

Versal Deformation of the Lie Algebra L_2

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We investigate deformations of the infinite-dimensional vector-field Lie algebra spanned by the fields $e_i = z^{i+1}d/dz$, where $i \geq 2$. The goal is to describe the base of a “versal” deformation; such a versal deformation induces all the other nonequivalent deformations and solves the deformation problem completely.

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INTRODUCTION

In the last decade the interest in deformation theory has grown in many areas of mathematics and physics. The deformation question is completely solved by describing a “versal” deformation of the given object; such a deformation induces all the other deformations. This problem turns out to be hard and a general procedure for solving this was given only recently in [FF2] for Lie algebras. It is still not trivial to apply this construction to specific examples.

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In this paper we apply that procedure to the following important example. Let W be the infinite-dimensional Lie algebra of polynomial vector fields in \mathbb{C} spanned by the fields $e_i = z^{i+1}d/dz$, $i \in \mathbb{Z}$. This Lie algebra is called the Witt algebra. The commutator in W is defined by the formula

$$[e_i, e_j] = (j - i)e_{i+j}.$$

Consider the subalgebras L_k of W spanned by the vector fields e_i , $i = k, k + 1, \dots$, where $k \geq 1$. The infinite-dimensional Lie algebras L_k turn out to be essential in mathematical physics and one wants to solve the deformation problem completely for those.

In [FF1] the Lie algebra L_1 was studied. At that time there was no general procedure for computing versal deformation and a special method was used which does not seem to generalize. In fact, that special method helped the authors to understand the general procedure. In [FF2] a general computational method for a versal deformation is described and it is applied to the Lie algebra L_1 . The next step after L_1 is the Lie algebra L_2 spanned by the fields e_i , $i = 2, 3, \dots$. Infinitesimal deformations and the Massey 2-products for L_2 were computed in [KP]. There is some confusion in the literature when one describes deformations, however. While infinitesimal ones and Massey products are important to compute, these computations are only the beginning. As there was no general procedure for constructing a versal deformation, no further developments for the Lie algebra L_2 have appeared so far. In particular, the following questions remained open:

- How many nonequivalent deformations have the same infinitesimal part?
- Are there any singular nontrivial deformations, i.e., deformations with zero infinitesimal part?

In this paper we give an explicit construction for the base of a versal deformation of the Lie algebra L_2 . In Section 1 we compute the cohomology spaces $H^2(L_2; L_2)$ and $H^3(L_2; L_2)$ that are necessary for the procedure. In Section 2 we summarize the notion of deformations and give the definition of a versal deformation. In Section 3 we introduce obstructions to extending deformations with the help of Harrison cohomology. In Section 4 we give examples of one-parameter deformations, both singular and nonsingular. Some of these are examples for any of the Lie algebras L_k . Section 5 describes the construction of the base of a versal deformation for the Lie algebra L_2 .

Let us mention that computationally the case L_2 is much harder than for L_1 . The general case L_k is yet more difficult. So far we only have some examples of deformations in those cases (see Section 4).

1. COHOMOLOGY COMPUTATIONS

Our Lie algebra L_2 is \mathbb{Z} -graded beginning with degree 2, $\deg e_i = i$. This gives rise to a grading in the cochain complex $C^*(L_2; L_2)$ and in $H^*(L_2; L_2)$: $\deg \alpha = k$ for $\alpha \in C^q(L_2; L_2)$ if $\alpha(e_{i_1}, \dots, e_{i_q})$ is in $\mathbb{C}e_{i_1 + \dots + i_q - k}$ for all i_1, \dots, i_q . In this case we write $\alpha \in C_k^q(L_2; L_2)$.

The coboundary operation $\partial: C^q \rightarrow C^{q+1}$ is

$$\begin{aligned} \partial_\varphi(x_1, \dots, x_{q+1}) &:= \sum_{1 \leq s < t \leq q+1} (-1)^{s+t-1} \varphi([x_s, x_t], x_1, \dots, \hat{x}_s, \hat{x}_t, \dots, x_{q+1}) \\ &+ \sum_{1 \leq s \leq q+1} (-1)^s [x_s, \varphi(x_1, \dots, \hat{x}_s, \dots, x_{q+1})], \end{aligned}$$

where \hat{x}_s means that the element \hat{x}_s is omitted.

For the cohomology space we have correspondingly

$$H^q(L_2; L_2) = \bigoplus_k H_k^q(L_2; L_2).$$

Here are the results for the cohomology spaces $H^2(L_2; L_2)$ and $H^3(L_2; L_2)$. The following theorem was stated without proof in [KP].

THEOREM 1.1.

$$H_k^2(L_2; L_2) \equiv \begin{cases} \mathbb{C} & \text{if } k = 2, 6, \\ \mathbb{C}^2 & \text{if } k = 3, 4, 5, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We introduce 1-cochains of the right degree in $C^1(L_2; W)$, rather than in $C^1(L_2; L_2)$. These are

$$\mu_k^{(s)}(e_i) = \delta_{s,i} e_{i-k}.$$

We calculate the differentials of these cochains and find all linear combinations such that the image is in L_2 . This gives

$$\begin{aligned} c_2 &= \partial\mu_2^{(3)} \\ c_{3,1} &= \partial\mu_3^{(3)}, & c_{3,2} &= \partial\mu_3^{(4)}, \\ c_{4,1} &= \partial\mu_4^{(3)} - 3\partial\mu_4^{(5)}, & c_{4,2} &= \partial\mu_4^{(4)}, \\ c_{5,1} &= 2\partial\mu_5^{(4)} - 3\partial\mu_5^{(6)}, & c_{5,2} &= \partial\mu_5^{(3)} - 4\partial\mu_5^{(5)}, \end{aligned}$$

and

$$c_6 = 5\partial\mu_6^{(3)} + 2\partial\mu_6^{(4)} - 25\partial\mu_6^{(5)} - 4\partial\mu_6^{(6)} + 25\partial\mu_6^{(7)}.$$

Clearly, for these cochains $c_{k,i}$ we have $\partial c_{k,i} = 0$. Moreover, one can check by finite calculations that the cocycles $c_{k,i}$ are independent in $H_k^2(L_2; L_2)$. In fact, we already removed a dependency due to

$$\partial\left(\sum_{s \geq 2} (s+k)\mu_k^{(s)}\right) = \partial^2(1 \otimes e_{-k}) = 0.$$

Using the Feigin–Fuchs spectral sequence (see [FeFu]) we can prove that these $c_{k,i}$ constitute all of $H_k^2(L_2; L_2)$.

