

## A phenomenological description of soliton splitting during run up

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**ABSTRACT.** In this paper a simple model is proposed to describe the splitting of an initial single solitary wave that runs into shallower water into two solitary waves. Different from results in the literature that use inverse scattering theory for the Korteweg - de Vries equation to find the splitting once a single (deformed) wave has arrived at a shallower region of constant depth, in this paper a quasi-static approximation is proposed to capture also the changes during run up. The model is completely based on qualitative properties of the energy and mass of single solitary waves as function of amplitude. With these relations, the splitting process can be described qualitatively in complete agreement with results from numerical calculations.

### 1. Introduction

In this contribution we investigate one of these fascinating phenomena in wave propagation that appear as a low dimensional dynamical system. Consider a solitary wave like profile, running to the right at constant speed above a flat part of the bottom. When the wave enters a region of variations in the bottom, it will deform since nonlinear and dispersive effects will not balance each other any longer as they did above the flat bottom. For increasing depth, dispersion will become more dominant, and constituent waves will disperse away. We will consider here the opposite effect: the deformation of a wave during run up. Depending on details of the topography, various deformations can be expected and have partly been described in the literature.

In fact, the experimental and numerical results of Beji & Battjes, [2] were a motivation for this investigation. They studied a periodic wave train over a bar. The combination of a decrease and subsequent increase of depth makes it difficult to understand their intriguing results. Therefore we restrict ourselves and study the run-up on its own.

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To understand the basic underlying process, we consider the simplest possible topography: a smooth (monotone) transition between two constant depths, decreasing in the direction of propagation of the wave. Although the slope will be assumed to be small, the total variation in depth will be of order one. As a motivation for the mathematical approach of this paper, and to have a partial, practical justification of the results, consider Fig. 1. This shows the wave profiles from numerical results obtained by Van Daalen who carefully calculated the deformation for the full set of nonlinear gravity waves using boundary elements methods, see [3]. The same qualitative phenomenon is described by the model that will be presented in Section 3; see Fig. 9.

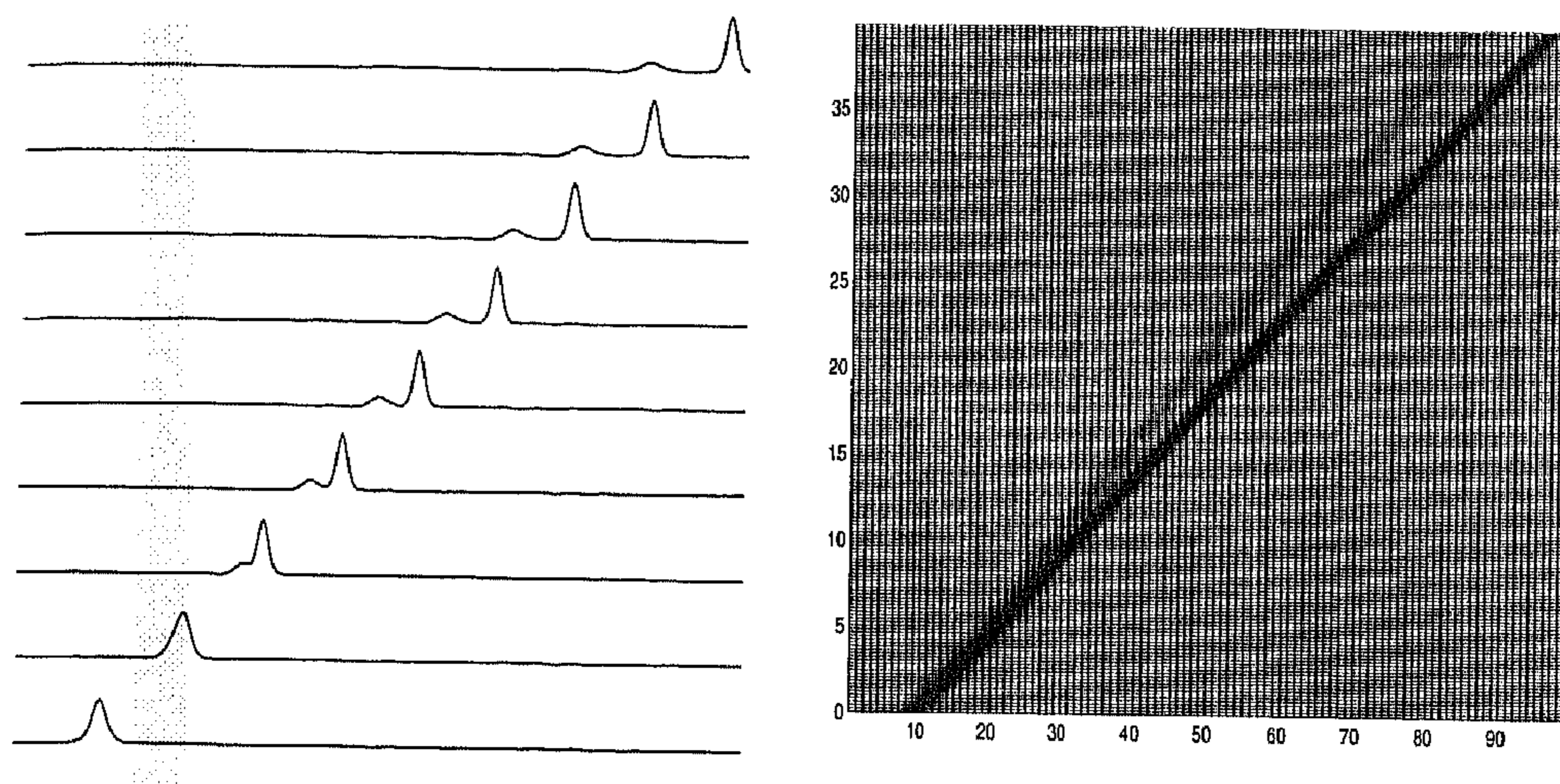


FIGURE 1. Numerical calculations of an initial solitary wave of .1 m during run up from a depth of .5 m to a depth of .35 m. The calculations are based on the complete set of surface wave equations for inviscid fluid without surface tension. At the left the wave profiles are shifted vertically with increasing time and the region of linear decrease in depth is indicated; at the right the splitting is shown in grey tones.

The graphic representation of the calculations shows that for the given topography the initial single solitary wave splits into two (noticeable) waves, each of which resembles quite well a solitary wave. Although experimental or numerical results that show the smooth separation process in such a clean way cannot be found in the literature, the deformation of waves by variable bottom has attracted the attention of many scientists, and both numerical and analytical results are available, in particular from the seventies: Madsen & Mei, [9], Tappert & Zabusky [16], Johnson [6, 7], Ono [13], Miles [10, 11]; see also Newell [12]. The description presented in this paper differs from these papers in the following way.

For the distortion when entering shallower water, the general approach in the literature consists of two steps. First it is assumed that during run up, a single initial hump of solitary wave type above a bottom at depth  $h_0$  deforms within a family of *single* hump profiles. Two parameters are used to describe the details of the hump (amplitude and width, for instance), and the change of these parameters are



calculated. Then a single hump is obtained at the smaller depth  $h$ . Secondly, the distortion of this single hump, that differs from a pure solitary wave profile, is investigated on the new constant depth  $h$ . Using the Korteweg-de Vries equation as an analytical model, inverse scattering techniques are applied to predict the number of solitary waves that emerge from this hump ("fission" of solitons). The result of this analysis is that the number of emerging solitons  $N$  depends on the depth ratio according to the following formula:

$$\frac{2}{N(N+1)} < \left(\frac{h}{h_0}\right)^{9/4} < \frac{2}{N(N-1)}.$$

In particular, two solitons are expected when  $(h/h_0)^{9/4} > 1/3$ , i.e. when  $h/h_0 > .614$ .

