Blocking Probabilities In A Loss System with Arrivals in Geometrically Distributed Batches and Heterogeneous Service Requirements

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***Abstract*—We analyze a generalization of the classical Erlang loss model. Customers of several types contend for access to a service facility consisting of a finite number of servers. Each customer requires a fixed number of servers simultaneously during an exponentially distributed service time, and is blocked on arrival if this requirement cannot be met. Customers of each type arrive in geometrically distributed batches, while the arrival of batches of each type is governed by a Poisson process. All relevant parameters may be type-dependent. We obtain the steady-state distribution of the number of customers of each type in the system (which turns out to have product form), and the blocking probabilities experienced by each customer type. In addition, we bring to light the connection between the model at hand and a method proposed by Delbrouck for estimating blocking probabilities in an incompletely specified setting.**

I. Introduction

We consider a system consisting of N service units at which K independent streams of customers arrive. Customers in stream i (which will be called type-i customers) require c_i service units simultaneously during an exponentially distributed time of mean μ_i^{-1}, $i = 1, 2, \cdots, K$. Customers who find on arrival that their requirements cannot be met disappear from the system and are said to be blocked.

When customers in each stream arrive according to a Poisson process we are dealing with a model which has been studied quite extensively. In particular, efficient procedures have been developed for computing the distribution of the number of occupied units and the blocking probabilities experienced by the individual streams, see [5], [7], [9] and the references there. Salient features of the model at hand are the product form of the solution of the Markovian state equations and its insensitivity to service time distributions apart from their means.

In this note we discuss a generalization of the above model in which type-i customers arrive in batches of size b_i, where

$$\Pr\{b_i = j\} = (1 - r_i)r_i^{j-1}, \quad j = 1, 2, \ldots,$$

and $0 \le r_i < 1$, while the arrival of batches is governed by a Poisson process with intensity ν_i. The customers in a batch are ordered randomly and dealt with individually. Hence, if a batch cannot be accommodated completely, the part which can be fitted is allowed in and the rest is discarded. Surprisingly, several features of the model in which customers arrive singly are maintained in this generalization. In particular, the Markovian state equations have a product-form solution, and there exist efficient methods for computing the state and blocking probabilities, as will be shown in the Sections II and III.

Apart from its interest *per se* our generalization to batch arrivals with geometrically distributed batch sizes is of interest because it gives us a model which fits in a setting where the arrival streams are characterized by their means and peakedness factors (≥ 1) only. Indeed, it will be shown in Section IV that this fitting approach leads to an algorithm which is completely identical to Delbrouck's [2] scheme for estimating blocking probabilities in such a setting. As an aside we remark that the original idea of fitting a batch Poisson process with geometrically distributed batch sizes in a setting where only mean and peakedness factor are known, seems to be due to Jensen [6].

As Delbrouck's paper [2] generalizes his earlier paper [1] to heterogeneous service requirements, so constitutes the present paper a generalization of [3], which deals with the special case of homogeneous service requirements, that is, $c_i = 1$ and $\mu_i = \mu$ for all i.

II. Steady-State Probabilities

The state of the system will be represented by the vector $\boldsymbol{x} \equiv (x_1, x_2, \cdots, x_K)$, where x_i denotes the number of type-i customers in the service system. The set of all possible states is denoted by Ω, that is,

$$\Omega \equiv \left\{ \boldsymbol{x} \equiv (x_1, x_2, \cdots, x_K) \mid \sum_{i=1}^{K} c_i x_i \le N \right\}.$$

It will be convenient to introduce the notation

$$\boldsymbol{x}_i^{\pm j} \equiv (x_1, \cdots, x_{i-1}, x_i \pm j, x_{i+1}, \cdots, x_K),$$

and

$$\delta_i(\boldsymbol{x}) \equiv \begin{cases} 1 & \text{if } \boldsymbol{x}_i^{+1} \in \Omega \\ 0 & \text{otherwise} \end{cases}.$$

With these conventions, and with $p(\boldsymbol{x})$ denoting the steady-state probability that the system is in state $\boldsymbol{x}$, the balance

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equations for the K-dimensional Markov chain representing the state of the system can be written down at once as

$$\left\{\sum_{i=1}^{K}(\nu_i\delta_i(\boldsymbol{x})+\mu_i x_i)\right\}p(\boldsymbol{x})= \sum_{i=1}^{K}\mu_i(x_i+1)\delta_i(\boldsymbol{x})p(\boldsymbol{x}_i^{+1})+ + \sum_{i=1}^{K}\sum_{j=1}^{x_i}\nu_i(1-r_i\delta_i(\boldsymbol{x}))r_i^{j-1}p(\boldsymbol{x}_i^{-j}),\quad \boldsymbol{x}\in\Omega. \tag{1}$$

