A unified proof of dynamic stability of interior ESS for projection dynamics

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Abstract

We present a unified proof of dynamic stability for interior evolutionarily stable strategies for two recently introduced projection dynamics using the angle between certain vectors as a Lyapunov function.
Key words: evolutionary games, projection dynamics, dynamic stability and evolutionary stability.
JEL-Codes: C62; C72; C73

1 Introduction

The aim of this note is to present a unified proof of asymptotic stability of interior evolutionarily stable strategies or states under two recently introduced evolutionary dynamics which have in common that they project the relative fitness function (cf., Joosten [1996]) called excess payoff function elsewhere (cf., Sandholm [2005]) unto the unit simplex. We do so by designing a Lyapunov function which depends on angles between certain vectors, rather than on distances, a prominent tactic in proving dynamic stability of evolutionary concepts, see e.g., Lahkar & Sandholm [2008], Hofbauer & Sandholm [2009], Joosten [1996,2006,2009], Joosten & Roorda [2011a].

To provide an intuition for our method of proof, think of two Mikado[1] sticks. Join the sticks at one end and then let the angle between the sticks decrease monotonically. Clearly, the sticks come closer and closer. We associate one (fixed) Mikado stick with a vector connected to an evolutionarily stable strategy and the (moving) other with vectors connected to points generated by a dynamic process related to the relative fitness function.

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[1] Popular in the pre-Nintendo age, chop sticks may serve as a substitute mental image.

2 On projection dynamics

Lahkar & Sandholm [2008] introduced the projection dynamics in evolutionary game theory. These dynamics can be seen as the orthogonal projection of a particular dynamical system taken sufficiently close to the unit simplex, unto the latter. Joosten & Roorda [2011a] pursued a similar idea, but used another type of projection and obtained the ray-projection dynamics. Here, projection occurs along a ray connecting the point at hand and the origin. Joosten & Roorda [2011a] proposed the terminology orthogonal projection dynamics for the dynamics of Lahkar & Sandholm [2008].

Joosten & Roorda [2011a,b] treat general types of projections to be used as evolutionary dynamics. Here, we restrict our attention to the following subclass of projection dynamics. Let $a = \alpha \cdot 1^{n+1} = \alpha \cdot (1, ..., 1)$, $\alpha \leq 0$ and let relative fitness function $f : S^n \to \mathbb{R}^{n+1}$ be given. Then, $h^a : int\ S^n \to \mathbb{O}^{n+1} = \{x \in \mathbb{R}^{n+1} | \sum_{i=1}^{n+1} x_i = 0\}$ is given by

$$h^a(x) = f(x) - \frac{\sum_{i=1}^{n+1} f_i(x)}{1 - \sum_{i=1}^{n+1} a_i}(x - a) \text{ for all } x \in int\ S^n.$$

$S^n = \{x \in \mathbb{R}_+^{n+1} | \sum_{i=1}^{n+1} x_i = 1\}$ is the n-dimensional unit simplex; $int\ S^n$ refers to its relative interior. It can be confirmed that on the relevant domain

$$\begin{aligned} h^{0^{n+1}}(x) &= \lim_{\alpha \to 0} h^a(x) = f(x) - \left(\sum_{i=1}^{n+1} f_i(x) \right) x, \\ h^{-\infty^{n+1}}(x) &= \lim_{\alpha \to -\infty} h^a(x) = f(x) - \left(\sum_{i=1}^{n+1} f_i(x) \right) \frac{1}{n+1} 1^{n+1}. \end{aligned}$$

The former function determines the ray-projection dynamics of Joosten & Roorda [2011a], the latter the orthogonal-projection dynamics of Lahkar & Sandholm [2008]. Joosten & Roorda [2011a] gave the following interpretation[2]:

Assume that a spectator located at 0^{n+1}, the $(n+1)$-dimensional origin, looks into the positive orthant and observes Samuelson's process. Assume, as in Plato's famous cave allegory, that the spectator can not move his head, hence can not see depth. There is a intransparent hyperplane with only one window, the unit simplex. Then, the true process looks as if taking place on this window, the unit simplex, as if it were the ray-projection dynamics. If one moves the spectator away from the origin towards $(-\infty, ..., -\infty)$, then if the spectator were hypothetically to arrive there and observe the true process through a telescope, the real process would again look as if taking place on the window, as if it were the orthogonal-projection dynamics.

[2] Samuelson's process is given by $\frac{dx}{dt} = f(x)$ for all $x \in int\ \mathrm{R}_+^{n+1}$ where f satisfies continuity, homogeneity of degree zero in x, and complementarity ($x \cdot f(x) = 0$ for all x).

The interpretation for arbitrary a is then that we place our Platonian cave dweller at observation point a.