In this paper we will follow a somewhat different approach. Essentially the idea is to describe the wave running up not as a single hump but as a gradually developing separation process of two solitary waves. (Conditions on the maximal depth ratio for which two waves will appear are not required beforehand, but will follow from the analysis.) The governing mechanism that will determine the separation will be the basic physical laws of energy and mass conservation. The aim is to give a simple description of this phenomenon, using only variables that appear to be necessary from the phenomenon, being the amplitude and position of each of the waves.

In Section 2 the consequence of the conservation properties of energy and mass will be investigated solely from an analysis of the dependence of the energy and mass on the parameters of a one-soliton solution. In fact, only qualitative properties of these functions, like convexity and monotonicity, turn out to be essential to describe the qualitative properties of the splitting process. Explicit analytic expressions for these functions are given in Section 3 based on a model equation that has been proposed by Van Groesen & Pudjaprasetya [5] to describe uni-directional waves above a slowly varying bottom. This model equation is a variant of the KdV-equation (Korteweg & De Vries, 1895) [8], with coefficients that depend explicitly on the bottom topography and is equivalent with an equation derived by Newell [12]. Although the restrictions that underly the validity of this equation may not be fulfilled for steeper bottom variations, the qualitative properties of the energy and mass functions obtained are in agreement with results obtained from calculations for the complete set of surface wave equations. In section 4 we describe the elements of a mathematical justification of the simple approximation presented here. Section 5 contains some conclusions, and suggestions for further research.

## 2. Qualitative description

Before specifying any evolution equation, we present a simple description of the phenomenon in this section.

First consider the constituent waves appearing in the phenomenon. Denote a single solitary wave above a flat bottom by  $S_1$ . Except from the depth of the bottom, other parameters are the "position" of the wave (say the position of its crest) and its amplitude (or a related quantity), to be denoted by  $\gamma$ . In the description to follow we will use a *quasi-homogeneous* evolution. Based on the assumption that the wave is "concentrated" in space (vanishingly small outside a region around the crest), we will take as approximation a solitary wave above a flat bottom of which the depth is determined by the depth below the actual position of the crest. In

that way,  $\gamma$  and the depth  $h$  can be taken as independent variables. Instead of the depth  $h$ , we will use in the following the quantity  $c$  defined by

$$c = \sqrt{gh}$$

where  $g$  is the gravitational acceleration, and we will write  $S_1(c, \gamma)$  accordingly. This defines a two-dimensional "wave manifold", a collection of waveshapes: at each depth a one-parameter family with parameter indicating the height of the wave.

Each wave carries a certain amount of mass and has certain energy; we will denote these quantities in the following by  $\mathcal{M}(c, \gamma)$  and  $\mathcal{E}(c, \gamma)$  respectively.

To describe the run up, start with a wave above a flat bottom at depth  $h_0$ , that has mass  $m_0$  and energy  $e_0$ . For decreasing depth, assuming the wave to remain a single wave with adjusting value of  $\gamma$ , energy conservation alone would determine the governing evolution (parameterized by  $c$ ). In fact,

$$\mathcal{E}(c, \gamma) = e_0$$

describes a curve in the  $c, \gamma$ -plane, and so implicitly  $\gamma$  as a function of  $c$ ; see Fig. 2. (The figures presented in this section are based on the model to be described in the next section. However, the arguments given in this section depend only on qualitative properties to be described shortly.)

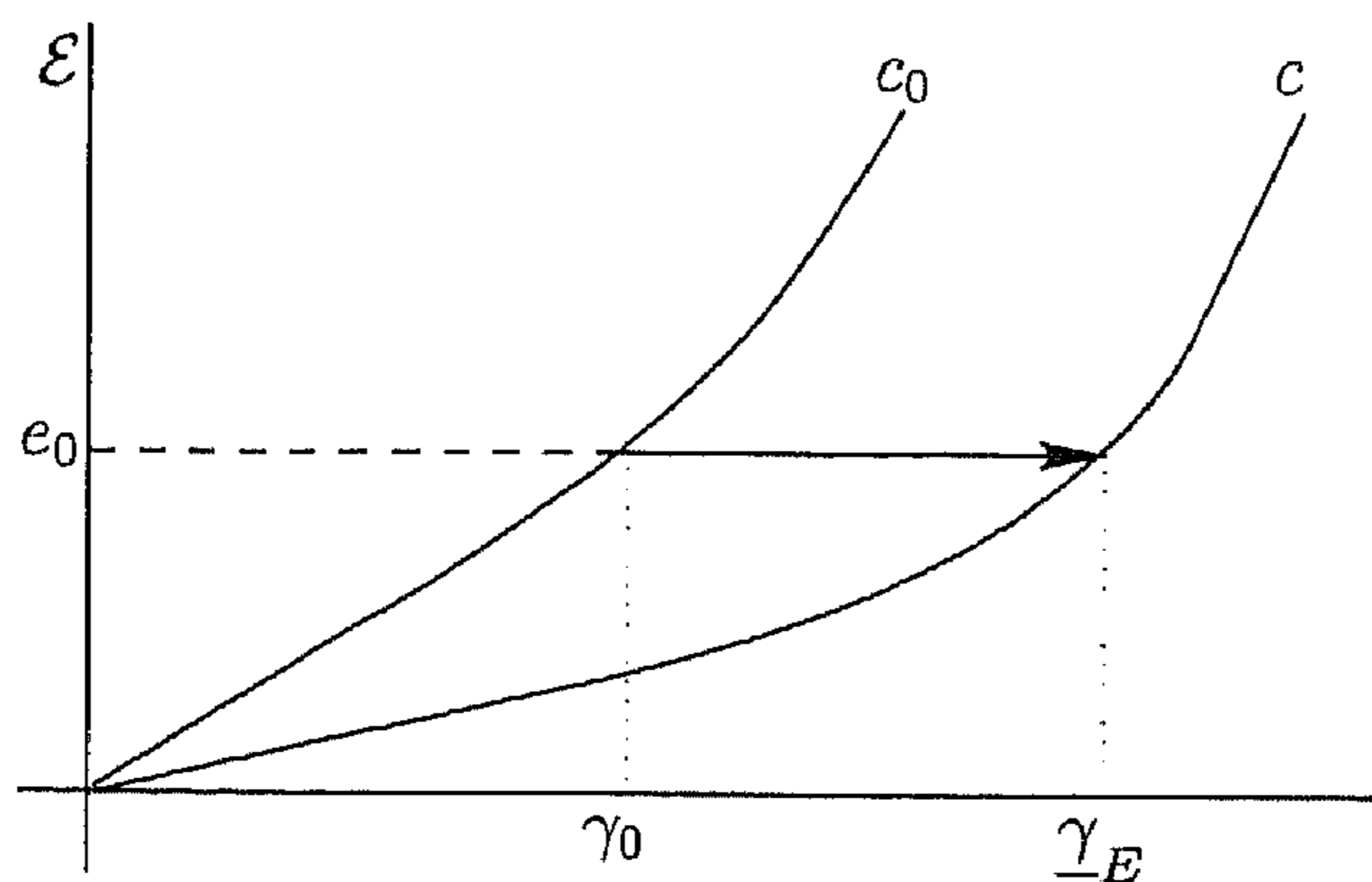


FIGURE 2. Energy vs momentum  $\gamma$  for two different values of  $c$ . The arrow would indicate the running up of one solitary wave of constant energy (for which the momentum has to increase), not taken into account mass conservation.

