note: In the case of a second layer in direction $\mathbf{b}^o$, the (invariant) stress–strain relation can easily be extended with a similar term. If the layers are *pin jointed*, we can apply a model with a single deformation tensor $\mathbf{F}$ which can be multiplicative decomposed in several orientation tensors and invariant deformation tensors:
$\mathbf{F} = \mathbf{R} \cdot \mathbf{G} = \mathbf{R} \cdot \mathbf{R}^a \cdot \mathbf{G}^a$
$\mathbf{F} = \mathbf{R} \cdot \mathbf{G} = \mathbf{R} \cdot \mathbf{R}^b \cdot \mathbf{G}^b$
The invariant stress reads then:
$\boldsymbol{\tau} = \mathbf{R}^a \cdot \boldsymbol{\tau}^a \cdot (\mathbf{R}^a)^T + \mathbf{R}^b \cdot \boldsymbol{\tau}^b \cdot (\mathbf{R}^b)^T$

2.4 A generalised elastic anisotropic model

The forgoing layer model can be extended to an arbitrary anisotropic elastic model by replacing $\mathbf{a}^o\mathbf{a}^o\mathbf{a}^o\mathbf{a}^o$ in (14) by an arbitrary fourth order material tensor ${}^4\overset{\mathbf{o}}{\mathbf{E}}$.
$\psi = \frac{1}{8\rho^o}(\mathbf{C_e} - \mathbf{1}) : {}^4\overset{\mathbf{o}}{\mathbf{E}} : (\mathbf{C_e} - \mathbf{1})$
For anisotropic solids that have axes of anisotropy with directions that are fixed relatively to each other, the fourth order elasticity tensor ${}^4\overset{\mathbf{o}}{\mathbf{E}}$ can be regarded as invariant, **and constant**.

With (11) the invariant stress reads:
$\boldsymbol{\tau} = \frac{\rho}{2\rho^o}(\mathbf{F}_e \cdot {}^4\mathbf{I} \cdot \mathbf{F}_e^{\mathrm{T}}) : {}^{4}\overset{\mathbf{o}}{\mathbf{E}} : (\mathbf{C_e} - \mathbf{1})$
In the next sections we consider the more general thermo-mechanically coupled elastic-plastic case.

2.4.a Thermo-mechanical rate equations
In order to obtain a rate equation for the invariant Cauchy stress, the material derivative of Eq. (11) is taken, resulting in:

$$\dot{\boldsymbol{\tau}} = \frac{\dot{\rho}}{\rho}\boldsymbol{\tau} + {}^4\mathbf{E} : \mathbf{L}_e + \mathbf{K}\dot{T} \tag{15}$$

in which the tensors ${}^4\mathbf{E}$ and $\mathbf{K}$ are defined as follows:
${}^4\mathbf{E} = {}^{4}\overset{*}{\mathbf{E}} + {}^4\mathbf{I} \cdot \boldsymbol{\tau} + \boldsymbol{\tau} \cdot {}^4\mathbf{I}$
${}^{4}\overset{*}{\mathbf{E}} = 4\rho(\mathbf{F}_e \cdot {}^4\mathbf{I} \cdot \mathbf{F}_e) : \frac{\partial^2\psi}{\partial \mathbf{C}_e \partial \mathbf{C}_e} : (\mathbf{F}_e^{\mathrm{T}} \cdot {}^4\mathbf{I} \cdot \mathbf{F}_e^{\mathrm{T}})$
$\mathbf{K} = 2\rho\mathbf{F}_e \cdot \frac{\partial^2\psi}{\partial \mathbf{C}_e \partial T} \cdot \mathbf{F}_e$
The stress and stress rate equations defined in this section hold for arbitrary size of the elastic part of the deformation including anisotropy . Moreover, no restriction is made to solids. Hence the forgoing also holds for (compressible) fluids.
To include plasticity we have to substitute (9) into (15):

$$\dot{\boldsymbol{\tau}} = \frac{\dot{\rho}}{\rho}\boldsymbol{\tau} + {}^4\mathbf{E} : (\mathbf{L}^g - \mathbf{L}_T - \mathbf{L}_p) + \mathbf{K}\dot{T} \tag{16}$$

Because of the symmetry of ${}^4\mathbf{E}$ the contribution of the anti–symmetric parts of $\mathbf{L}^g, \mathbf{L}_T, \mathbf{L}_p$ vanish. So without losing generality we can replace (16) by:

$$\dot{\boldsymbol{\tau}} = \frac{\dot{\rho}}{\rho}\boldsymbol{\tau} + {}^4\mathbf{E} : (\mathbf{d} - \mathbf{d}_T - \mathbf{d}_p) + \mathbf{K}\dot{T} \tag{17}$$

For metals the elastic strains are about 0.001 to 0.005, hence $\mathbf{F}_e$ is close to the unit tensor.