3 Angles and a candidate Lyapunov function

Denote $\mathbb{R}_+^{n+1} = \{x \in \mathbb{R}_+^{n+1} \backslash \{0^{n+1}\} |\ x_i \geq 0$ for all $i = 1, 2, ..., n+1\}$. Let $x, y, -a \in \mathbb{R}_+^{n+1} \cup \{0^{n+1}\}$, then the cosine of the angle between the vectors $(y-a)$ and $(x-a)$ in $\mathbb{R}_+^{n+1}$ is given by

$$\varpi^a(y, x) = \frac{(y-a) \cdot (x-a)}{||y-a|| \cdot ||x-a||}.$$

In Joosten & Roorda [2011a] we examined rather general dynamics in $\mathbb{R}_+^{n+1}$ and projected these unto the unit simplex. To make a connection to that contribution, we assume that the dynamics are determined by

$$\frac{dx}{dt} = h(x), \tag{D}$$

where $h : int\ \mathbb{R}_+^{n+1} \to \mathbb{R}^{n+1}$ is a continuous and homothetic function with respect to $\mathbb{R}_+^{n+1}$. The latter implies that for all $x \in int\ \mathbb{R}_+^{n+1}$ and $\lambda \geq 0$,

$$\frac{h(x) \cdot h(\lambda x)}{||h(x)|| \cdot ||h(\lambda x)||} = 1,$$

provided of course that the above is well-defined, i.e., $||h(x)|| \neq 0$; if $h(x) = 0^{n+1}$, then $h(\lambda x) = 0^{n+1}$. This means that on every ray through the origin the dynamics point in the same direction. Furthermore, we assume that every solution $\{x_t\}_{t \geq 0}$ to $x_0 \in int\ \mathbb{R}_+^{n+1}$ and (D), is unique and that a compact set $M(x_0)$ exists such that $\{x_t\}_{t \geq 0} \subset M(x_0) \subset \mathbb{R}_+^{n+1}$. We are not concerned with boundary behavior of the dynamics as we will focus on interior evolutionarily stable strategies. For more general statements, one would need that h is extended to the boundary of $\mathbb{R}_+^{n+1}$ such that h is homothetic and $x_i = 0$ implies $h_i(x) \geq 0$.[3] We also require that the zeroes of f and h coincide, i.e., $f(x) = 0^{n+1}$ is equivalent to $h(x) = 0^{n+1}$.

Clearly,

$$\begin{aligned} \varpi^a(y, x) &= 1 \text{ if } x = \lambda y \text{ for some } \lambda > 0, \\ \varpi^a(y, x) &< 0 \text{ otherwise, provided } x \in \mathbb{R}_+^{n+1}. \end{aligned}$$

The cosine of the angle between $y-a$ and $x-a$ changes as follows

$$\frac{d\varpi^a(y, x)}{dt} = \frac{(y-a) \cdot h(x)}{||y-a|| \cdot ||h(x)||} - \varpi^a(y, x) \frac{(x-a) \cdot h(x)}{||x-a|| \cdot ||h(x)||}.$$

[3] We refer to Joosten & Roorda [2011a] for possible definitions for extensions to the boundary and comparisons.

If $\varepsilon > 0$ exists such that for all $0 < \varepsilon' \leq \varepsilon$

$$1 - \varpi^a(y,x) < \varepsilon' \text{ implies } \frac{d\varpi^a(y,x)}{dt} > 0,$$

then under the conditions placed on the dynamics, we can use $\varpi^a(y,x)$ as a Lyapunov function. The interpretation is that the dynamics generate a trajectory $\{x_t\}_{t\geq 0}$ such that $\varpi^a(y, x_{t+k}) > \varpi^a(y, x_t)$ for all $t, k > 0$, i.e., the angle between the vectors $y - a$ and $x_t - a$ strictly decreases monotonically.

Rewriting the time derivative of the cosine of the angle as defined yields:

$$\begin{aligned}
&\frac{d\varpi^a(y,x)}{dt} \\
= \;& \frac{||x-a||\,(y-a)\cdot h(x) - ||y-a||\varpi(y,x)\,(x-a)\cdot h(x)}{||y-a||\cdot||h(x)||\cdot||x-a||} \\
= \;& \frac{||y-x||}{||y-a||}\frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} - \frac{(||y-a||\varpi(y,x) - ||x-a||)\,(x-a)\cdot h(x)}{||y-a||\cdot||h(x)||\cdot||x-a||} \\
= \;& \frac{||y-x||}{||y-a||}\frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} - \left(\varpi(y,x) - \frac{||x-a||}{||y-a||}\right)\frac{(x-a)\cdot h(x)}{||x-a||\cdot||h(x)||} \\
= \;& \frac{||y-x||}{||y-a||}\frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} - \left(\frac{(y-x)\cdot(x-a)}{||y-a||\cdot||x-a||}\right)\frac{(x-a)\cdot h(x)}{||x-a||\cdot||h(x)||} \\
= \;& \frac{||y-x||}{||y-a||}\left[\frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} - \left(\frac{(y-x)\cdot(x-a)}{||y-x||\cdot||x-a||}\right)\frac{(x-a)\cdot h(x)}{||x-a||\cdot||h(x)||}\right].
\end{aligned}$$