In general it cannot be expected that mass will be conserved in this energy-conserved evolution. It is shown in Fig.3. (depicted for the model of the next section) that the mass decreases and that there is a mass-surplus when modelling with a single solitary wave. One way to overcome the violation in mass conservation for a single wave is to introduce some additional tail at the back of the wave that is designed to account for the mass surplus. In literature various models have been used: shelves, non-flat equilibrium surfaces, etc.; see e.g. Newell, [12], Pudjaprasetya & Van Groesen [15].

Here we will take another point of view, and assume that instead of a single wave (plus tail), the profile will evolve during run up into the superposition of *two* waves which together satisfy energy and mass conservation. (Assuming that there



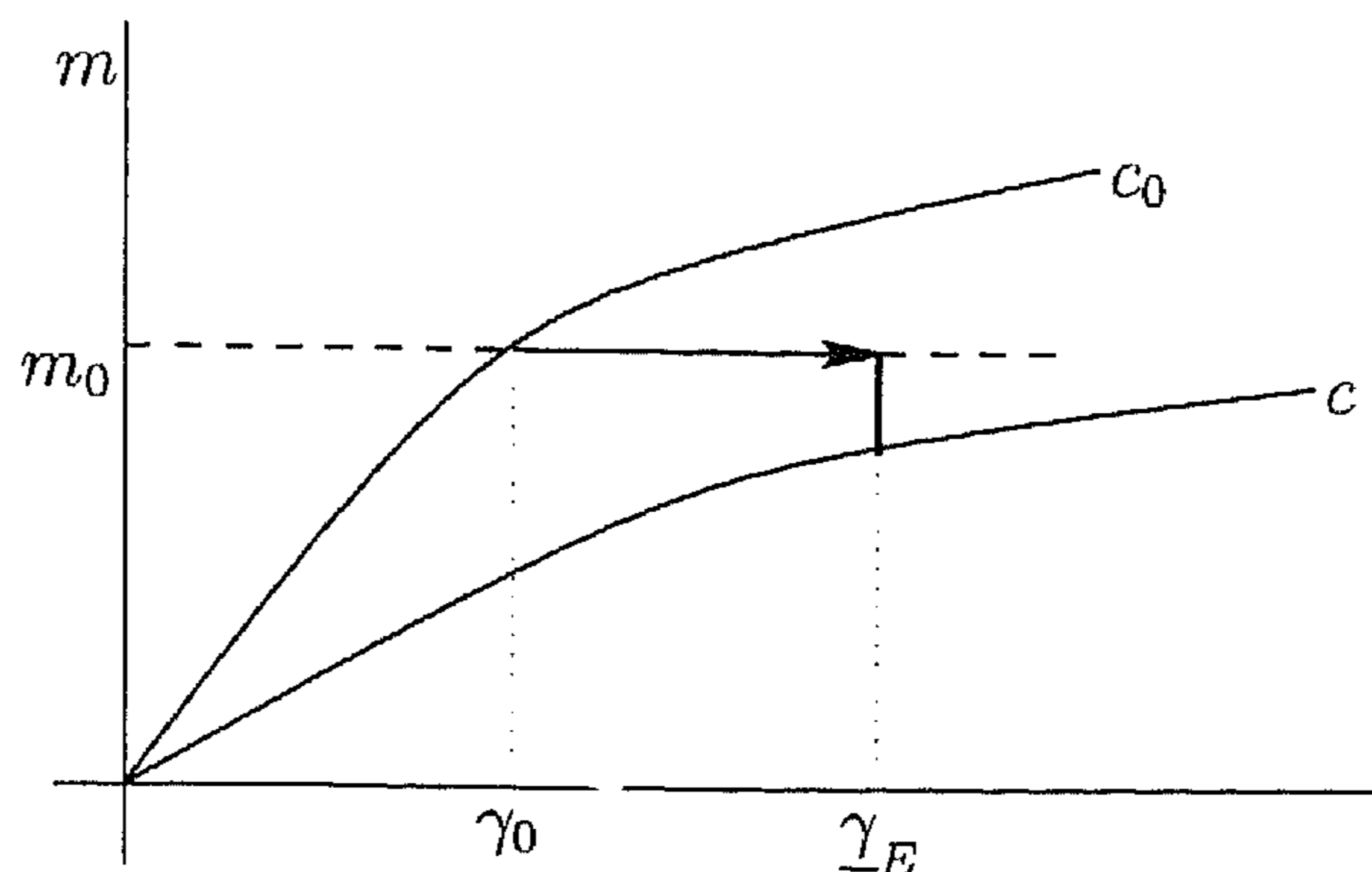


FIGURE 3. Mass vs momentum; the two curves for different value of  $c$  indicate that during run up of a single solitary wave that conserves energy, there is mass surplus.

will result only two waves will put some conditions on the topography; in general a breaking up into more than two waves can be expected.)

More precisely, we make the following two simplifying assumptions (which seem to be somewhat contradictory at first sight):

- both constituent waves are taken as if they are at each instant above the same bottom depth,
- the energy and mass of the combined waves are the sum of these quantities of the two constituent waves.

The last assumption is a consequence of the finite extension of the waves when the constituent waves are sufficiently separated. In that case the energy and mass, which are integrals over local densities, are additive.

Closely related to the additivity argument is the motivation for the choice of the parameter  $\gamma$  that, for a single wave, should be related to its amplitude. However, a quantity like amplitude is not well defined for an arbitrary wave profile, and it cannot be expected to be additive. A parameter that is globally defined, and is additive when waves separate, is preferred. This motivates to use as parameter  $\gamma$  some integral of a local density. As will become clear in the specific model, a natural choice for  $\gamma$  is the value of the horizontal momentum, proportional to the  $L_2$ -norm but depending on the topography. Motivating the nomenclature, above a horizontal bottom this quantity is conserved as a consequence of translation invariance.

The first assumption and the choice of  $\gamma$  together imply that the phenomenon of splitting can symbolically be written as

$$S = S_1(c, \gamma_+) + S_1(c, \gamma_-)$$

where  $\gamma_-, \gamma_+$  are parameters that characterize the two waves. Note that we do not aim to specify the precise positions of the constituent waves. This will need an expression for the velocity of the wave, which will be found in the model to be presented in Section 3.

The second assumption provides the equations that should determine the values  $\gamma_-$  and  $\gamma_+$  as a function of  $c$ . From energy and mass conservation there result the

conservation conditions:

$$\begin{cases} \mathcal{E}(c, \gamma_+) + \mathcal{E}(c, \gamma_-) & = e_0 \equiv \mathcal{E}(c_0, \gamma_0) \\ \mathcal{M}(c, \gamma_+) + \mathcal{M}(c, \gamma_-) & = m_0 \equiv \mathcal{M}(c_0, \gamma_0) \end{cases}$$

where  $c_0, \gamma_0$  are the values of the initial single wave.

Given the assumptions made above, the whole description will be based on an analysis of these two equations. *The physical properties of the water wave problem are reflected in qualitative information about the functions  $\mathcal{E}(c, \gamma)$  and  $\mathcal{M}(c, \gamma)$ .*

We state the properties of these functions as found for the model problem in the next section; these properties are also verified numerically for the complete set of equations.

In investigating the two conditions above, it is simplest to take  $\gamma = \gamma_- + \gamma_+, \xi = \gamma_-$  as parameters; the conservation conditions then require to find  $\gamma, \xi$  from

$$\begin{cases} \mathcal{E}(c, \gamma - \xi) + \mathcal{E}(c, \xi) & = e_0, \\ \mathcal{M}(c, \gamma - \xi) + \mathcal{M}(c, \xi) & = m_0. \end{cases}$$

We first investigate each condition separately and then combine the results.