2.5 Anisotropic plastic deformations
In plasticity theory the plastic deformation rate is commonly related to the gradient of a *plastic potential* χ:
$\mathbf{d}_p = \dot{\lambda}\frac{\partial\chi}{\partial\boldsymbol{\tau}}$
Plastic deformation occurs if the elastic limit or *yield surface* $\varphi = \varphi(\boldsymbol{\tau}, \boldsymbol{\alpha}^i, \rho, T, (\dot{\boldsymbol{\alpha}}^i)) = 0$
will be exceeded. The tensors $\boldsymbol{\alpha}^i$ are a number $(i = 1, n)$ of strain hardening parameters. In the following elaboration we will for convenience omit the superscripts i and consider $\boldsymbol{\alpha}$ as a second order tensor. The elaboration is similar when several tensors of different order are included. The term $(\dot{\boldsymbol{\alpha}})$ accounts for strain rate dependency.
Plastic loading / elastic (un)loading is related to the Kuhn-Tucker condition: $\dot{\lambda} \geq 0, \;\; \varphi \leq 0, \;\; \dot{\lambda}\varphi = 0$
Generally, for metals *associated flow* is assumed , which means that the plastic potential and the yield surface are isomorphic.
Substituting the relations for the plastic strain rate into the constitutive rate equation (17) Leads to:

$$\dot{\boldsymbol{\tau}} = \frac{\dot{\rho}}{\rho}\boldsymbol{\tau} + \mathbf{E} : \left(\mathbf{d} - \dot{\lambda}\frac{\partial\chi}{\partial\boldsymbol{\tau}} - \mathbf{d}_T\right) + \mathbf{K}\dot{T} \tag{18}$$

Now the plastic multiplier $\dot{\lambda}$ must be determined. If the yield criterion is independent of $\dot{\boldsymbol{\alpha}}$ then $\dot{\lambda}$ can be eliminated by requiring that $\dot{\varphi}$ vanishes. In the rate dependent case it is more convenient to switch to an incremental formulation. At time $t_o + \Delta t$ the yield condition can be written as:
$\varphi = \varphi(\boldsymbol{\tau}_o + \Delta\boldsymbol{\tau}, \boldsymbol{\alpha}_o + \Delta\boldsymbol{\alpha}, \rho_o + \Delta\rho, T_o + \Delta T, \frac{\Delta\boldsymbol{\alpha}}{\Delta t}) =$
$\phi(\boldsymbol{\tau}_o, \boldsymbol{\alpha}_o, \rho_o, T_o, \Delta\boldsymbol{\tau}, \Delta\boldsymbol{\alpha}, \Delta\rho, \Delta T, \Delta t) = 0$
For a fixed time increment Δt we can linearise the expression to:

$$\varphi = \phi_o + \frac{\partial\phi}{\partial\boldsymbol{\tau}} : \Delta\boldsymbol{\tau} + \frac{\partial\phi}{\partial\boldsymbol{\alpha}} : \Delta\boldsymbol{\alpha} + \frac{\partial\phi}{\partial\rho}\Delta\rho + \frac{\partial\phi}{\partial T}\Delta T = 0 \tag{19}$$

Furthermore, it is assumed that the rate of change of the hardening tensor $\boldsymbol{\alpha}$ is only nonzero if plastic deformation occurs, so $\Delta\boldsymbol{\alpha}$ can be expressed in the (average) plastic rate of deformation during the time increment as: $\Delta\boldsymbol{\alpha} = {}^4\mathbf{P} : \tilde{\mathbf{d}}_p\Delta t = ({}^4\mathbf{P} : \frac{\partial\chi}{\partial\boldsymbol{\tau}})\Delta\lambda$
so that the third term of Eq. (19) can be written as

$$\frac{\partial\phi}{\partial\boldsymbol{\alpha}} : \Delta\boldsymbol{\alpha} = (\frac{\partial\phi}{\partial\boldsymbol{\alpha}} : {}^4\mathbf{P} : \frac{\partial\chi}{\partial\boldsymbol{\tau}})\Delta\lambda = \Phi_1\Delta\lambda \tag{20}$$