THEOREM 1.2.

$$H_k^3(L_2; L_2) \equiv \begin{cases} \mathbb{C} & \text{if } k = 8, 15, \\ \mathbb{C}^2 & \text{if } k = 9, 14, \\ \mathbb{C}^3 & \text{if } k = 10, 11, 12, 13, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let us introduce for $s < r$ the cochains $\mu_k^{(s,r)} \in C^2(L_2; W)$ by

$$\mu_k^{(s,r)}(e_i, e_j) = \delta_{s,i}\delta_{r,j}e_{i+j-k}.$$

The representative cocycles for $H^3(L_2; L_2)$ are given in the appendix. The proof is similar to that of Theorem 1.1, although much harder in computational complexity.

2. DEFORMATIONS

We recall the basic definitions on Lie algebra deformations. Let L be a Lie algebra over a characteristic 0 field \mathbb{K} , and let A be a commutative algebra with identity over \mathbb{K} with a fixed augmentation $\varepsilon: A \rightarrow \mathbb{K}$, $\varepsilon(1) = 1$; we set $\text{Ker } \varepsilon = \mathfrak{m}$. To avoid transfinite induction, we assume that $\dim(\mathfrak{m}^k/\mathfrak{m}^{k+1}) < \infty$ for all k .

DEFINITION 2.1. A *deformation* λ of L with base (A, \mathfrak{m}) , or simply with base A , is a Lie A -algebra structure on the tensor product $A \otimes_{\mathbb{K}} L$ with bracket $[\ ,]_\lambda$, such that

$$\varepsilon \otimes \text{id}: A \otimes L \rightarrow \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism (see [Fi1, Fi2]). Specifically, it means that for all $a, b \in A$ and $x, y \in L$,

1. $[a \otimes x, b \otimes y]_\lambda = (ab \otimes \text{id})[1 \otimes x, 1 \otimes y]_\lambda$ (A -linearity),
2. $[\cdot, \cdot]$ is skew-symmetric and satisfies Jacobi's identity,
3. $\varepsilon \otimes \text{id}([1 \otimes x, 1 \otimes y]_\lambda) = 1 \otimes [x, y]$.

Properties 1 and 2 say that $A \otimes L$ is a Lie A -algebra. Property 3 says that this Lie algebra should be compatible with the augmentation ε .

From Property 1 above it is clear that to describe a deformation it is enough to fix the elements $[1 \otimes x, 1 \otimes y]_\lambda$ for all $x, y \in L$. According to Property 3, this Lie product has the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_i a_i \otimes z_i \tag{2.1}$$

where $a_i \in A$ and $z_i \in L$.

DEFINITION 2.2. Two deformations λ_1 and λ_2 of a Lie algebra L with the same base A are called *equivalent* if there exists a Lie algebra isomorphism between the two copies of $A \otimes L$ with the two Lie algebra structures, compatible with $\varepsilon \otimes \text{id}$.

A deformation with base A is called *local* if the algebra A is local, and it is called *infinitesimal* if, in addition to this, $\mathfrak{m}^2 = 0$.

DEFINITION 2.3. Let A be a complete local algebra (completeness means that $A = \varprojlim_{n \rightarrow \infty} (A/\mathfrak{m}^n)$, where \mathfrak{m} is the maximal ideal in A). A *formal deformation* of L with base A is a Lie A -algebra structure on the completed tensor product $A \hat{\otimes} L = \varprojlim_{n \rightarrow \infty} ((A/\mathfrak{m}^n) \otimes L)$ such that

$$\varepsilon \hat{\otimes} \text{id}: A \hat{\otimes} L \rightarrow \mathbb{K} \otimes L = L$$

is a Lie algebra homomorphism (see [Fi2]).

The above notion of equivalence is extended to formal deformations in an obvious way.

Let A' be another commutative algebra with identity over \mathbb{K} with a fixed augmentation $\varepsilon': A' \rightarrow \mathbb{K}$, and let $\varphi: A \rightarrow A'$ be an algebra homomorphism with $\varphi(1) = 1$ and $\varepsilon' \circ \varphi = \varepsilon$.

DEFINITION 2.4. If a deformation λ of L with base (A, \mathfrak{m}) is given, then the *push-out* $\mu = \varphi_* \lambda$ of λ by φ is the deformation μ of L with base $(A', \mathfrak{m}' = \text{Ker } \varepsilon')$, which is the Lie algebra structure

$$\begin{aligned} [a'_1 \otimes_A (a_1 \otimes l_1), a'_2 \otimes_A (a_2 \otimes l_2)]_\mu &= a'_1 a'_2 \otimes_A [a_1 \otimes l_1, a_2 \otimes l_2]_\lambda, \\ a'_1, a'_2 \in A', a_1, a_2 \in A, l_1, l_2 \in L, \end{aligned}$$

on $A' \otimes L = (A' \otimes_A A) \otimes L = A' \otimes_A (A \otimes L)$. Here A' is regarded as an A -module with the structure $a'a = a'\varphi(a)$.

