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Best data-dependent triangulations

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Dedicated to Prof. Larry L. Schumaker on the occasion of his 60th birthday

Abstract

When reconstructing a surface from irregularly spaced data we need to decide how to identify a good triangulation. As a measure of quality we consider various differential geometrical properties, namely integral absolute Gaussian curvature, integral absolute mean curvature and area. A comparison is made with data-dependent triangulation methods that exist in the literature. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Triangulating a data set is an essential technique in solving problems in surface reconstruction, i.e., in scattered data interpolation and approximation, or in applications where one needs to recover 3D objects. The surface reconstruction problem can be formulated as follows:

Given a discrete data set $\{(x^i, y^i, z^i) \in \mathbb{R}^3\}_{i=1}^N$, fit a surface to it.

The data sites are supposed to be irregularly located.

In general, it is not possible to interpolate or approximate the data without preprocessing. One first needs to organise the data or, in other words, to put a structure on the data. Therefore, the first step in evaluating a surface is to obtain a triangulation of the data. Determining a triangulation, the simplest C^0-approximation, is the quickest and cheapest way to take an initial look at the data before turning to higher-order interpolation/approximation methods. A ‘good’ triangulation can help to solve many problems. These problems are not limited to finding a smooth interpolant, but also

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concern the definition of the shape of the object [26,23,4,5,17,10], or, the other way around, the reduction of the number of points without much damaging the actual shape of the object [12,6], the control of the automatic processing of surfaces [8], or the estimates of some geometric properties, such as area, volume, axes of inertia, or the extraction of elementary shapes [16,11].

Using pure geometric criteria, Alboul and van Damme have introduced new triangulations for irregularly located 3D data [25,1,2].

The first triangulation is called the Tight triangulation and is based on optimising a discrete analogon of integral absolute Gaussian curvature. Tight triangulations are evidently better than Delaunay triangulations, as for example, the Tight triangulation automatically preserves convexity.

The second optimality criterion that was proposed in [2] concerns triangulations that minimise integral absolute mean curvature.

Up to now it was not clear which of those two criteria was the ‘best’ (if any), and how they compare with known methods in the literature such as

- minimising the area of the resulting object [18],
- heuristic criteria, based on minimising a measure of roughness of the resulting object. One such measure is the jump in normal derivative (JND) and the other measure is the angle between normals (ABN) [10,9,17],
- methods based on minimising a certain functional, like the energy of a bending plate, constrained by the interpolation conditions. Such a method can, e.g., be found in [19–21] (with the obvious disadvantage of this method that it only works for *functional* data).

This article is organised as follows. We first briefly review the triangulation methods proposed in [25,1,2], based on minimising absolute Gaussian and mean curvature. Next, we perform numerical experiments for closed objects with these criteria from which it turns out that the latter in general performs better, although this method may not be shape preserving: it is not known whether the method conserves convexity in general. Both perform much better than minimising the area of the object as it comes to visual pleasingness.

In the second part of the paper we investigate the quality of the methods for the special case of functional data: both tight as well as minimal absolute mean curvature methods can be compared with methods known in the literature. Also in these cases the mean criterion or the two methods ABN and JND (at least as good) seem to be superior when it comes to precision.

2. Geometric criteria for best triangulations

In this section we review the geometric notion *integral* Gaussian and mean curvature for triangulations, as well as *absolute* Gaussian and mean curvature. For smooth objects Gaussian curvature K and mean curvature H are defined in terms of the principal curvatures λ_1, λ_2, being the eigenvalues of the differential of the Gauss map [7]:

$$K = \lambda_1 \lambda_2, \quad H = (\lambda_1 + \lambda_2)/2.$$

The integral version of these quantities can also be defined for triangulations, and they lead to different criteria to define the best triangulation. We consider two of these criteria and compare them with the aid of experiments.