#### Energy-conservation.

Based on the model and numerical calculations we will use the fact that *the function  $\mathcal{E}(c, \gamma)$  is (strictly) convex in  $\gamma$ , and, in the relevant parameter interval, an increasing function of  $c$ .*

As a consequence, there exists an interval,  $[\underline{\gamma}_E, \bar{\gamma}_E]$  and a monotonically increasing function  $\xi_E$  such that energy conservation is satisfied. In fact,

$$\xi_E(\underline{\gamma}_E) = 0, \quad \xi_E(\bar{\gamma}_E) = \frac{1}{2}\bar{\gamma}_E,$$

where  $\underline{\gamma}_E, \bar{\gamma}_E$  are defined by

$$\mathcal{E}(c, \underline{\gamma}_E) = e_0, \quad \mathcal{E}(c, \frac{1}{2}\bar{\gamma}_E) = \frac{1}{2}e_0$$

These results are obvious by noting that the function

$$\xi \mapsto \mathcal{E}(c, \gamma - \xi) + \mathcal{E}(c, \xi)$$

is (strictly) convex and symmetric around  $\frac{1}{2}\gamma$  at which it attains its least value  $2\mathcal{E}(c, \frac{1}{2}\gamma)$  (see Fig 4).

#### Mass conservation.

Based on the model and numerical calculations we will use the fact that *the function  $\mathcal{M}(c, \gamma)$  is strictly concave in  $\gamma$  and increasing in  $c$ .*

From this we find that

$$\xi \mapsto \mathcal{M}(c, \gamma - \xi) + \mathcal{M}(c, \xi)$$

is strictly concave. Hence there exists an interval  $[\underline{\gamma}_M, \bar{\gamma}_M]$  and a monotonically decreasing function  $\xi_M$  such that mass conservation is satisfied. The values are given by

$$\mathcal{M}(c, \frac{1}{2}\underline{\gamma}_M) = \frac{1}{2}m_0, \quad \mathcal{M}(c, \bar{\gamma}_M) = m_0,$$

and the function  $\xi_m$  satisfies

$$\xi_M(\underline{\gamma}_M) = \frac{1}{2}\underline{\gamma}_M \quad \text{and} \quad \xi_M(\bar{\gamma}_M) = 0;$$

see Fig. 5.

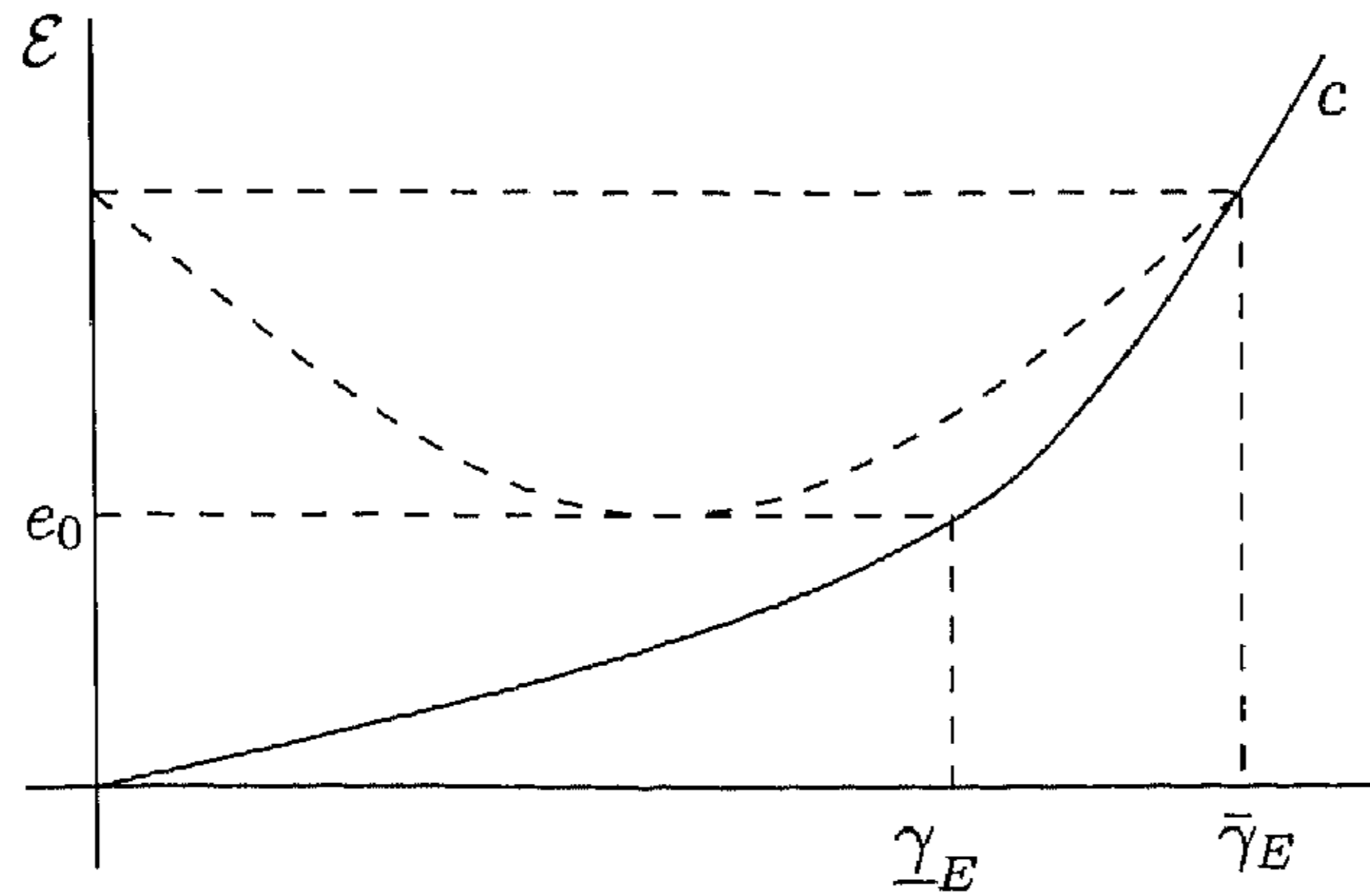


FIGURE 4. Definition of the limiting values of  $\gamma$  for which there exists a superposition of two constituent waves of total energy  $e_0$ .