Suppose λ is given by the formula (2.1). Then

$$[1 \otimes x, 1 \otimes y]_\mu = 1 \otimes [x, y] + \sum_i \varphi(a_i) \otimes z_i.$$

The push-out of a formal deformation is defined in a similar way.

The Universal Infinitesimal Deformation. Assume that the Lie algebra L satisfies the condition

$$\dim H^2(L; L) < \infty.$$

Denote the space $H^2(L; L)$ by \mathbb{H} .

Consider the algebra

$$A = \mathbb{K} \oplus \mathbb{H}'$$

(' means *dual*) by setting

$$(\alpha_1, h_1) \cdot (\alpha_2, h_2) = (\alpha_1 \alpha_2, \alpha_1 h_2 + \alpha_2 h_1).$$

We are particularly interested in a special deformation μ_L of L with this base A . Fix some homomorphism

$$\mu: \mathbb{H} \rightarrow C^2(L; L) = \text{Hom}(\Lambda^2 L, L)$$

which takes a cohomology class into a cocycle representing this class. Define a Lie algebra structure on

$$A \otimes L = (\mathbb{K} \otimes L) \oplus (\mathbb{H}' \otimes L) = L \oplus \text{Hom}(\mathbb{H}, L)$$

by the formula

$$[(l_1, \varphi_1), (l_2, \varphi_2)] = ([l_1, l_2], \psi),$$

where

$$\begin{aligned} \psi(\alpha) &= \mu(\alpha)(l_1, l_2) + [\varphi_1(\alpha), l_2] + [l_1, \varphi_2(\alpha)], \\ l_1, l_2 &\in L, \varphi_1, \varphi_2 \in \text{Hom}(\mathbb{H}, L), \alpha \in \mathbb{H}. \end{aligned}$$

(The Jacobi identity for this operation is implied by $\partial\mu(\alpha) = 0$.) This determines a deformation of L with base A which is clearly infinitesimal.

Let c_1, \dots, c_m be a basis of \mathbb{H} and t_1, \dots, t_m the dual basis in \mathbb{H}' . Define the deformation η_L of L with base A given by the formula

$$[1 \otimes x, 1 \otimes y]_{\eta_L} = 1 \otimes [x, y] + \sum_{i=1}^m t_i \otimes c_i(x, y).$$

It is easy to see that this deformation does not depend on the choice of the basis c_1, \dots, c_m .

The main property of η_L is its universality in the class of infinitesimal deformations (see [FF2]).

PROPOSITION 2.5. *For any infinitesimal deformation λ of the Lie algebra L with a finite-dimensional base A there exists a unique homomorphism $\varphi: \mathbb{K} \oplus \mathbb{H}' \rightarrow A$ such that λ is equivalent to the push-out $\varphi_* \eta_L$.*

It is not possible to construct a local or formal deformation of a Lie algebra with a similar universality property in the class of local or formal deformations. But it becomes possible for an appropriate weakening of this property.

DEFINITION 2.6. A formal deformation η of a Lie algebra L with base B is called *versal* if

- (i) for any formal deformation λ of L with any (local) base A there exists a homomorphism $f: B \rightarrow A$ such that the deformation λ is equivalent to $f_* \eta$;
- (ii) in the notation of (i), if A satisfies the condition $m^2 = 0$, then f is unique (see [Fi1]).

3. OBSTRUCTIONS TO EXTENDING DEFORMATIONS

Let λ be a deformation of a Lie algebra L with a finite-dimensional local base A , and let $0 \rightarrow \mathbb{K} \rightarrow^i B \rightarrow^p A \rightarrow 0$ be a one-dimensional extension of A . It is known [H] that such an extension corresponds to a Harrison 2-cohomology class $f \in H^2_{\text{Harr}}(A; \mathbb{K})$. Let $I = i \otimes \text{id}: L = \mathbb{K} \otimes L \rightarrow B \otimes L$ and $P = p \otimes \text{id}: B \otimes L \rightarrow A \otimes L$. Let also $E = \hat{e} \otimes \text{id}: B \otimes L \rightarrow \mathbb{K} \otimes L = L$, where \hat{e} is the augmentation of B . The Lie algebra structure $[\ ,]_\lambda$ in $A \otimes L$ can be lifted to a B -bilinear operation $\{ \ , \}$: $\Lambda^2 B \rightarrow B$ such that

- (i) $P\{l_1, l_2\} = [P(l_1), P(l_2)]_\lambda$ for any $l_1, l_2 \in B \otimes L$,
- (ii) $\{I(l), l_1\} = I[l, E(l_1)]$ for any $l \in L, l_1 \in B \otimes L$.

This operation $\{ \ , \}$ partially satisfies the Jacobi identity

$$\varphi(l_1, l_2, l_3) := \{l_1, \{l_2, l_3\}\} + \{l_2, \{l_3, l_1\}\} + \{l_3, \{l_1, l_2\}\} \in \text{Ker } P.$$

Remark that φ is multilinear and skew-symmetric, and $\varphi(l_1, l_2, l_3) = 0$ if $l_1 \in \text{Ker } E$. Hence φ determines a multilinear form

$$\bar{\varphi}: \Lambda^3 L = \Lambda^3((B \otimes L)/\text{Ker } E) \rightarrow \text{Ker } P = L,$$

that is, an element $\bar{\varphi}$ of $C^3(L; L)$. It is easy to check that $\partial \bar{\varphi} = 0$.

Let $\{ , \}'$ be another B -bilinear operation $\Lambda^2 B \rightarrow B$ satisfying the conditions (i), (ii).

It is easy to check that if $\bar{\varphi}, \bar{\varphi}' \in C^3(L; L)$ are the cochains corresponding to $\{ , \}, \{ , \}'$, then

$$\bar{\varphi}' - \bar{\varphi} = \partial\psi$$

with some $\psi \in C^2(L; L)$.