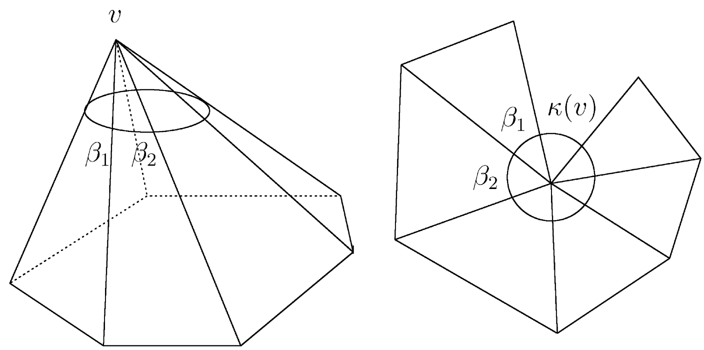

Fig. 1. Curvature around a vertex: $\kappa(v) = 2\pi - \sum \beta_i = 2\pi - \theta(v)$.

2.1. Gaussian curvature

The notion of Gaussian curvature is one of the central concepts in differential geometry and is strictly related to the concept of angle. On the basis of the notion of angle, we can define the following curvatures for a triangulation $\triangle$ considered as a polyhedral surface:

- The (integral) curvature κ (an analogue of the integral Gaussian curvature).
  The total angle $\theta(v)$ around the vertex v is the sum of angles of all the plane polygons incident to v, called the star of v, Star(v). For any point $x \in \triangle$: $\kappa(x) = 2\pi - \theta(x)$. The quantity κ is also known as the angle deficit. Only for vertices we have $\kappa(x) \neq 0$ (see Fig. 1).
- The positive (extrinsic) curvature $\kappa^+(v)$.
  Suppose that through the vertex v there passes some (local) supporting plane of a triangulation $\triangle$. Then this vertex lies on the boundary of the convex hull of Star(v). We denote the star of v in the boundary of this convex hull by Star$^+(v)$ and will call it the star of the convex cone of a vertex. The curvature $\kappa^+(v)$ of Star$^+(v)$ is called the positive (extrinsic) curvature of v. If there is no supporting plane through v then we put $\kappa^+(v)$ equal to zero.
- The negative (extrinsic) curvature $\kappa^-(v)$: $\kappa^-(v) = \kappa^+(v) - \kappa(v)$.
- The absolute (extrinsic) curvature $\hat{\kappa}(v)$: $\hat{\kappa}(v) = \kappa^+(v) + \kappa^-(v)$.

We can identify the following sets of vertices:

1. Proper convex vertices: $\kappa(v) = \kappa^+(v) = \hat{\kappa}(v)$ and $\kappa^-(v) = 0$. Geometrically this means that Star(v) coincides with Star$^+(v)$.
2. Proper saddle vertices: $\hat{\kappa}(v) = \kappa^-(v) = -\kappa(v)$ and $\kappa^+(v) = 0$. The Gaussian curvature κ of a proper saddle vertex is less than zero and there exists no supporting plane, i.e., there does not exist a plane which passes through vertex v such that all the neighbouring vertices lie on the same side of (or in) this plane.
3. Mixed vertices: $\kappa^-(v) > 0$, $\kappa^+(v) > 0$. A vertex is mixed if it is neither convex nor a saddle. Therefore it has a supporting plane, but there exist two successive edges incident on the vertex which span a plane that divides the set of adjacent vertices.

Examples of the three above-described vertices are given in Fig. 2.

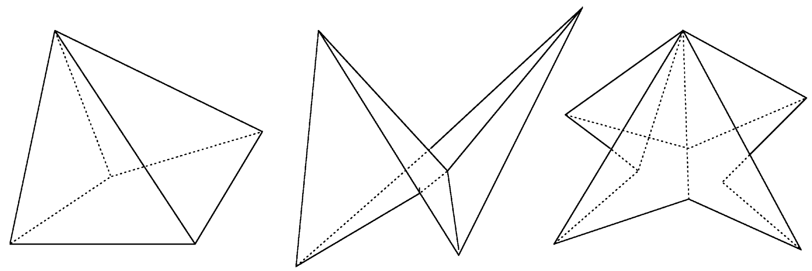

Fig. 2. The three types of vertices: proper convex, proper saddle and mixed.