Theorem 1: The steady-state distribution is given by

$$p(\boldsymbol{x})=(G(\Omega))^{-1}\prod_{i=1}^{K}p_i(x_i),\quad \boldsymbol{x}\in\Omega, \tag{2}$$

where

$$p_i(x)\equiv\binom{\varrho_i+x-1}{x}r_i^x=\frac{\Gamma(\varrho_i+x)}{\Gamma(\varrho_i)}\frac{r_i^x}{x!},\quad x=0,1,\ldots, \tag{3}$$

with

$$\varrho_i\equiv\nu_i/(\mu_i r_i), \tag{4}$$

and

$$G(\Omega)\equiv\sum_{\boldsymbol{x}\in\Omega}\prod_{i=1}^{K}p_i(x_i). \tag{5}$$

Proof: It is easily seen that if $p(\boldsymbol{x})$, $\boldsymbol{x}\in\Omega$, satisfies (2) – (5), then, for $i=1,2,\cdots,K$,

$$\mu_i(x_i+1)\delta_i(\boldsymbol{x})p(\boldsymbol{x}_i^{+1})=(\nu_i+r_i\mu_i x_i)\delta_i(\boldsymbol{x})p(\boldsymbol{x}) \tag{6}$$

and, for $j=1,2,\cdots,x_i$,

$$\nu_i r_i^{j-1}p(\boldsymbol{x}_i^{-j})=\mu_i\varrho_i\binom{x_i}{j}\binom{\varrho_i+x_i-1}{j}^{-1}p(\boldsymbol{x}). \tag{7}$$

Exploiting the identity

$$\sum_{j=1}^{n}\binom{n}{j}\binom{\varrho+n-1}{j}^{-1}=\frac{n}{\varrho}, \tag{8}$$

see, e.g., [8, p. 154], it subsequently follows that $p(\boldsymbol{x})$, $\boldsymbol{x}\in\Omega$, satisfies the partial balance equations

$$(\nu_i\delta_i(\boldsymbol{x})+\mu_i x_i)p(\boldsymbol{x})= \mu_i(x_i+1)\delta_i(\boldsymbol{x})p(\boldsymbol{x}_i^{+1})+ + \sum_{j=1}^{x_i}\nu_i(1-r_i\delta_i(\boldsymbol{x}))r_i^{j-1}p(\boldsymbol{x}_i^{-j}),\quad \boldsymbol{x}\in\Omega,$$

and hence the global balance equations (1).

We note that, apart from the normalizing factor $(1-r_i)^{\varrho_i}$, the quantities $p_i(x)$, $x=0,1,\ldots$ constitute a negative binomial distribution with parameters ϱ_i and $1-r_i$. Since $\Gamma(\varrho+x)/\Gamma(\varrho)\sim\varrho^x$ for $\varrho\to\infty$, we have

$$\lim_{r\downarrow 0}\binom{\nu/(\mu r)+x-1}{x}r^x=\frac{(\nu/\mu)^x}{x!},$$

so that Theorem 1 reduces to the well-known result for single arrivals if we let $r_i\downarrow 0$ for all i.

The efficient algorithm developed in [7] and [9] for computing the steady-state distribution of the total number of occupied service units in the case of single arrivals is based on a recurrence relation. This relation (as well as the algorithm) can be generalized to the present context, as can be seen from the next theorem. We let $q(j)$ denote the probability that the number of occupied service units in steady state equals j, $j=0,1,\cdots,N$, and $q(j)\equiv 0$ for $j<0$; also, for any real number a, $[a]$ denotes the largest integer less than or equal to a.

Theorem 2: The steady-state probabilities $q(j)$, $j=0,1,\cdots,N$, satisfy the relations

$$jq(j)=\sum_{i=1}^{K}\frac{c_i\nu_i}{\mu_i}\sum_{m=1}^{[j/c_i]}r_i^{m-1}q(j-mc_i),\quad j=1,2,\cdots,N, \tag{9}$$

and

$$\sum_{j=0}^{N}q(j)=1.$$

Proof: We have

$$q(j)=\sum_{\boldsymbol{x}\in\Omega_j}p(\boldsymbol{x}),\quad j=0,1,\cdots,N,$$

where

$$\Omega_j\equiv\left\{\boldsymbol{x}\equiv(x_1,\cdots,x_K)\mid\sum_{i=1}^{K}x_i c_i=j\right\}.$$