If both eqs. (18) and (20) are substituted into the yield function (19), a return mapping formula for the plastic multiplier $\Delta\lambda$ is derived to read:
$\Delta\lambda = \frac{(\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\mathbf{E}:(\mathbf{d}-\mathbf{d}_T)+(\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\boldsymbol{\tau}+\rho\frac{\partial\phi}{\partial\rho})\frac{\dot{\rho}}{\rho}+(\frac{\partial\phi}{\partial T}+\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\mathbf{K})\dot{T})\Delta t+\phi_o}{\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\mathbf{E}:\frac{\partial\chi}{\partial\boldsymbol{\tau}}-\Phi_1}$
Substitution of this expression for $\Delta\lambda$ into the incremental form of Eq. (18) and the relative density rate is replaced by $-\operatorname{tr}(\mathbf{d})$, yields:

$$\begin{aligned}\Delta\boldsymbol{\tau} &= (\mathbf{E} - (1-h)\mathbf{Y}) : (\mathbf{d} - \mathbf{d}_T)\Delta t \\ &+ (\mathbf{K} - (1-h)\mathbf{Y}^\theta)\Delta T - \\ &(\boldsymbol{\tau}\mathbf{1} - (1-h)\mathbf{Y}^\rho) : \mathbf{d}\Delta t + \Phi_o\end{aligned} \tag{21}$$

where *h* is the hardening parameter,
$h = \frac{\Phi_1}{\Phi_1 - N_y}$with $\Phi_1 = (\frac{\partial\phi}{\partial\boldsymbol{\alpha}} : {}^4\mathbf{P} : \frac{\partial\chi}{\partial\boldsymbol{\tau}})$
$N_y = \frac{\partial\phi}{\partial\boldsymbol{\tau}} : \mathbf{E} : \frac{\partial\chi}{\partial\boldsymbol{\tau}}$and $\Phi_o = \frac{\mathbf{E}:\frac{\partial\chi}{\partial\boldsymbol{\tau}}\phi_o}{N_y - \Phi_1}$
and the fourth order tensors $\mathbf{Y}$ and $\mathbf{Y}^\rho$ and the second order tensor $\mathbf{Y}^\theta$ are given by:
$\mathbf{Y} = \frac{\mathbf{E}:\frac{\partial\chi}{\partial\boldsymbol{\tau}}\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\mathbf{E}}{N_y}, \quad \mathbf{Y}^\theta = \frac{\frac{\partial\phi}{\partial T}+\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\mathbf{K}}{N_y}\mathbf{E} : \frac{\partial\chi}{\partial\boldsymbol{\tau}}$
$\mathbf{Y}^\rho = \frac{\frac{\partial\phi}{\partial\rho}\rho+\frac{\partial\phi}{\partial\boldsymbol{\tau}}:\boldsymbol{\tau}}{N_y}\mathbf{E} : \frac{\partial\chi}{\partial\boldsymbol{\tau}}\mathbf{1}$
If desired an objective rate of the Cauchy stress is found by rewriting (7) using (10) to:
$\overset{\diamond}{\boldsymbol{\sigma}} = (\mathbf{R}\cdot{}^4\mathbf{I}\cdot\mathbf{R}) : \dot{\boldsymbol{\tau}}$ and similarly writing (5) as: $\mathbf{d} = (\mathbf{R}^{\mathrm{T}} \cdot {}^4\mathbf{I} \cdot \mathbf{R}^{\mathrm{T}}) : \mathbf{D}$. Substitution of (21) yields in the isothermal and time independent case for the objective rate of the Cauchy stress:
$\overset{\diamond}{\boldsymbol{\sigma}} = (\mathbf{R} \cdot {}^4\mathbf{I} \cdot \mathbf{R}) : {}^4\mathbf{E}^{\mathbf{ep}} : (\mathbf{R}^{\mathrm{T}} \cdot {}^4\mathbf{I} \cdot \mathbf{R}^{\mathrm{T}}) : \mathbf{D}$

3 APPLICATIONS

The formulation is implemented in the in house Fem code DiekA. The first application concerns forming of

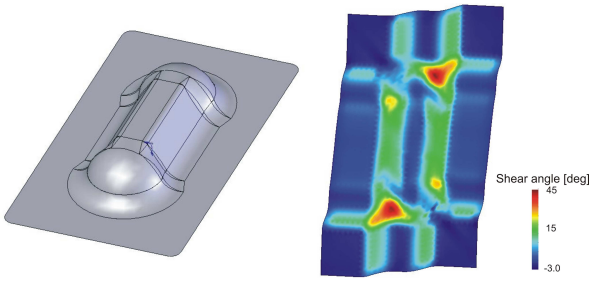

Figure 1: Draping simulations of a biaxial non crimp fabric on a double dome geometry

fabric reinforced composites. Fig 1 shows the predicted angle change of initially orthogonal fabric fibres. Details can be found in [4] and [5].
The second application concerns anisotropic softening.