Let $\mathcal{O}_\lambda(f) \in H^3(L; L)$ be the cohomology class of the cochain $\bar{\varphi}$. It is obvious that

$$\mathcal{O}_\lambda: H_{\text{Harr}}^2(A; \mathbb{K}) \rightarrow H^3(L; L), f \mapsto \mathcal{O}_\lambda(f)$$

is a linear map.

We get the following.

PROPOSITION 3.1. *The deformation λ with base A can be extended to a deformation of L with base B if and only if $\mathcal{O}_\lambda(f) = 0$.*

DEFINITION 3.2. The cohomology class $\mathcal{O}_\lambda(f)$ is called the *obstruction* to the extension of the deformation λ from A to B .

Suppose now that $\mathcal{O}_\lambda(f) = 0$, that is, the deformation λ is extendible to a deformation with base B . We are going to study the set of all possible extensions.

Let μ, μ' be deformations of L with base B such that $p_* \mu = p_* \mu' = \lambda$. Then we know that the difference $[,]_{\mu'} - [,]_\mu$ determines and is determined by a certain cochain $\psi \in C^2(L; L)$. Since $[,]_{\mu'}$ and $[,]_\mu$ both satisfy the Jacobi identity, $\partial\psi = 0$. Moreover, it is easy to check that if we replace any of the structures $[,]_\mu, [,]_{\mu'}$ with an equivalent one, then the cocycle ψ will be replaced by a cohomologous one. That means that \mathbb{H} operates transitively on the set of these equivalence classes.

It is known [H] that $H_{\text{Harr}}^1(A; M)$ is the space of derivations $A \rightarrow M$. From this it follows that if A is local algebra with maximal ideal \mathfrak{m} , then $H_{\text{Harr}}^1(A; \mathbb{K}) = (\mathfrak{m}/\mathfrak{m}^2)' = TA$.

On the other hand, $H_{\text{Harr}}^1(A; M)$ may also be interpreted as the group of automorphisms of any extension $0 \rightarrow M \rightarrow^i B \rightarrow^p A \rightarrow 0$ of A . (An automorphism of this extension is an algebra automorphism $f: B \rightarrow B$ such that $f \circ i = i, p \circ f = p$. See [H].) The group of automorphisms of the extension also operates on the set of equivalence classes of deformations.

PROPOSITION 3.3. *If $r: B \rightarrow B$ determines an automorphism of the extension $0 \rightarrow \mathbb{K} \rightarrow^i B \rightarrow^p A \rightarrow 0$ which corresponds to an element $h \in H_{\text{Harr}}^1(A; \mathbb{K}) = TA$, then for any deformation μ of L with base B such that $p_* \mu = \lambda$, the difference between $[,]_{r_* \mu}$ and $[,]_\mu$ is a cocycle of the cohomology class $d\lambda(h)$.*

In other words, these two operations are related to each other by the differential $d\lambda: TA \rightarrow \mathbb{H}$.

The proof is obvious.

COROLLARY 3.4. *Suppose that the differential $d\lambda: TA \rightarrow \mathbb{H}$ is onto. Then the group of automorphisms of the extensions $0 \rightarrow \mathbb{K} \rightarrow^i B \rightarrow^p A \rightarrow 0$ operates transitively on the set of equivalence classes of deformations μ of L with base B such that $p_* \mu = \lambda$. In other words, μ is unique up to an isomorphism and an automorphism of the extension $0 \rightarrow \mathbb{K} \rightarrow B \rightarrow A \rightarrow 0$.*

Remark. The relation between the second Harrison cohomology group of a finite-dimensional local commutative algebra A and extensions of A may be described in terms of one extension. This is the extension,

$$0 \rightarrow H_{\text{Harr}}^2(A; \mathbb{K})' \rightarrow C \rightarrow A \rightarrow 0, \tag{3.1}$$

where the operation of A on $H_{\text{Harr}}^2(A; \mathbb{K})'$ is induced by the operation of A on \mathbb{K} , and the cocycle

$$f_A: S^2 A \rightarrow H_{\text{Harr}}^2(A; \mathbb{K})'$$

is defined as the dual of a homomorphism

$$\mu: H_{\text{Harr}}^2(A; \mathbb{K}) \rightarrow (S^2, A)',$$

which takes a cohomology class to a cocycle from this class. This extension does not depend, up to an isomorphism, on the choice of μ and possesses the following partial (co-)universality property.

PROPOSITION 3.5. *Let M be an A -module with $\mathfrak{m}M = 0$. Then the extension (3.1) admits a unique homomorphism into an arbitrary extension $0 \rightarrow M \rightarrow B \rightarrow A \rightarrow 0$ of A .*

4. EXAMPLES OF ONE-PARAMETER DEFORMATIONS

In this section we will study one-parameter deformations λ , i.e., the local algebra $A = \mathbb{C}[[t]]$ and $\mathfrak{m} = t\mathbb{C}[[t]]$; cf. Definition 2.1. Hence a one-parameter deformation λ is of the form

$$[1 \otimes x, 1 \otimes y]_\lambda = 1 \otimes [x, y] + \sum_{i=1}^{\infty} t^i \otimes c_i(x, y)$$

where $c_i \in C^2(L_2; L_2)$. Usually we omit the tensors and replace λ by t ; hence we write

$$[x, y]_t = [x, y] + \sum_{i=1}^{\infty} t^i c_i(x, y). \tag{4.1}$$

DEFINITION 4.1. (a) The one-parameter deformation (4.1) is called a *homogeneous deformation* of degree m if $c_i \in C_m^2(L_2; L_2)$. In this case we write $\deg(t) = m$.

