Convection by means of least squares projection for ALE calculations

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ABSTRACT: Element result data are in general discontinuous across element boundaries. In the ALE method convection of these data with respect to the element grid is required.

In this paper we present a convection method, which is based on a least squares projection. For moderate convective displacements it is reformulated in terms of a field integral and boundary flux terms. For one dimensional problems the method can be shown to be third order in space. For two dimensional problems the method is stable for Courant numbers up to 1.05.

Based on this method a simplification is suggested where the calculation of gradients of fields in upwind elements is not needed. This simplification is paid for by a slight decrease in stability. A maximum Courant number of 0.72 is still attainable.

The methods are outlined in one dimension, results are shown of two dimensional calculations.

Key words: ALE, transient convection, least squares

1 INTRODUCTION

In the Arbitrary Lagrangian Eulerian (ALE) method the finite element mesh is allowed to move independently from the material. In this way it is possible to retain a good mesh quality even at large material distortions [1], [2]. The outline of the method is shown in figure 1. The material moves with a velocity \( v_m \). A grid velocity is denoted as \( v_g \). The convective velocity can be defined as \( v_c = v_m - v_g \). The benefit of retaining a good mesh quality is obtained at the cost of having to convect (usually discontinuous) finite element result data with respect to the mesh.

An often applied procedure is to first construct a continuous field based on nodal averaging and then apply the convection equation to this field. The construction of the continuous field causes a considerable amount of diffusion. This on the one hand enhances the stability, but on the other hand deteriorates the accuracy [3]. A discretisation of the convection, which is able to cope with the discontinuous character of the data which have to be convected, is required.

A possible choice for discretisation of the convection equation is the Discontinuous Galerkin method [4], [5]. Many variants have appeared in literature in recent years. Different choices have been made for the order of spatial discretisation as well as for the time integration [6], [8], [7].

In this paper we develop a method, which is based on a least squares projection. It can be regarded as a generalization of an idea posed in [9] and applied in [10]. For moderate Courant numbers (ratio of the convective step length and the element reference length) this method can be written in a way which resembles the Discontinuous Galerkin method with third order spatial discretisation. Based on the full least squares projection a slightly cheaper but also less accurate method is derived, which gives good results for small
2 LEAST SQUARES PROJECTION

It is assumed that the convective displacement \( u_c = v_c \Delta t \) is known, based on the results of the previous Lagrangian calculation and the mesh regularization algorithm. In that case the value after convection of any element variable \( f \) can be written as:

\[
f_{\text{new}}(x) = f_{\text{old}}(x - u_c)
\]

Then the least squares projection of the old \( f \)-field to the new configuration of the \( i \)th element \([x_i, x_{i+1}]\) can be written as:

\[
\int_{x_i}^{x_{i+1}} \tilde{f} f_{\text{new}} \, dx = \int_{x_i}^{x_{i+1}} (\tilde{f} + u_t \frac{\partial \tilde{f}}{\partial x}) f_{\text{old}} \, dx
\]

This is depicted in figure 2. When the convective displacement remains smaller than the element length, a projection scheme as shown in figure 3 can be developed (skip the superscript \( \text{old} \)).

\[
\int_{x_i}^{x_{i+1}} \tilde{f} f_{\text{new}} \, dx = \int_{x_i}^{x_{i+1}} (\tilde{f} + u_t \frac{\partial \tilde{f}}{\partial x}) f_{\text{old}} \, dx + \int_{x_i}^{x_{i+1}} \tilde{f} f_{\text{upw}} \, dx - \int_{x_{i+1}}^{x_{i+1} - u_{i+1}} \tilde{f} f_{\text{upw}} \, dx
\]

This result bears much resemblance to the typical Discontinuous Galerkin discretisation with a field integral and boundary fluxes. For elements with linear interpolation of the data fields the boundary flux terms can be written in terms of values and gradients at the boundaries.

\[
\int_{x_i}^{x_{i+1}} \tilde{f} f_{\text{upw}} \, dx = u_t (\tilde{f}_i f_{i+1}^\text{upw} + \frac{1}{2} u_{i+1} \frac{\partial \tilde{f}_i}{\partial x}) f_{i+1}^\text{upw} - \frac{1}{2} u_t \tilde{f}_i \frac{d f_{\text{upw}}}{dx} - \frac{1}{6} u_t^2 \frac{d \tilde{f}_i}{dx} \frac{d f_{\text{upw}}}{dx}
\]

And a similar term at the outflow boundary. In one dimension this scheme is exact for Courant numbers less than 1.