The total absolute (extrinsic) curvature $K_{abs}(\triangle)$ of a triangulation $\triangle$ is given by the following expression:

$$K_{abs}(\triangle) = \sum_{v_\alpha \text{ convex}} \kappa^+(v_\alpha) + \sum_{v_\beta \text{ saddle}} \kappa^-(v_\beta) + \sum_{v_\gamma \text{ convex}} (\kappa^+(v_\gamma) + \kappa^-(v_\gamma)).$$

For any triangulation applies: proper convex, proper saddle and mixed vertices together form a partition of all vertices.

Definition 2.1. Given a data set $\{x^i\}$, $i = 1, \ldots, N$. A triangulation $\bar{\triangle}$ of the data set is said to be the Tight triangulation if it is proper, and if $K_{abs}(\bar{\triangle})$ is minimal, i.e.,

$$K_{abs}(\triangle) \geqslant K_{abs}(\bar{\triangle})$$

for all possible other proper triangulations $\triangle$.

Some properties of the Tight triangulation were given in [3], the most important being that it preserves convexity. Moreover, it can be proved that with the local swapping algorithm, first suggested in [15], this global optimum can actually be obtained if the data are convex [3].

Unfortunately, this optimisation criterion has some undesirable effects, as it seems to create unwanted long thin triangles. This already happens on a relatively well-sampled object as shown in Fig. 3. The initial triangulation (left) is very good, but it was obtained in a laborious fashion, namely by hand.

2.2. Mean curvature

Another curvature that can be defined for polyhedra and consequently for a triangulation, is mean curvature. One easily shows that a proper definition for integral mean curvature on a strip along an edge e is given by

$$H(e) = \alpha ||x_i - x_j||.$$

Here x_i, x_j are the coordinates of the vertices of e, and α is the angle between the normals of the two triangles which have e in common. (This can be derived by considering a C^1 approximation of the triangulation by replacing edges with cylinder of small radius.)

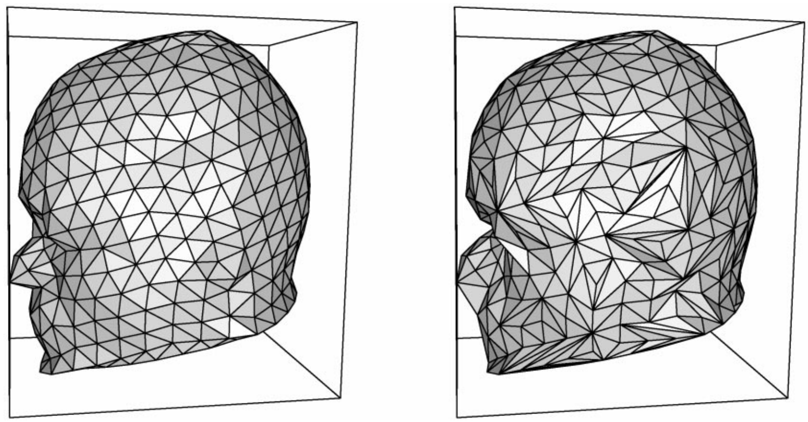

Fig. 3. The scalp before (left) and after (right) applying the tight criterion.

The criterion of minimising absolute mean curvature is given by

$$H_{\text{abs}} = \sum_e |H(e)|$$

and consequently a triangulation of a data set is said to be the triangulation of minimal absolute mean curvature if it is proper and minimises H_{abs}.

This definition implies that we look for triangulations that minimise

$$\int_K (|\lambda_1 + \lambda_2|)\,\mathrm{d}S \tag{2.1}$$

as well as

$$\int_K (|\lambda_1| + |\lambda_2|)\,\mathrm{d}S, \tag{2.2}$$

because

- vertices have no contribution in the integral, since they have zero measure,
- points inside triangles have no contribution, since both principal curvatures are zero,
- edges do have contribution, but only of one principal curvature, the other being zero.

Since only one of the two principal curvatures is nonzero, the following quantities are also minimised:

$$\int_\Omega \max(|\lambda_1|, |\lambda_2|)\,\mathrm{d}S, \quad \int_\Omega \sqrt{|\lambda_1^2 + \lambda_2|^2}\,\mathrm{d}S. \tag{2.3}$$

This criterion gives very promising results (see Fig. 4). This result is obtained using a local swap algorithm with as initial condition the Tight triangulation, obtained in the previous subsection: it almost recovers the good initial result which was used there (see Fig. 3).