Hence, for $j>0$,

$$\begin{aligned} jq(j)&=\sum_{i=1}^{K}c_i\sum_{\boldsymbol{x}\in\Omega_j}x_i p(\boldsymbol{x})\\ &=\sum_{i=1}^{K}c_i\sum_{\boldsymbol{x}\in\Omega_j}\varrho_i\sum_{m=1}^{x_i}\binom{x_i}{m}\binom{\varrho_i+x_i-1}{m}^{-1}p(\boldsymbol{x})\\ &=\sum_{i=1}^{K}(c_i\nu_i/\mu_i)\sum_{\boldsymbol{x}\in\Omega_j}\sum_{m=1}^{x_i}r_i^{m-1}p(\boldsymbol{x}_i^{-m})\\ &=\sum_{i=1}^{K}(c_i\nu_i/\mu_i)\sum_{m=1}^{[j/c_i]}\sum_{\boldsymbol{x}\in\Omega_{j-mc_i}}r_i^{m-1}p(\boldsymbol{x})\\ &=\sum_{i=1}^{K}(c_i\nu_i/\mu_i)\sum_{m=1}^{[j/c_i]}r_i^{m-1}q(j-mc_i), \end{aligned}$$

where we have used (7) and (8).

III. Blocking Probabilities

We will now express the steady-state blocking probabilities B_i experienced by type-i customers, $i=1,2,\cdots,K$, in terms of the probabilities $q(j)$ of the previous section.

Exploiting PASTA (Poisson Arrivals See Time Averages) we notice that the stream of blocked type-i customers constitutes a batch Poisson process in which the arrival rate of batches is ν_i (as in the original stream), but the batch size $\tilde{b}_i$ is distributed as

$$\Pr\{\tilde{b}_i = j\} = \begin{cases} \sum_{m=0}^{N-1} q(m)(1 - r_i^{[\frac{N-m}{c_i}]}) & \text{if } j = 0 \\ (1 - r_i) \sum_{m=0}^{N} q(m) r_i^{[\frac{N-m}{c_i}]+j-1} & \text{if } j > 0 \end{cases}.$$

It follows that the average batch size is given by

$$E\tilde{b}_i = \sum_{j=0}^{\infty} j \Pr\{\tilde{b}_i = j\} = (1 - r_i)^{-1} \sum_{m=0}^{N} q(m) r_i^{[(N-m)/c_i]}.$$

Since $B_i = E\tilde{b}_i / Eb_i$, while $Eb_i = (1 - r_i)^{-1}$, the next result emerges.

Theorem 3 The steady-state blocking probability B_i experienced by type-i customers satisfies

$$B_i = \sum_{m=0}^{N} q(m) r_i^{[(N-m)/c_i]}, \quad i = 1, 2, \cdots, K. \tag{10}$$

We note that the time congestion C_i for type-i customers is given by

$$C_i = \sum_{m=N-c_i+1}^{N} q(m).$$

Hence, letting $r_i \downarrow 0$ in (10) leads to $B_i = C_i$ which, by PASTA, is indeed the result prevailing when arrivals occur singly.

IV. Concluding Remarks

Defining m_i and v_i as the mean and variance, respectively, of the number of type-i customers in the system when $N = \infty$, it is easily seen that

$$m_i = (1 - r_i)^{-1} \nu_i / \mu_i \tag{11}$$

and

$$z_i \equiv v_i / m_i = (1 - r_i)^{-1}. \tag{12}$$

The quantities m_i and z_i are called the *mean* and *peakedness factor*, respectively, of stream i.

Now suppose that we are dealing with a system in which the arrival streams have known means m_i and peakedness factors $z_i \geq 1$, but are otherwise unspecified. From (11) and (12) it then follows that by choosing

$$r_i \equiv 1 - z_i^{-1} \tag{13}$$

and

$$\nu_i \equiv \mu_i m_i z_i^{-1}, \tag{14}$$

we can fit our batch Poisson model of the previous sections into this setting. Subsequently, we can obtain estimates for the blocking probabilities experienced by each stream in the original, incompletely specified model by applying the results of the previous sections. Surprisingly however, a little algebra reveals the equivalence of (9) and (10) on the one hand, and a scheme proposed by Delbrouck [2] for estimating blocking probabilities in this partially specified setting on the other hand. So we have shown that for given means m_i and peakedness factors $z_i \geq 1$, $i = 1, 2, \cdots, K$, arrival streams can be constructed, which, when offered to a common system, experience blocking probabilities that are exactly determined by Delbrouck's recursive scheme. Upon some reflection this result is less surprising than it seems at first sight, since the assumption underlying Delbrouck's scheme is essentially the system of balance equations (6), which is equivalent to the system (1).

Finally we remark that Delbrouck's estimative scheme has been generalized to fixed-routing circuit-switched networks by Dziong and Roberts [4]. It is not difficult to see that our interpretation of Delbrouck's scheme in the single-station case can also be generalized to this wider context.

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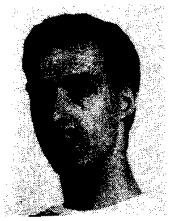

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