(b) the one-parameter deformation (4.1) is called *nonsingular* if c_1 is nonzero in $H^2(L_2; L_2)$. (Automatically it holds that $\partial c_1 = 0$.) In the opposite case we call the deformation *singular*.

Solving the problem of finding all nonequivalent one-parameter deformations is equivalent to “solving” the relations in the versal deformation; see the next section. This is too hard to accomplish. We will discuss the problem of finding all nonsingular nonequivalent one-parameter deformations. Since $H_m^2(L_2; L_2) \neq 0$ only for $m = 2, 3, 4, 5, 6$, the degree is restricted to these values.

We discuss two general constructions, valid in all L_k with $k \geq 1$. The first deformation is given by

$$[e_i, e_j]_t = (j - i)e_{i+j} + (j - i)t e_{i+j-m}. \quad (4.2)$$

An easy check shows that for each degree $m \in \{1, 2, \dots, k + 1\}$ this defines a (possibly trivial) deformation. For L_2 the cases $m = 2$ and $m = 3$ define nontrivial deformations. In fact, these can be combined to obtain a nontrivial two-parameter deformation

$$[e_i, e_j]_t = (j - i)e_{i+j} + (j - i)t_2 e_{i+j-2} + (j - i)t_3 e_{i+j-3}. \quad (4.3)$$

The second group of deformations is of the form

$$[e_i, e_j]_t = (j - i)e_{i+j} + j\delta_{i,m}t e_j - i\delta_{j,m}t e_i \quad (4.4)$$

for $m = k, k + 1, \dots, 2k$. In L_2 all the cases $m = 2, 3, 4$ yield nontrivial deformations.

Note that in the case of L_1 all nonequivalent deformations are of these types, namely (4.2) for $m = 1$ and (4.4) for $m = 2$ and $m = 3$; see [Fi2]. In the case of L_2 we noted already that this is not the case; see (4.3). However, apart from this obvious extension there exist more deformations. By hard but straightforward calculations one can prove

PROPOSITION 4.2. *There exist exactly seven (nonequivalent, nontrivial) nonsingular homogeneous one-parameter deformations for L_2 , namely*

- (a) *the deformations (4.2) for $m = 2$ and $m = 3$,*
- (b) *the deformations (4.4) for $m = 2$, $m = 3$, and $m = 4$, and*
- (c) *two exceptional ones of degree 2, described below.*

The first exceptional deformation of degree 2 is given by $c_i = 0$ $i > 1$ and

$$\begin{aligned} c_1(e_{2k}, e_{2l}) &= -2(2l - 2k)e_{2l+2k-2}; \\ c_1(e_{2k}, e_{2l+1}) &= -2(2l - 2k + 1)e_{2l+2k-1}; \\ c_1(e_{2k+1}, e_{2l+1}) &= 0. \end{aligned}$$

This c_1 is a coboundary in $C^2(L_2; L_0)$; we have $c_1 = \partial b$ with

$$b(e_{2l}) = 2e_{2l-2} \quad \text{and} \quad b(e_{2l+1}) = e_{2l-1}.$$

Hence $\partial c = 0$. By a direct calculation one finds that the Massey square $[c_1, c_1] = 0$.

The second exceptional deformation is most easily presented using an embedding φ from L_2 into L_0 . We define

$$\varphi(e_2) = e_2 + te_0; \quad \varphi(e_3) = e_3 + 2te_1; \quad \varphi(e_i) = e_i \quad (i \geq 4).$$

Putting

$$[x, y]_t = \varphi^{-1}([\varphi(x), \varphi(y)])$$

we obtain that $[e_i, e_j]_t = (j - i)e_{i+j} + tc_1(e_i, e_j)$ with

$$\begin{aligned} c_1(e_2, e_j) &= -je_j \quad (j \geq 4); & c_1(e_3, e_j) &= -2(j - 1)e_{j+1} \quad (j \geq 4); \\ c_1(e_2, e_3) &= -e_3; & c_1(e_i, e_j) &= 0 \quad (i, j \geq 4). \end{aligned}$$

Note that (4.3) gives lots of examples of singular deformations. In particular, we have the homogeneous ones of degree 1 ($\alpha \in \mathbb{C}$),

$$[e_i, e_j]_t = (j - i)e_{i+j} + (j - i)t^2e_{i+j-2} + \alpha(j - i)t^3e_{i+j-3}.$$

5. CONSTRUCTION OF A VERSAL DEFORMATION

In [FF2] an inductive process for obtaining a versal deformation is described. Here we will give a short summary of it, without proofs. Suppose that $\dim \mathbb{H} < \infty$.

Let $C_0 = \mathbb{K}$, $C_1 = \mathbb{K} \oplus \mathbb{H}'$, and let

$$0 \rightarrow \mathbb{H}' \xrightarrow{i_1} C_1 \xrightarrow{p'_1} \mathbb{K} \rightarrow 0$$

be the canonical splitting extension. Denote the universal infinitesimal deformation η_L of L with base C_1 constructed in Section 2 by η_1 . Suppose

that for some $k \geq 1$ we have already constructed a finite-dimensional commutative algebra C_k and a deformation η_k of L with base C_k . Consider the extension

$$0 \rightarrow H^2_{\text{Harr}}(C_k; \mathbb{K})' \xrightarrow{i_{k+1}} \bar{C}_{k+1} \xrightarrow{\bar{p}'_{k+1}} C_k \rightarrow 0 \tag{5.1}$$

using the cocycle f_{C_k} . The obstruction to the extension of η_k is

$$\mathcal{O}_{\eta_k}(f_{C_k}) \in H^2_{\text{Harr}}(C_k, \mathbb{K})' \otimes H^3(L; L).$$

This gives a map

$$\omega_k : H^2_{\text{Harr}}(C_k, \mathbb{K}) \rightarrow H^3(L; L).$$

Set

$$C_{k+1} = \bar{C}_{k+1}/\bar{i}_{k+1} \circ \omega'_k(H^3(L; L)').$$

Obviously, the extension (5.1) factors to an extension

$$0 \rightarrow (\text{Ker } \omega_k)' \xrightarrow{i_{k+1}} C_{k+1} \xrightarrow{p'_{k+1}} C_k \rightarrow 0. \tag{5.2}$$

Here all algebras C_k are local. Since the algebra C_k is finite dimensional, the cohomology $H^2_{\text{Harr}}(C_k; \mathbb{K})$ is also finite dimensional, and hence C_{k+1} is finite dimensional.