3. Other data-dependent triangulations

O'Rourke suggested to define polyhedra of minimum area [18] as the best; however, this criterion may yield strange results, even for very simple data (see Fig. 5). Here the data are drawn from two

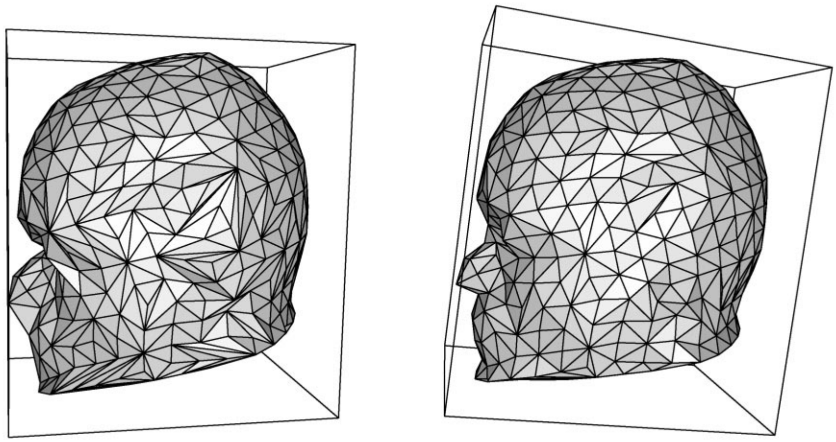

Fig. 4. The scalp before (Tight, left) and after (right) applying the mean curvature criterion.

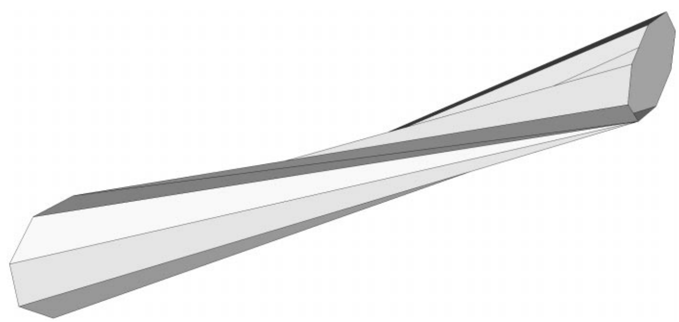

Fig. 5. Two slices skewed cylinder.

parallel circles. Both the Tight triangulation as well as the triangulation of minimal absolute mean curvature lead to the good result in this case, namely the convex triangulation.

There are more criteria which, in a certain sense, are related to minimisation of mean curvature, such as (see [10]).

- angle between normals (ABN). Let $n^{(1)}, n^{(2)}$ be the normals to the two adjacent planes (triangles) with common edge e.

  Then the cost function is the acute angle θ between those two vectors, i.e.,

  $$\mathscr{E}(e) = \theta = \cos^{-1}(A)$$

  with

  $$A = \frac{n^{(1)} \cdot n^{(2)}}{||n^{(1)}||\,||n^{(2)}||}.$$

- jump in normal derivatives (JND): in this case the cost for an edge e is given by

  $$\mathscr{E}(e) = |n_x(n_x^{(1)} - n_x^{(2)}) + n_y(n_y^{(1)} - n_y^{(2)})|$$

  with $(n_x, n_y)^{\mathrm{T}}$ is a unit vector, orthogonal to the direction of the edge e.

The variational criterion, proposed in [21], is connected with the integral absolute mean curvature. They use as energy measure the expression

$$\mathscr{E}(s,\triangle)=\sum_{\{T_j\}}\int_{T_j}\left[\left(\frac{\partial^2}{\partial x^2}s(x,y)\right)^2+2\left(\frac{\partial^2}{\partial x\partial y}s(x,y)\right)^2+\left(\frac{\partial^2}{\partial y^2}s(x,y)\right)^2\right]\mathrm{d}x\,\mathrm{d}y, \tag{3.4}$$

where $\{T_j\}$ denotes the set of triangles of the triangulation $\triangle$, and s is an interpolating cubic C^1 spline using a Clough–Tocher split [15].