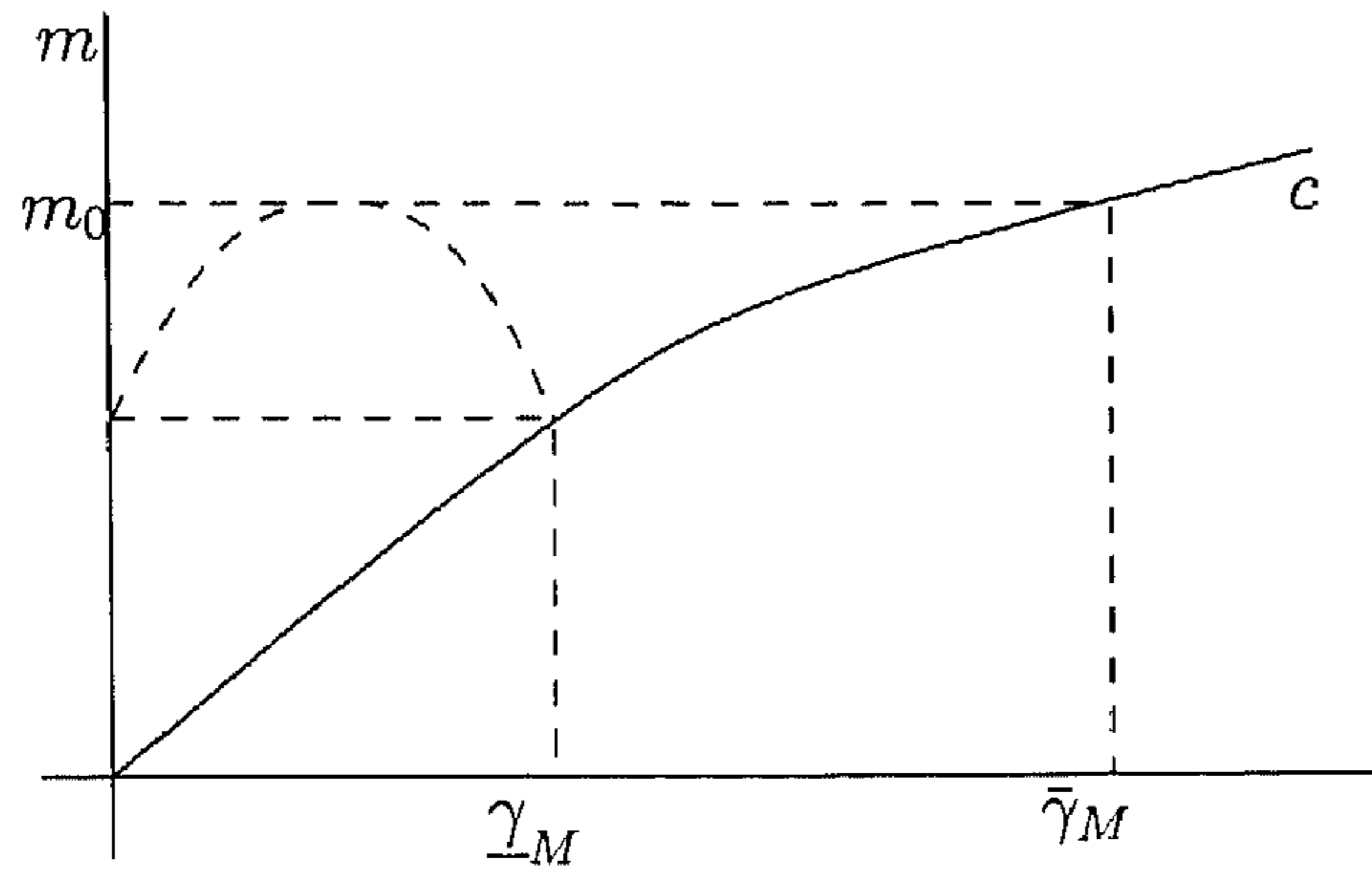


FIGURE 5. Definition of the limiting values of  $\gamma$  for which there exists a superposition of two constituent waves of total mass  $m_0$ .

**Combined energy and mass conservation**

Taken together, the two resulting curves  $\xi_E$ , and  $\xi_M$  found above (depicted schematically in Fig. 6) imply that a unique common value  $\hat{\xi}$  exists, which determines the values  $\gamma_-$  and  $\gamma_+$  of the constituent waves with total mass and energy as desired, if the following conditions are satisfied

$$\underline{\gamma}_M < \bar{\gamma}_E < \bar{\gamma}_M.$$

For the model to be presented, the last inequality is always satisfied. The first one leads to a lower bound for the depth and corresponding value of  $c$ . When an approximation for the energy is taken ( $\varepsilon = 0$  in the expression to be given in the next section) there results as condition approximately  $h/h_0 > .54$ .

These results indicate that in this model, no matter how little the depth decrease, a second wave starts to split off from the initial single wave. From the description here, this process continues until the difference in depth exceeds a certain value,  $h/h_0 \approx .54$ . In approaching this critical value, the two constituent waves would become almost equal ( $\gamma_- \approx \gamma_+ \approx \frac{1}{2}\bar{\gamma}_E$ ) and a symmetric situation would result.

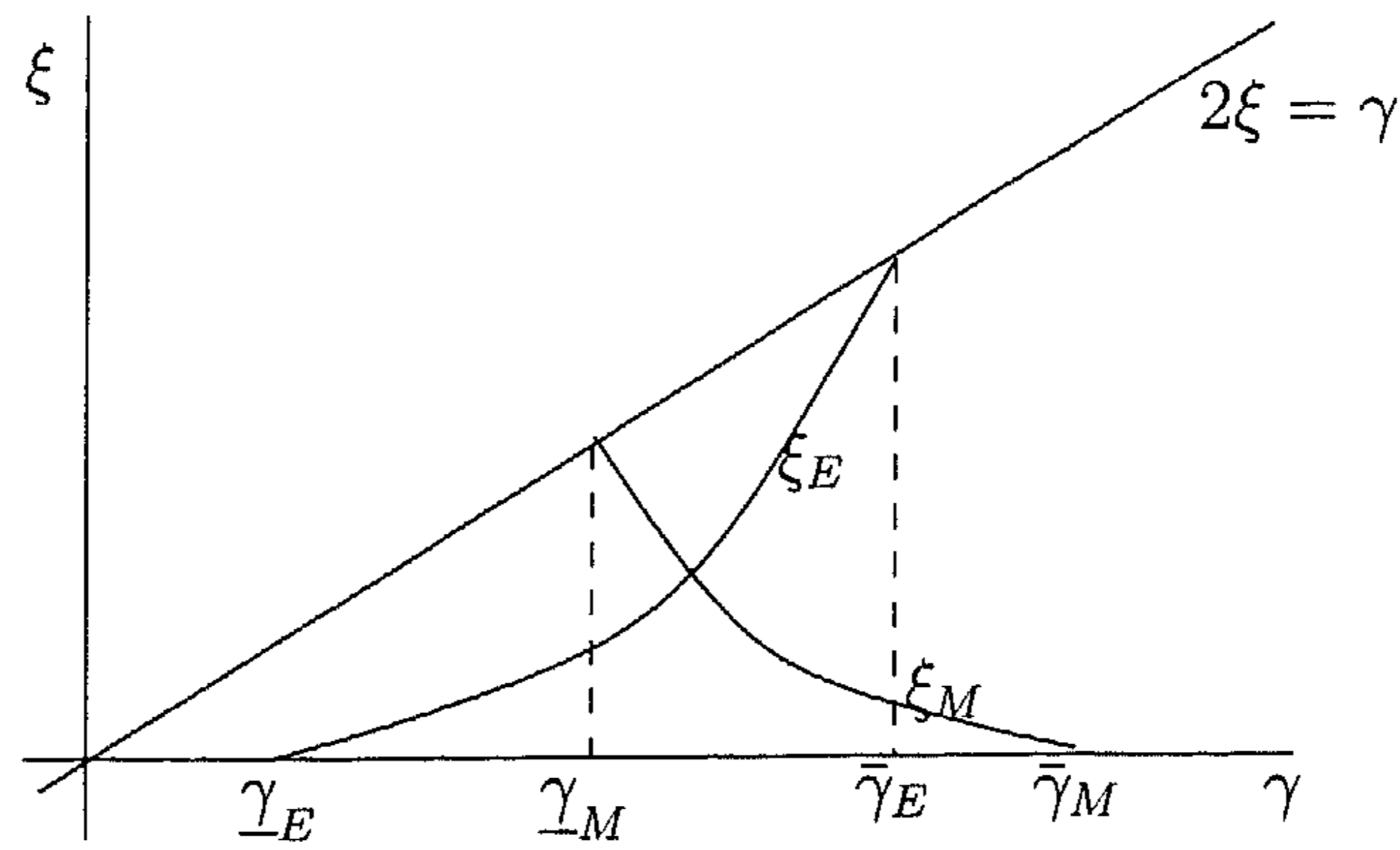


FIGURE 6.  $\xi$  vs  $\gamma$  curves for energy and mass conservation.