PROPOSITION 5.1. *The deformation η_k admits an extension to a deformation with base C_{k+1} , and this extension is unique up to an isomorphism and an automorphism of an extension (5.2).*

Let us choose an extended deformation and denote it by η_{k+1} .

The induction yields a sequence of finite-dimensional algebras

$$\mathbb{K} \xleftarrow{p'_1} C_1 \xleftarrow{p'_2} \dots \xleftarrow{p'_k} C_k \xleftarrow{p'_{k+1}} C_{k+1} \xleftarrow{p'_{k+2}} \dots,$$

and a sequence of deformations η_k of L such that $(p'_{k+1})_* \eta_{k+1} = \eta_k$.

Taking the projective limit, we obtain a formal deformation η of L with base $C = \varprojlim_{k \rightarrow \infty} C_k$.

Let \mathfrak{m} be the maximal ideal in $\mathbb{K}[[H']]$.

PROPOSITION 5.2. $C_k = \mathbb{K}[[H']]/I_k$ where

$$\mathfrak{m}^2 = I_1 \supset I_2 \supset \dots, I_k \supset \mathfrak{m}^{k+1}.$$

PROPOSITION 5.3. $C = \mathbb{K}[[H']]/I$, where I is an ideal contained in \mathfrak{m}^2 . Since $\mathbb{K}[[H']]$ is Noetherian, then I is finitely generated.

THEOREM 5.4. *If $\dim \mathbb{H} < \infty$, then the formal deformation η is a versal deformation of L .*

THEOREM 5.5. *If $\dim \mathbb{H} < \infty$, then the base of the versal formal deformation of L is formally embedded in \mathbb{H} , that is, it may be described in \mathbb{H} by a finite system of formal equations.*

The proof follows directly from Proposition 5.3.

We will describe now this inductive process for the Lie algebra L_2 . The space \mathbb{H} is \mathbb{Z} -graded and finite-dimensional. Let

$$\mathbf{t} = \{t_2, t_{31}, t_{32}, t_{41}, t_{42}, t_{51}, t_{52}, t_6\}$$

be a basis of \mathbb{H}' . According to Theorem 5.5 there exists a versal deformation η with local algebra $C = \mathbb{C}[[\mathbf{t}]]/I$. Let \mathfrak{m} denote the unique maximal ideal in $\mathbb{C}[[\mathbf{t}]]$ consisting of all formal power series with zero constant term. Then $I \subset \mathfrak{m}^2$. We will use the following notation, some of which we have already introduced:

- $C_k = C/\mathfrak{m}^{k+1}$;
- $\mathfrak{m}_k = \mathfrak{m}/\mathfrak{m}^{k+1}$;
- $\varphi^{(k)}: C \rightarrow C_k$, the canonical projection;
- $I_k = (I + \mathfrak{m}^{k+1})/\mathfrak{m}^{k+1}$;
- $\eta_k = \varphi_*^{(k)}\eta$.

The cohomology computations in Section 1 have several consequences for the ideal I . Since \mathbb{H} is \mathbb{Z} -graded, I is also \mathbb{Z} -graded. Since $H^3(L_2; L_2)$ is nonzero only in the degrees 8, 9, ..., 15, I is generated by elements of these degrees. (We assume that t_{ki} has degree k .) Like any ideal in $\mathbb{C}[[\mathbf{t}]]$, I is finitely generated. When writing down relations, we implicitly mean that these relations are homogeneous and minimal. Hence the relations (homogeneous and minimal) are *polynomials* in \mathbf{t} ; since $H_k^3(L_2; L_2) = 0$ for $k \geq 16$, we a priori know that the maximal *polynomial* degree occurring in a relation is 7. Hence our process below will give I (and not only I_7) after seven steps!

Let us start with C_1 . Since $I \subset \mathfrak{m}^2$ we have

$$C_1 = \mathbb{C} \oplus \mathbb{H}' \quad \text{and} \quad I_1 = 0,$$

and the corresponding deformation $\eta_1 = \varphi_*^{(1)}\eta$ is simply the universal infinitesimal deformation (see Section 2).

Now we explain how to construct the deformation η_2 with base (C_2, I_2) . First we extend C_1 to $\overline{C}_2 = \mathbb{C}[\mathbb{H}']/\mathfrak{m}^3$. The Lie bracket on $C_1 \otimes L_2$ is extended to a bilinear operation on $\overline{C}_2 \otimes L_2$ in the obvious way. However, there are obstructions for this operation to be a Lie bracket: this yields the

relations I_2 and $C_2 = \overline{C}_2/I_2$. These relations correspond to the nontrivial Massey products between the elements of \mathbb{H} . In our case one finds

$$\begin{aligned}
 I_2 = & (1066t_2t_6 + 46t_{31}t_{51} + 3474t_{31}t_{52} - 135t_{32}t_{51} - 459t_{32}t_{52} \\
 & - 6858t_{41}t_{42} + 59454t_{42}^2, \\
 & 108264t_{31}t_6 - 125t_{41}t_{51} - 15165t_{41}t_{52} + 75390t_{42}t_{51} + 243918t_{42}t_{52}, \\
 & 108264t_{32}t_6 - 64177t_{41}t_{51} - 266769t_{41}t_{52} + 1146486t_{42}t_{51} \\
 & + 1669734t_{42}t_{52}, \\
 & 6669t_{41}t_6 + 345202t_{51}t_{52} + 565686t_{52}^2, \\
 & 40014t_{42}t_6 + 152867t_{51}t_{52} + 249579t_{52}^2, \\
 & 19t_{51}^2 + 3372t_{51}t_{52} + 5321t_{52}^2, t_{51}t_6, t_{52}t_6, t_6^2).
 \end{aligned}$$

If these obstructions are resolved, we can choose an arbitrary extension to proceed to the next step.