Expression (3.4) represents the energy of a so-called 'thin plate'. Due to the specific nature of a 'thin plate', it does not seem convenient to apply the above-mentioned criterion directly to a closed surface (see [2]). However, if we regard it from a geometric point of view, we can see a criterion, that generalises the given one and that can also be applied to a closed surface. Indeed, for a 'thin plate' the functional in (3.4) is approximately equal to $\int_\Omega(\lambda_1^2+\lambda_2^2)\,\mathrm{d}A$ where λ_1,λ_2 are the principal curvatures of a surface. The last functional was used in some work on surface approximation as well. For example, in [14] it is minimised for smoothing an approximating surface. The expression $(\lambda_1^2+\lambda_2^2)$ can be rewritten as follows:

$$\lambda_1^2+\lambda_2^2=\lambda_1^2+\lambda_2^2+2\lambda_1\lambda_2-2\lambda_1\lambda_2=(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2=4H^2-2K.$$

Thus, we have

$$\int_\Omega(\lambda_1^2+\lambda_2^2)\,\mathrm{d}S=4\int_\Omega H^2\,\mathrm{d}S-2\int_\Omega K\,\mathrm{d}S. \tag{3.5}$$

Note that this expression also makes sense in the case of closed surfaces.

We can try to minimise the integral in the left-hand side of (3.5). Only the first integral is subject to minimisation, since the Gauss–Bonnet theorem tells us that the second integral is constant. This latter statement holds for closed data, as well as for functional data.

Consequently, minimisation of the expression $\int_\Omega(\lambda_1^2+\lambda_2^2)\,\mathrm{d}S$ is equivalent to minimisation of the expression $\int_\Omega H^2\,\mathrm{d}S$ and vice versa.

It is interesting to note that this is also equivalent to minimising $|H|$ (albeit in a different norm), and not of $|K|$.

4. Examples for the functional case

In this section, we make a comparison of all the methods described in this article for the special case of functional data, as we also want to compare all methods with the method of [19–21]. Although our final findings and conclusions hold for many more examples we only present here two typical examples.

The test-functions that we use are

$$f^{(1)}(x,y)=\exp(-81/4((x-\tfrac{1}{2})^2+(y-\tfrac{1}{2})^2)),$$

$$f^{(2)}(x,y)=\sin\pi(x-y). \tag{4.6}$$

For $f^{(1)}$ we use a regular 20×30 grid, called $\triangle^{(1)}$, and for $f^{(2)}$ we use $\triangle^{(2)}$ which is one of the standard grids from [13] with 36 points.

Table 1
Results for $f^{(1)}$ on data $\triangle^{(1)}$

	$\|H\|$	$\|K\|$	QS	JND	$\varepsilon(\triangle)$	ε(subdiv)
$\|H\|$	0.392	0.888	48.5	88	$4.91 \cdot 10^{-3}$	$1.39 \cdot 10^{-4}$
$\|K\|$	0.458	0.882	48.0	101	$5.82 \cdot 10^{-3}$	$3.57 \cdot 10^{-4}$
QS	0.495	0.905	47.4	108	$7.67 \cdot 10^{-3}$	$1.66 \cdot 10^{-4}$
JND	0.473	0.888	52.4	82	$4.91 \cdot 10^{-3}$	$1.42 \cdot 10^{-4}$
Del	0.465	0.895	47.7	113	$4.91 \cdot 10^{-3}$	$1.39 \cdot 10^{-4}$

For finding the minimum we use the following strategy.

- We use the swapping procedure of Lawson [15].
- Secondly, in order to get as close to the real minimum as possible, we also use simulated annealing (see [22]). This method of course does not guarantee that we obtain the global minimum.

Furthermore, we measure the quality of a method in two ways.