Hence $\frac{d\varpi^a(y,x)}{dt}$ is positive for $x \neq y$ if and only if

$$\frac{(y-x)\cdot h(x)}{||y-x||\cdot||h(x)||} - \left(\frac{(y-x)\cdot(x-a)}{||y-x||\cdot||x-a||}\right)\frac{(x-a)\cdot h(x)}{||x-a||\cdot||h(x)||} > 0.$$

The latter inequality is for $x \neq y$ equivalent to

$$\left(y - x\left[1 + \frac{(y-x)\cdot(x-a)}{(x-a)\cdot(x-a)}\right] + \frac{(y-x)\cdot(x-a)}{(x-a)\cdot(x-a)}a\right)\cdot h(x) > 0. \qquad \text{(P)}$$

4 The unified proof for two projection dynamics

We are interested in evolutionarily stable strategies and the properties of certain evolutionary dynamics nearby. The concept of the evolutionarily stable strategy is due to Maynard Smith & Price [1973] and several equivalent definitions have become available. We (have) use(d) one particular version which serves our purposes well (cf., Joosten [1996]). We call $y \in S^n$ an evolutionarily stable strategy (ESS) if an open neighborhood $U \subset S^n$ exists which contains y and that for every x in $U\backslash\{y\}$

$$(y-x)\cdot f(x) > 0.$$

Now, we list two 'obvious' observations, proofs omitted for brevity's sake.

a For $a = 0^{n+1}$, Inequality (P) reduces to

$$\Big(y - x\frac{y \cdot x}{x \cdot x}\Big) \cdot h(x) > 0.$$

b For $a \to -\infty^{n+1}$, (P) reduces to

$$\big(y - x + (y - x) \cdot (x - a)1^{n+1}\big) \cdot h(x) > 0.$$

We have completed all preliminaries for the following result. The respective parts have been proven elsewhere (e.g., Lahkar & Sandholm [2008], Joosten & Roorda [2011a]).

Proposition 1 *Every interior evolutionarily stable strategy is an asymptotically stable fixed point of the ray-projection dynamics and the orthogonal projection dynamics.*

Proof. Let f denote the relative fitness function at hand and let $y \in int\ S^n$ be an evolutionarily stable strategy, then an open neighborhood U in the relative interior of S^n exists such that U contains y and for all $x \in U \backslash \{y\}$:

$$(y - x) \cdot f(x) > 0.$$

We now show that (P) holds, so $\varpi^a(y, x)$ is a Lyapunov function for projection dynamics h^a where $a = 0^{n+1}$ and $a = (\lim_{\alpha \to -\infty} \alpha) \cdot 1^{n+1}$.

So, if $h = h^{0^{n+1}}$ and $x, y \in int\ S^n$, then Observation (a) leads to

$$\Big(y - x\frac{y \cdot x}{x \cdot x}\Big) \cdot h(x) =$$

$$\Big(y - x\frac{y \cdot x}{x \cdot x}\Big) \cdot \left[f(x) - \left(\sum_{i=1}^{n+1} f_i(x)\right) x\right] =$$

$$y \cdot f(x) - \frac{y \cdot x}{x \cdot x} x \cdot f(x) - \left(\sum_{i=1}^{n+1} f_i(x)\right) y \cdot x + \left(\sum_{i=1}^{n+1} f_i(x)\right) \frac{y \cdot x}{x \cdot x} x \cdot x =$$

$$y \cdot f(x) = (y - x) \cdot f(x).$$

The final equality sign follows from complementarity of the relative fitness function f. So, the statement that y is an ESS guarantees that $\varpi^{0^{n+1}}(y, x)$ is a Lyapunov function for $h^{0^{n+1}}$, i.e., the ray-projection dynamics.

Note that, if $h = h^{-\infty^{n+1}}$ and $x, y \in int\ S^n$ then Observation (b) leads to

$$(y - x) \cdot h(x) = (y - x) \cdot \left[f(x) - \frac{\left(\sum_{i=1}^{n+1} f_i(x)\right)}{n+1} 1^{n+1}\right] = (y - x) \cdot f(x).$$

So, the statement that y is an ESS guarantees that $\varpi^{-\infty^{n+1}}(y, x)$ is a Lyapunov function for $h^{-\infty^{n+1}}$, i.e., the orthogonal-projection dynamics. ■

Remark. Adequate rescaling of the axes may yield strategies of proof for convergence to ESS for other (projection) dynamics.

5 References

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