Now we construct C_3 and I_3 . The procedure is similar to that above: first we define $\overline{C}_3 = \mathbb{C}[\mathbb{H}']/(mI_2 + m^4)$. The Lie bracket on $C_2 \otimes L_2$ is extended to a bilinear operation on $\overline{C}_3 \otimes L_2$. There are obstructions for this bilinear operation to be a Lie bracket. As a result we find the old relation of I_2 but now updated with cubic terms and possibly with new relations without quadratic term. For example, in degree 8 the following polynomial defines a relation:

$$\begin{aligned}
 & -2730t_2^2t_{41} - 22554t_2^2t_{42} + 6120t_2t_{31}^2 + 2248t_2t_{31}t_{32} \\
 & - 126854t_2t_6 - 5474t_{31}t_{51} - 413406t_{31}t_{52} + 16065t_{32}t_{51} \\
 & + 54621t_{32}t_{52} + 8161026t_{41}t_{42} - 7075026t_{42}^2.
 \end{aligned}$$

We remark that this relation will not change in the next steps, i.e., this is a relation in I . This can be seen as follows. Looking at the right degree, we see that the only term that can be added is t_2^4 . However, by putting all t_{ki} equal to 0, except t_2 , we have a global one-parameter deformation. Hence the term t_2^i for any $i \geq 1$ cannot appear in the relations.

We calculated by computer all relations up to level 7. We summarize the results in the Table I, where—denotes that the relation at one level above is in final form.

Note that the number of relations does not exceed the dimension of $H^3(L_2; L_2)$ in any degree.

TABLE I
Number of Relations by Degree and Level

Degree:	8	9	10	11	12	13	14	15	All
Level 2	1	2	3	2	1	0	0	0	9
Level 3	1	2	3	3	3	3	0	0	15
Level 4	—	2	3	3	3	3	1	0	16
Level 5	—	—	—	3	3	3	1	0	16
Level 6	—	—	—	—	—	3	1	0	16
Level 7	—	—	—	—	—	—	—	0	16

APPENDIX: REPRESENTING COCYCLES FOR $H^3(L_2; L_2)$

$$\begin{aligned}
 c_8 &= \partial\mu_8^{(3,5)} \\
 c_{9,1} &= \partial\mu_9^{(2,6)} \\
 c_{9,2} &= \partial\mu_9^{(4,6)} \\
 c_{10,1} &= \partial\mu_{10}^{(4,6)} \\
 c_{10,2} &= 3\partial\mu_{10}^{(3,5)} - 4\partial\mu_{10}^{(3,7)} \\
 c_{10,3} &= 12\partial\mu_{10}^{(5,6)} + 8\partial\mu_{10}^{(4,7)} - 3\partial\mu_{10}^{(3,8)} \\
 c_{11,1} &= -12\partial\mu_{11}^{(5,6)} - 8\partial\mu_{11}^{(4,7)} + 3\partial\mu_{11}^{(3,8)} \\
 c_{11,2} &= -15\partial\mu_{11}^{(5,7)} - 6\partial\mu_{11}^{(4,8)} + 3\partial\mu_{11}^{(3,9)} + 8\partial\mu_{11}^{(4,6)} \\
 c_{11,3} &= 18\partial\mu_{11}^{(5,6)} + 12\partial\mu_{11}^{(4,7)} + 7\partial\mu_{11}^{(4,6)} + 3\partial\mu_{11}^{(2,6)} \\
 c_{12,1} &= 725\partial\mu_{12}^{(6,7)} + 690\partial\mu_{12}^{(5,8)} - 134\partial\mu_{12}^{(4,9)} - 115\partial\mu_{12}^{(3,10)} - 234\partial\mu_{12}^{(4,8)} \\
 &\quad - 195\partial\mu_{12}^{(5,6)} - 260\partial\mu_{12}^{(4,7)} + 234\partial\mu_{12}^{(4,6)} + 78\partial\mu_{12}^{(4,5)} + 39\partial\mu_{12}^{(3,6)} \\
 c_{12,2} &= 136\partial\mu_{12}^{(6,7)} + 219\partial\mu_{12}^{(5,8)} + 8\partial\mu_{12}^{(4,9)} - 56\partial\mu_{12}^{(3,10)} - 156\partial\mu_{12}^{(5,6)} \\
 &\quad - 104\partial\mu_{12}^{(4,7)} + 39\partial\mu_{12}^{(3,8)} \\
 c_{12,3} &= 100\partial\mu_{12}^{(6,7)} + 75\partial\mu_{12}^{(5,8)} - 40\partial\mu_{12}^{(4,9)} + 20\partial\mu_{12}^{(3,10)} - 104\partial\mu_{12}^{(5,7)} \\
 &\quad - 52\partial\mu_{12}^{(4,8)} + 52\partial\mu_{12}^{(4,6)} + 26\partial\mu_{12}^{(3,7)} + 39\partial\mu_{12}^{(2,8)} \\
 c_{13,1} &= -1170\partial\mu_{13}^{(6,8)} + 390\partial\mu_{13}^{(4,10)} + 544\partial\mu_{13}^{(6,7)} + 876\partial\mu_{13}^{(5,8)} + 32\partial\mu_{13}^{(4,9)} \\
 &\quad - 224\partial\mu_{13}^{(3,10)} - 468\partial\mu_{13}^{(5,6)} - 312\partial\mu_{13}^{(4,7)} + 117\partial\mu_{13}^{(3,8)} \\
 c_{13,2} &= 11700\partial\mu_{13}^{(6,8)} - 3900\partial\mu_{13}^{(4,10)} - 1150\partial\mu_{13}^{(6,7)} - 4080\partial\mu_{13}^{(5,8)}
 \end{aligned}$$