1. The error as the maximal distance of the resulting triangulation to the original function, in the tables denoted as $\varepsilon(\triangle)$.
2. We apply the subdivision scheme from [24] to the data supplemented with exact derivatives, and again measure the error compared with the original function. In the tables this error is denoted with ε(subdiv).

For the first example, using $f^{(1)}$ on data $\triangle^{(1)}$ we obtain the results shown in Table 1: in the first row the results from minimising absolute mean curvature ($|H|$) are shown, the second row contains the results when applying the tight criterion ($|K|$), the third and fourth are the results by applying the method from [19–21] (QS) resp. the JND criterion from [10] (the ABN criterion gives almost identical results). For the sake of completeness, the last row shows the data for the Delanay triangulation (Del). It should be remarked that, probably due to the fact that the test-function is relatively smooth *and* the grid is rather fine, the use of simulated annealing had no effect on the resulting triangulations, for all criteria.

In the first four columns one finds the resulting energy measured according to the same criteria (and as a consequence the minimum in each column is on the diagonal). In Fig. 6 the four resulting triangulations are shown.

From this we can draw the following conclusions.

- The criteria minimising $|H|$ and JND (as well as ABN) yield almost identical results. The methods of Dyn, Rippa and Levin have the tendency to create longer edges than the first one.
  Of course we have to be cautious with such a remark as this conclusion might depend on the specific subdivision scheme (or any other approximation method) we use.
- The tight criterion gives a triangulation that has the same characteristics as the triangulations of $|H|$ and JND, but less pronounced.
  From Table 1 it is clear that the tight criterion performs considerably worse, which is the same conclusion as we arrived at in the previous section.
- The criterion QS is even more conservative if we consider Fig. 6. The triangulation shows a totally different behaviour than the other three around the peak of the function, near the point