The geometric picture is depicted in Figs. 7 and 8. Observe that, while  $\gamma_-$  increases monotonically, the change in  $\gamma_+$  is not monotone: after an initial increase, the value decreases with decreasing depth before the symmetric situation is attained. This is not realistic from a modelling point of view; in another paper ([14]) this is investigated in more detail. If we take the approximation to be valid until  $\gamma_+$  attains its maximum value, this leads to another bound for the maximal depth ratio, approximately  $h/h_0 > .7$ . Note that this last critical value is somewhat larger than the result described in the introduction obtained with the model of a changing single-hump with subsequent fission by inverse scattering theory.

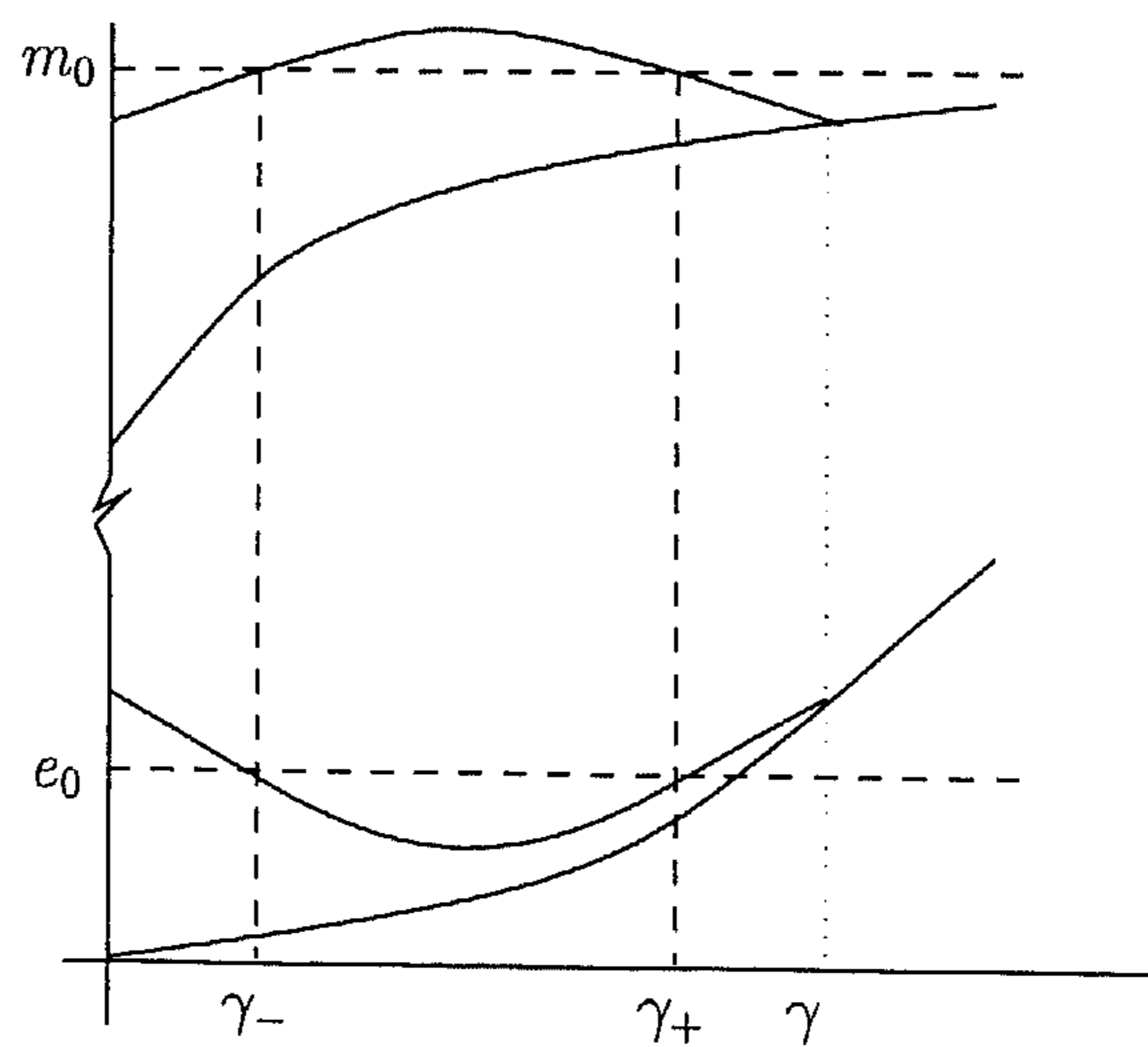


FIGURE 7. Combined conservation of mass and energy.

### 3. KdV-top model

The properties of the determining functions for energy and mass as stated above will now be obtained for a model equation that describes uni-directional