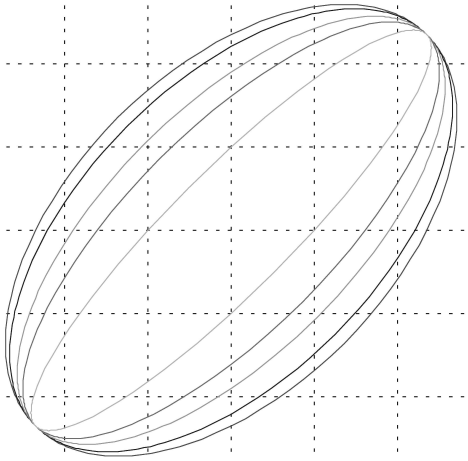

Figure 2: Anisotropic softening yield loci.

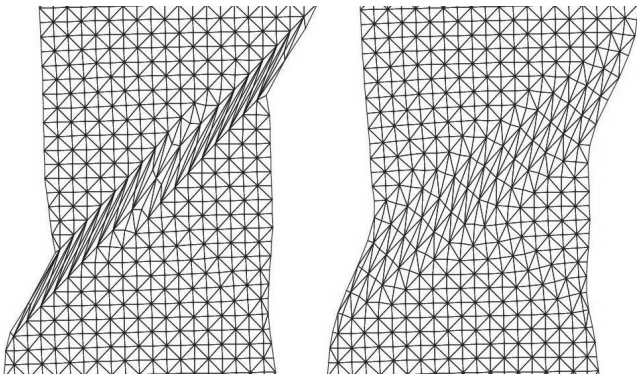

Figure 3: Tensile test simulation, isotropic softening (l), anisotropic softening (r).

The Hill'48 yieldcriterion is used for plane stress:
$\phi = (G+H)\tau_{xx}^2 - 2H\tau_{xx}\tau_{yy} + (F+H)\tau_{yy}^2 + 2N\tau_{xy}^2 - 2\sigma_{\mathrm{f}}^2 = 0$
Anisotropic softening is implemented by an evolution equations for F, G, H, N:
$F = G = 2 + 4 * \varepsilon, \quad H = 4 + 48 * \varepsilon, \quad N = 6 + 12\varepsilon$
This rule represents anisotropic damage evolution.
Fig 2 shows the flattening of the yieldsurface due to this anisotropic softening. In fig 3 the results of a tensile test simulation are shown. The left figure shows the results with planar isotropic softening, the right figure with anisotropic softening. It appears that the severe mesh dependency in the case of isotropic softening (localization in a band of one element thickness), does not occur in this case of anisotropic softening.

4 CONCLUSIONS

The proposed alternative multiplicative decomposition of the deformation tensor can easily be linked to the additive decomposition of the velocity gradient into a spin tensor and a rate of deformation tensor. The decomposition is not unique and hence decomposition in several orientation tensors and (non symmetric) invariant deformation tensors is possible. This is an advantage for modelling of fabric reinforced composites in which the angle between the fibres are changing.
Any appropriate objective rate of the Cauchy stress can be expressed as the material rate of the invariant stress tensor $\boldsymbol{\tau}$, and subsequently pre and post multiplied by the orientation tensor $\mathbf{R}$ and $\mathbf{R}^{\mathrm{T}}$ respectively. By some additional elaboration the objective rate of the Cauchy stress can be expressed as the multiplication (double contraction) of a fourth order stiffness tensor and the rate of deformation tensor. This fourth order stiffness tensor is objective and refers to the global basis and hence can directly be applied in assembling the global stiffness matrix in FEM simulations. However, it is more convenient in programming practice to establish the element stiffness matrices and reaction forces with respect to local (element oriented) coordinates and then transform these to the global directions before assembling. By anisotropic softening the mesh dependency is reduced. This is apparently due to rotation of the most softened direction, which does not stay in line with the loading direction.

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