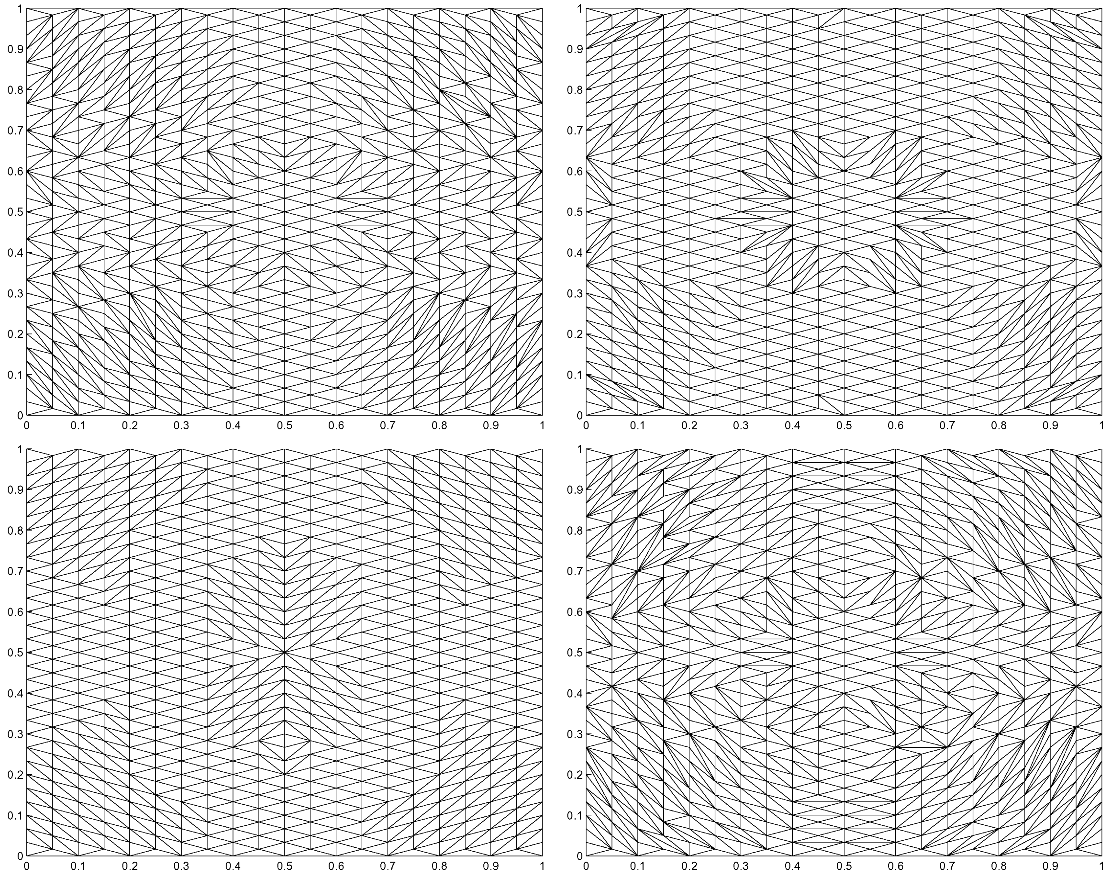

Fig. 6. Best triangulations according to the mean and tight (up) and QS and JND criterion (down) for $f^{(1)}$ in (4.6).

Table 2
Results for $f^{(2)}$ on data $\triangle^{(2)}$, without simulated annealing

	$\|H\|$	$\|K\|$	QS	JND	$\varepsilon(\triangle)$	$\varepsilon(\text{subdiv})$
$\|H\|$	0.370	1.34	164	16.1	$1.18 \cdot 10^{-1}$	$2.51 \cdot 10^{-3}$
$\|K\|$	0.340	1.30	162	11.9	$1.04 \cdot 10^{-1}$	$1.93 \cdot 10^{-3}$
QS	0.496	1.42	157	21.6	$1.19 \cdot 10^{-1}$	$2.84 \cdot 10^{-3}$
JND	0.342	1.30	164	10.6	$7.03 \cdot 10^{-2}$	$9.06 \cdot 10^{-4}$
Delaunay	0.657	1.47	161	27.1	$1.55 \cdot 10^{-1}$	$5.12 \cdot 10^{-3}$

$(\frac{1}{2}, \frac{1}{2})$. It is interesting to note that if we change the energy of (3.4) into

$$\mathscr{E}(s, \triangle) = \sum_{\{T_j\}} \int_{T_j} \left| \frac{\partial^2}{\partial x^2} s(x,y) \cdot \frac{\partial^2}{\partial y^2} s(x,y) - \left(\frac{\partial^2}{\partial x \, \partial y} s(x,y) \right)^2 \right| \mathrm{d}x \, \mathrm{d}y \approx \int_{\Omega} |K| \, \mathrm{d}S,$$

Table 3
Results for $f^{(2)}$ on data $\triangle^{(2)}$, with the use of simulated annealing

	$\|H\|$	$\|K\|$	QS	JND	$\varepsilon(\triangle)$	$\varepsilon(\text{subdiv})$
$\|H\|$	0.331	1.30	163	10.1	$7.03 \cdot 10^{-2}$	$9.06 \cdot 10^{-4}$
$\|K\|$	0.343	1.29	164	11.7	$1.03 \cdot 10^{-1}$	$1.93 \cdot 10^{-3}$
QS	0.411	1.39	157	17.9	$1.04 \cdot 10^{-1}$	$1.93 \cdot 10^{-3}$
JND	0.331	1.30	163	10.1	$7.03 \cdot 10^{-2}$	$9.06 \cdot 10^{-4}$
Delaunay	0.657	1.47	161	27.1	$1.55 \cdot 10^{-1}$	$5.12 \cdot 10^{-3}$