$$\begin{aligned}
& - 20362 \partial \mu_{13}^{(4,9)} + 680 \partial \mu_{13}^{(3,10)} + 4680 \partial \mu_{13}^{(5,6)} + 3120 \partial \mu_{13}^{(4,7)} \\
& + 1053 \partial \mu_{13}^{(4,6)} + 585 \partial \mu_{13}^{(2,6)} \\
c_{13,3} = & - 1300 \partial \mu_{13}^{(4,10)} - 7680 \partial \mu_{13}^{(6,7)} - 5760 \partial \mu_{13}^{(5,8)} + 2058 \partial \mu_{13}^{(4,9)} \\
& + 960 \partial \mu_{13}^{(3,10)} + 2275 \partial \mu_{13}^{(4,8)} + 1040 \partial \mu_{13}^{(4,7)} - 1989 \partial \mu_{13}^{(4,6)} \\
& - 650 \partial \mu_{13}^{(4,5)} + 65 \partial \mu_{13}^{(2,4)} \\
c_{14,1} = & 459480 \partial \mu_{14}^{(7,8)} + 530208 \partial \mu_{14}^{(6,9)} - 128100 \partial \mu_{14}^{(5,10)} - 71232 \partial \mu_{14}^{(4,11)} \\
& + 29295 \partial \mu_{14}^{(3,12)} - 439164 \partial \mu_{14}^{(6,8)} - 238392 \partial \mu_{14}^{(5,9)} + 66924 \partial \mu_{14}^{(4,10)} \\
& + 163152 \partial \mu_{14}^{(3,11)} - 271040 \partial \mu_{14}^{(6,7)} - 203280 \partial \mu_{14}^{(5,8)} + 71456 \partial \mu_{14}^{(4,9)} \\
& - 35420 \partial \mu_{14}^{(3,10)} + 297990 \partial \mu_{14}^{(5,7)} + 119196 \partial \mu_{14}^{(4,8)} - 59598 \partial \mu_{14}^{(3,9)} \\
& + 36960 \partial \mu_{14}^{(4,7)} + 83160 \partial \mu_{14}^{(3,8)} 110880 \partial \mu_{14}^{(2,9)} - 79464 \partial \mu_{14}^{(4,6)} \\
c_{14,2} = & 91680 \partial \mu_{14}^{(7,8)} + 72288 \partial \mu_{14}^{(6,9)} - 46500 \partial \mu_{14}^{(5,10)} - 39360 \partial \mu_{14}^{(4,11)} \\
& + 5895 \partial \mu_{14}^{(3,12)} - 57420 \partial \mu_{14}^{(6,8)} - 60984 \partial \mu_{14}^{(5,9)} + 5148 \partial \mu_{14}^{(4,10)} \\
& + 3168 \partial \mu_{14}^{(3,11)} + 1760 \partial \mu_{14}^{(6,7)} + 1320 \partial \mu_{14}^{(5,8)} + 4840 \partial \mu_{14}^{(4,9)} \\
& - 220 \partial \mu_{14}^{(3,10)} + 76230 \partial \mu_{14}^{(5,7)} + 20988 \partial \mu_{14}^{(4,8)} - 15246 \partial \mu_{14}^{(3,9)} \\
& - 5544 \partial \mu_{14}^{(4,7)} - 16632 \partial \mu_{14}^{(2,9)} - 13992 \partial \mu_{14}^{(4,6)} - 7128 \partial \mu_{14}^{(3,6)} \\
& + 1584 \partial \mu_{14}^{(3,4)} \\
c_{15} = & 1909215 \partial \mu_{15}^{(7,9)} - 49962 \partial \mu_{15}^{(6,10)} - 782595 \partial \mu_{15}^{(5,11)} - 130944 \partial \mu_{15}^{(4,12)} \\
& + 79035 \partial \mu_{15}^{(3,13)} - 236928 \partial \mu_{15}^{(7,8)} - 1027584 \partial \mu_{15}^{(6,9)} - 244800 \partial \mu_{15}^{(5,10)} \\
& + 390144 \partial \mu_{15}^{(4,11)} - 3192 \partial \mu_{15}^{(3,12)} + 99924 \partial \mu_{15}^{(6,8)} - 83160 \partial \mu_{15}^{(5,9)} \\
& + 382492 \partial \mu_{15}^{(4,10)} + 23760 \partial \mu_{15}^{(3,11)} + 1047552 \partial \mu_{15}^{(6,7)} + 544896 \partial \mu_{15}^{(5,8)} \\
& - 168960 \partial \mu_{15}^{(4,9)} + 67584 \partial \mu_{15}^{(3,10)} + 83160 \partial \mu_{15}^{(5,7)} - 498960 \partial \mu_{15}^{(4,8)} \\
& - 16632 \partial \mu_{15}^{(3,9)} + 160512 \partial \mu_{15}^{(5,6)} - 208384 \partial \mu_{15}^{(4,7)} - 158400 \partial \mu_{15}^{(3,8)} \\
& + 285120 \partial \mu_{15}^{(4,6)} + 118272 \partial \mu_{15}^{(4,5)} + 59136 \partial \mu_{15}^{(3,6)}
\end{aligned}$$

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