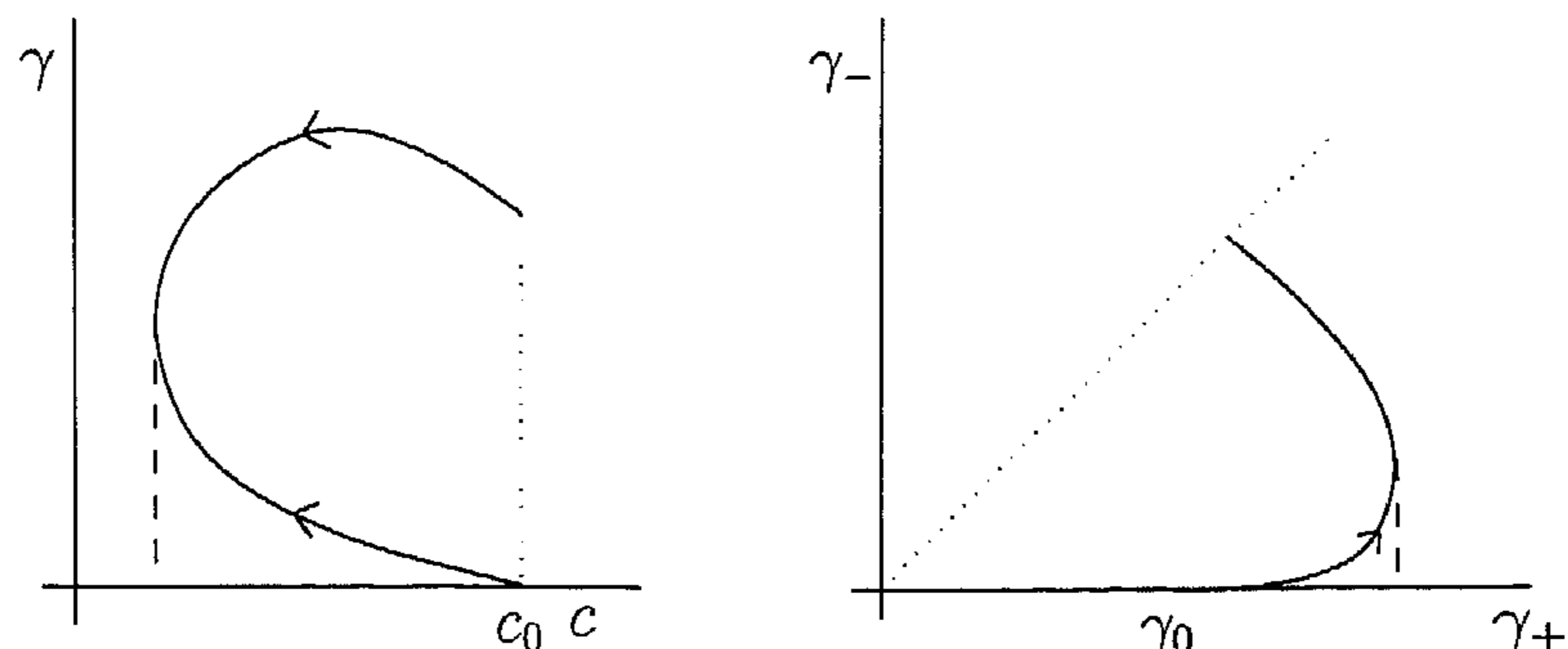


FIGURE 8. Values of  $\gamma_+$  and  $\gamma_-$ , at the left as function of  $c$  and at the right parameterized by  $c$ , during evolution for which energy and mass are conserved. Note that during run up (decreasing  $c$ ), the value of  $\gamma_+$  attains a maximal value before the symmetric situation (the turning point where  $\gamma_- = \gamma_+$ ) is reached.

waves above a slowly varying bottom. The equation is given here in the form as derived in [5]; it is equivalent to an equation derived by Newell [12].

The equation for the wave elevation  $\eta$  has been derived under the usual assumptions of Boussinesq properties of the wave shapes (long, low waves) and small bottom variations so that reflections can be neglected; a small parameter  $\varepsilon$  measures the Boussinesq and bottom effects. The equation is easiest described (and reflects the derivation) in the form of a Hamiltonian system like

$$\partial_t \eta = -\Gamma \delta H(\eta)$$

where the structure map is given by  $\Gamma = \sqrt{c(x)} \partial_x \sqrt{c(x)}$ , and the Hamiltonian, which is an approximation for the total energy, by

$$H(\eta) = \int \left[ \frac{1}{2} \eta^2 + \varepsilon \left( -\frac{1}{12} h^2 \eta_x^2 + \frac{1}{4h} \eta^3 \right) \right].$$

This equation is called *KdV-top*, since it reduces for waves above constant depth to the familiar Korteweg-de Vries (KdV)-equation but incorporates the effect of topography.

The skew symmetry of the operator  $\Gamma$  implies the Hamiltonian structure of the equation. As one consequence, energy is conserved during the evolution of any solution:  $\partial_t H(\eta) = 0$ . Also, since  $\Gamma$  is degenerate,  $\Gamma 1 / \sqrt{c(x)} = 0$ , there is a Casimir functional that is the generalization of the total mass functional above a flat bottom:

$$C(\eta) = \int \frac{\eta}{\sqrt{c(x)}}$$

The generalization of the horizontal momentum is given by

$$I(\eta) = \frac{1}{2} \int \frac{\eta^2}{c(x)};$$

above a constant bottom this functional is conserved, which property is lost when translation symmetry is destroyed by bottom variations.

These properties of this model equation resemble the corresponding properties for the full wave equations. Therefore we based the previous analysis on the energy and mass conservation, while the horizontal momentum (not constant) was used merely

as a convenient parameter, which has a clear physical interpretation nonetheless. The specific form of the operator  $\Gamma$  is interesting in itself. Even excluding Boussinesq effects, the linear equation

$$\partial_t \eta = -\Gamma \eta$$

describes already the main properties of a wave (of arbitrary form) running up: decrease of wave length and speed, and increase of amplitude; this follows from the exact solution that is easily written down; see [5].

Using the KdV-top model, Van Beckum [1] calculated the running up of a single solitary wave using consistent Fourier truncation, see Fig 9, and obtained the same qualitative result for the splitting as in Fig. 1.

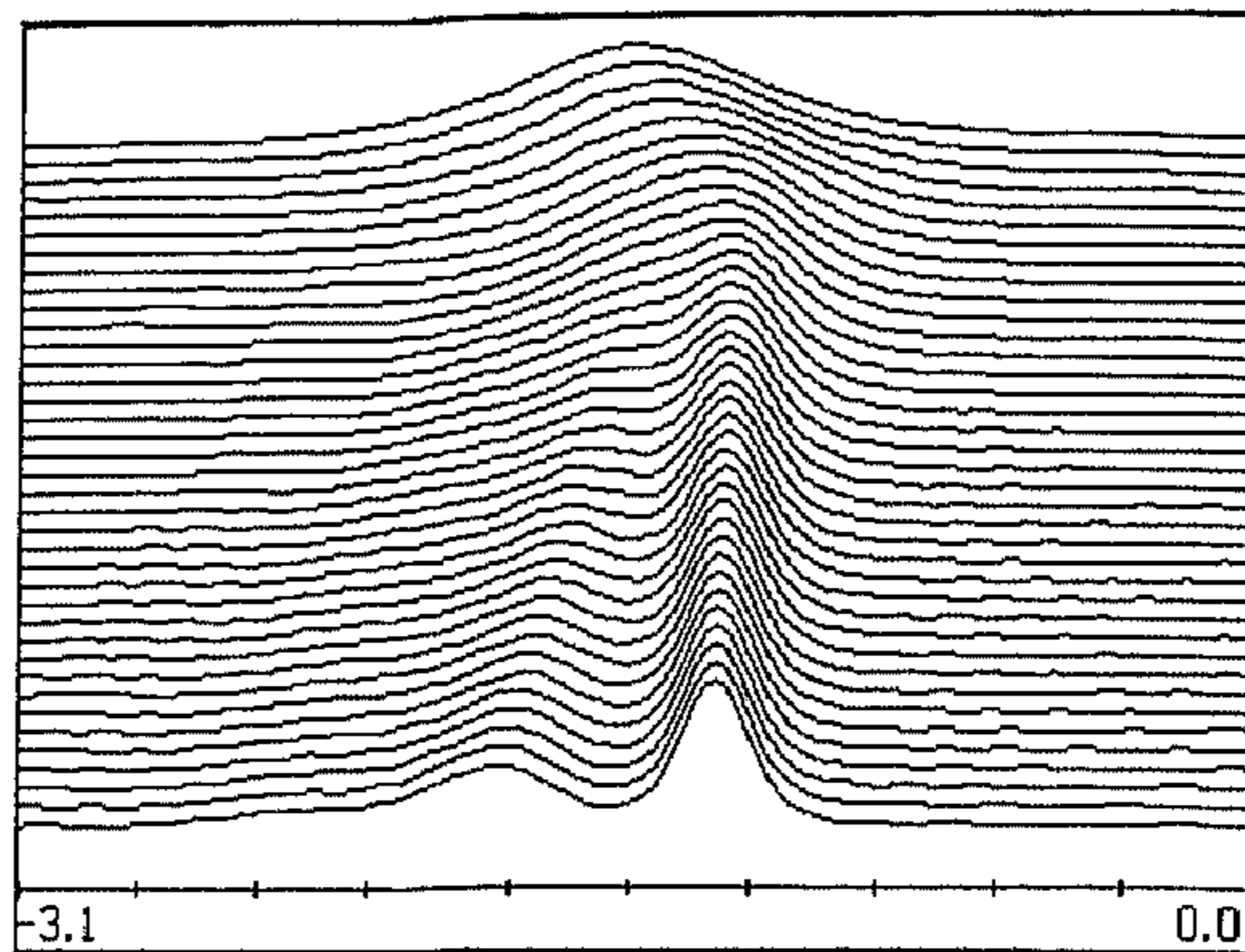


FIGURE 9. Soliton splitting as calculated with the KdV-top equation. For clarity, the process is depicted in a frame moving with the waves, and profiles are translated vertically downwards for increasing time.