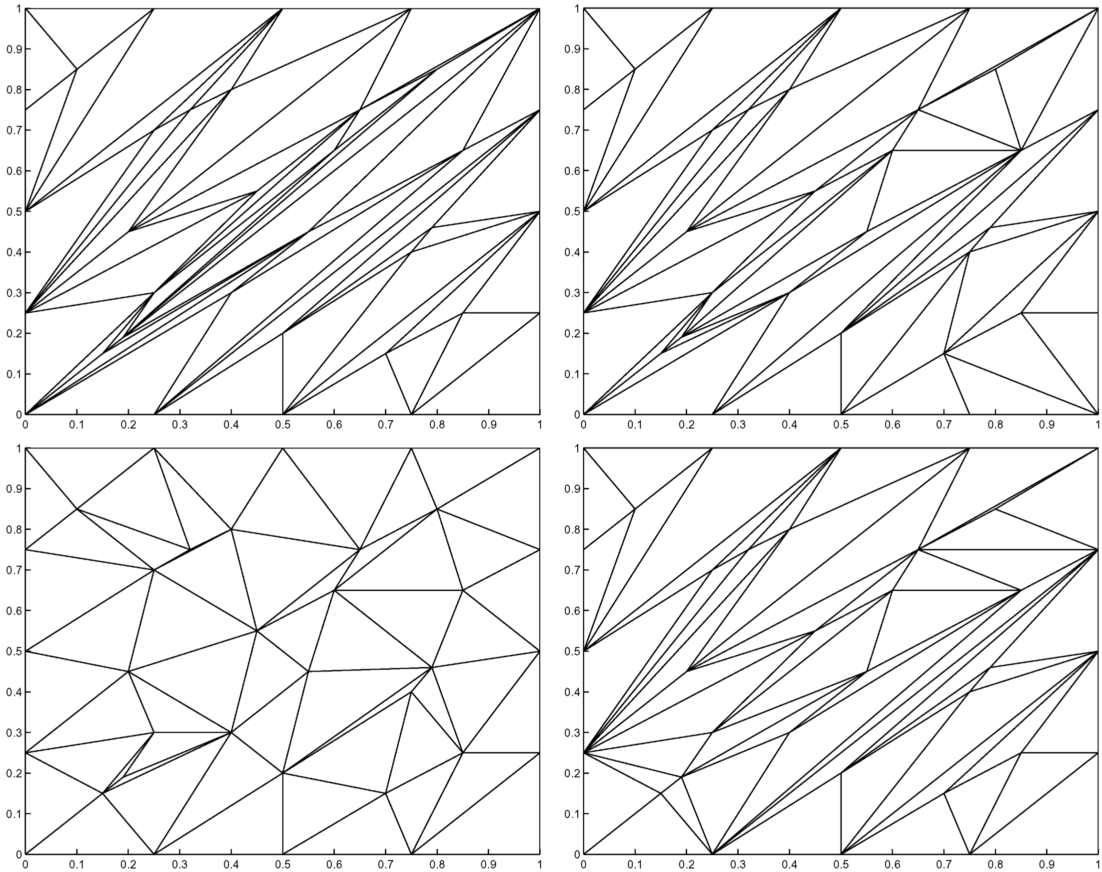

Fig. 7. Best triangulations according to the mean and tight (up) and QS and JND criterion (down) ford $f^{(2)}$ in (4.6 with the use of simulated annealing).

we obtain a triangulation that is very similar to the Tight triangulation. However, we could not find an energy (like in (2.1), (2.2) or (2.3)) that reproduced the triangulation of minimal absolute mean curvature.

- As to the complexity we can say that the 'discrete' criteria (minimising Mean or Gaussian curvature, as well as the JND and ABN criterion) are equivalent. Only QS is a little more expensive, as with every possible swap the change in integral (3.4) has to be evaluated.
- All results in this table could not be improved with the aid of simulated annealing: it seems that the function is so smooth and the grid so fine, that we (almost) found the global minimum.

Tables 2 and 3 and Fig. 7 show the same results, but for data given by $\triangle^{(2)}$ drawn from test-function $f^{(2)}$. The conclusions are the same as in the previous example, except that in this case simulated annealing does have a considerable effect — in general, the simulated annealing algorithm usually works the best by freezing very slowly, but we could not find any strategy which always worked the best.

5. Conclusions

We can conclude that minimising the absolute Gaussian curvature, i.e., the tight criterion, is not very applicable to general datasets. The area criterion has been shown to lead to poor results even in the globally optimal case. The most promising criterion seems to be minimising absolute mean curvature, or (at least almost as good) the two methods ABN and JND from [10]. The method of Quak and Schumaker gives too conservative triangulations, in general. Apart from that, their method is more expensive as all other criteria can be computed at the C^0-level.

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