3 SIMPLIFICATION

The method outlined above requires evaluation of the gradients of the data fields in the upwind elements. In general this is time consuming and if possible it should be avoided. The obvious remedy is to take equation (4) and leave out all terms containing gradients in upwind elements. See also figure 4. In order to balance the inflow and outflow fluxes, the corresponding terms have to be discarded in the outflow flux term also. When this is not done, conservation is not guaranteed anymore and the phase velocity will
Figure 5: Generalization of Eq. (3) to 2D

be inaccurate.

\[
\int_{x_i-\bar{u}_i}^{x_{i+1}+\bar{u}_{i+1}} \tilde{f} f^u_{pw} \, dx \approx u_i(\tilde{f}_i f^u_{pw} + \frac{1}{2} u_i \frac{d\tilde{f}}{dx} f^u_{pw})
\]

\[
\int_{x_{i+1}}^{x_{i+1}+\bar{u}_{i+1}} \tilde{f} f \, dx \approx u_{i+1}(\tilde{f}_{i+1} f_{i+1} + \frac{1}{2} u_{i+1} \frac{d\tilde{f}}{dx} f_{i+1})
\]

(5)

4 2D EXAMPLES

Both methods have been implemented in 3-node and 6-node triangles. The integration scheme of figures 3 and 4 is generalised as shown in figure 5. A number of tests have been run in order to assess the accuracy. The first test concerns the convection of a Gaussian bump by a constant velocity field on an unstructured mesh. The object of these calculations was to determine the stability limits of both methods in terms of the Courant number \( Cr \).

\[
Cr = \frac{1}{2a_i} \oint_{\Gamma_i} \| u \cdot n \| \, ds
\]

(6)

Also the relative merits of both methods in terms of cost and maximum Courant numbers are assessed. The second set of calculations concerns the convection of a block function by a rotating velocity field. The preservation of the shape of this block after a full rotation is assessed.

4.1 Maximum Courant numbers

A Gaussian bump, initially \( f(x,y) = 0.01^2(x^2+y^2) \), is convected by a constant velocity field \( \{ u_x, u_y \} = \{ \delta u_x, 0 \} \). The initial distribution is shown in figure 6. The final distribution using the full discretisation with a Courant number 0.9 is shown in figure 7. The final distribution using the simplification as outlined in section 3 with a Courant number 0.7 is shown in figure 8.

We find that the full discretisation still gives accurate results for Courant numbers up to 1.05, whereas the simplified version fails for \( Cr > 0.72 \). In both runs the maximum value of the bump decreases by 4\% , the undershoot is less than 0.5\%. In our implementation the simplified version requires approximately 33\% less time compared to the full version.

4.2 Shape preservation

The initial distribution of the element data to be convected is shown in figure 9. This is a least squares fit
of a unit block function covering the region $-\frac{3}{4} < x < -\frac{1}{4}$ and $-\frac{1}{4} < y < \frac{1}{4}$. It is rotated about \{x, y\} = \{0, 0\} for one full revolution in 320 steps. The final distribution is shown in figure 10. Some intermediate steps are shown in figure 11. The initial shape of the block is preserved reasonably well.

5 CONCLUSIONS

In this paper we present a discretization of the convection equation, which is based on a least squares projection. For moderate Courant numbers it can be written as a field integral and boundary fluxes. Numerical experiments in 2-D show that the method is stable and shows no spurious diffusion for $Cr < 1.05$.

A second method, which is a simplification of the first one, is shown to be stable and without numerical diffusion for $Cr < 0.72$.

References