As is well known, the KdV-equation (above flat bottom) admits solitary waves. These waves can actually be described as constrained minimizers of the energy at prescribed momentum. Denoting the energy, mass and momentum above a flat bottom by  $H(c, \eta)$ ,  $C(c, \eta)$ ,  $I(c, \eta)$  respectively, the profile  $S_1$  of a solitary wave is characterized for each  $\gamma > 0$  as a solution of

$$S_1(c, \gamma) \in \text{Min}\{H(c, \eta) \mid I(c, \eta) = \gamma\}.$$

The equation is found from Lagrange's multiplier rule  $\delta H = \lambda \delta I$ . The explicit expression for the solitary wave profile is given by

$$S_1(c, \gamma)(x) = A \operatorname{sech}(B(x - \varphi))$$

where the dependence on  $\gamma$  and  $c$  is explicitly given by

$$A = 3gc^{-4/3} \left(\frac{\gamma}{4}\right)^{2/3}, \quad B = \frac{3}{2}g^2 c^{-11/3} \left(\frac{\gamma}{4}\right)^{1/3}.$$

Observe that both coefficients depend monotonically on  $\gamma$  and on  $c$ . The dynamic solution is a translation of this profile with speed  $\lambda$ . The expression for the speed

$$\lambda(c, \gamma) = c + \frac{3}{2}\varepsilon g^2 c^{-7/3} \left(\frac{\gamma}{4}\right)^{2/3}$$



shows that it increases monotonically with respect to  $\gamma$ . The variable  $\varphi$  is an arbitrary phase factor that determines the position of (the crest of) the wave. In the quasi-homogeneous description presented above,  $\varphi$  is adapted such that the crest of the wave is above the point at the bottom where the depth is related to  $c$ , i.e.  $\sqrt{gh(\varphi)} = c$ .

From these expressions for the solitary waves, the functions for energy and mass can be found explicitly. They are given by

$$\mathcal{E}(c, \gamma) = c\gamma + ac^{-7/3}\gamma^{5/3}, \quad \mathcal{M}(c, \gamma) = bc^{11/2}\gamma^{1/3}$$

with  $a = 18\varepsilon g^2/4^{5/3} \approx 35\varepsilon$ ,  $b = 4^{2/3}/9 \approx .28$ .

As stated before, these functions have the qualitative properties of the function  $\mathcal{E}$  and  $\mathcal{M}$  in the previous section and were used to produce the graphs.

#### 4. Elements of the mathematical justification

At first sight it is rather remarkable that a simple description as presented in Section 2 can describe the actual process correctly. In fact it means that a two-dimensional description suffices to capture the most relevant properties.

However, there is a clear mathematical reason, that can only be described rudimentary here; for a more extensive treatment, see Fledderus & Van Groesen, [4].

The main problem is that although the bottom is weakly sloping, the actual deformation from the initial state is of order one since the decrease in depth is of order one. However, the use of the specific functions  $S_1$  in the quasi-homogeneous description guarantees that at each instant the approximation is but a slight deformation from an exact solution: the wave forms used in the description are exact solutions of the corresponding homogenized system. Moreover, as becomes obvious from the constrained minimizing property stated for these waves, these forms are stable in a suitable sense (taking into account the constraint and the degeneration from translation symmetry). Having this property, the next important fact is the correct choice of the parameter dynamics: the projection of the true solution into the two-dimensional manifold with parameters  $c, \gamma$ . Here the dynamics is simply based on the conservation properties of mass and energy. From a more convincing mathematical point of view, the relevant observation is that it can be shown that these conditions are equivalent to Fredholm necessary solvability conditions that should be satisfied in order that the error is uniformly small during the evolution where changes of the order one appear. This first order condition, combined with the stability property of the waves, suffice to prove the uniform validity of the approximation as constructed for the given KdV-top equation.

The relation with the complete set of surface wave equations should then be justified by referring to the construction of KdV-top as a consistent model for specific wave evolutions. In this process some quantitative differences can be expected since this can already be shown to be the case for soliton solutions from Boussinesq equations compared to solitons from the KdV equation; see also [1].

#### 5. Conclusions

Although the simple description predicts the splitting process quite well qualitatively, the underlying assumptions should be refined. In particular, the assumption that the two separated waves are taken at the same depth should be improved. In another paper, [14], this will be described in more detail, but the results merely

modify the analysis as presented here in an inessential way. Another point for further investigation is the critical value of the depth ratio at which the two-wave approximation breaks down. If it is conjectured that the critical depth is where the largest wave reaches its maximal amplitude, a subsequent splitting of this largest wave, and consequently also of the smaller one, should be expected. Unfortunately, reliable numerical calculations, or experiments, that can show this pairwise splitting are not available yet.

### References

- [1] F.P.H. van Beckum. "Hamiltonian consistent discretisation of wave equations". Thesis, University of Twente, 1995.
- [2] B. Beji and J.A. Battjes. "Numerical simulation of nonlinear wave propagation over a bar". *Coastal Eng.*, 23:1-16, 1994.
- [3] E.F.G. Daalen. "Solitary wave split-up due to a sloping bottom: Theory, numerics and experiments". *Comp. Math.*, 1995.
- [4] E.R. Fledderus and E. van Groesen. "Deformation of coherent structures". *Rep. Progr. Phys.*, 59:1-89, 1996.
- [5] E. van Groesen and S.R. Pudjaprasetya. "Uni-directional waves over slowly varying bottom. Part I: Derivation of a KdV-type of equation". *Wave Motion*, 18:345-370, 1993.
- [6] R.S. Johnson. "Some numerical solutions of a variable coefficient Korteweg - de Vries equation". *J. Fluid Mech.*, 54:81-91, 1972.
- [7] R.S. Johnson. "On the development of a solitary wave moving over an uneven bottom". *Proc. Camb. Phil. Soc.*, 73:183-203, 1973.
- [8] D.J. Korteweg and G. de Vries. "On the change of form of long waves advancing in a rectangular canal, and on a new type of long stationary waves". *Phil. Mag.*, 39:422-443, 1895.
- [9] O.S. Madsen and C.C. Mei. "The transformation of a solitary wave over an uneven bottom". *J. Fluid Mech.*, 39:781-791, 1969.
- [10] J.W. Miles. "On the Korteweg - de Vries equation for a gradually varying channel". *J. Fluid Mech.*, 91:181-190, 1979.
- [11] J.W. Miles. "Solitary waves". *Annual Rev. Fluid Mech.*, 12:11-44, 1980.
- [12] A.C. Newell. *Solitons in mathematics and physics*. SIAM, Philadelphia, 1985.
- [13] H. Ono. "Wave propagation in an inhomogeneous anharmonic lattice". *J. Phys. Soc. Jap.*, 32:332-336, 1972.
- [14] S.R. Pudjaprasetya and E. van Groesen. "Model for the splitting of a solitary wave running into shallower water,". *in preparation*, 1995.
- [15] S.R. Pudjaprasetya and E. van Groesen. "Uni-directional waves over slowly varying bottom, part II Deformation of travelling waves". *Wave Motion*, 1995.
- [16] F.D. Tappert and N.J. Zabusky. "Gradient-induced fission of solitons". *Phys. Rev. Lett.*, 27:1774-1776, 1971.

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