Sensitivity & Uncertainty Analysis of Markov-Reward Models

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Key words — **Coverage, Dependability, Markov-reward model, Monte Carlo simulation, Parameter uncertainty, Performability, Sensitivity analysis, Stochastic Petri net.**

Reader aids —
General purpose: Show the importance of sensitivity & uncertainty analysis
Special math needed for explanations: Probability theory, statistics, Markov-reward models
Special math needed to use results: Same
Results useful for: Reliability & performability analysts, and system designers

Summary & Conclusions — **Markov-reward models are often used to analyze the reliability & performability of computer systems. One difficult problem therein is the quantification of the model parameters. If they are available, *eg*, from measurement data collected by manufacturers, they are, a) generally regarded as confidential, and b) difficult to access.**

This paper addresses two ways of dealing with uncertain parameters: 1) sensitivity analysis, and 2) Monte Carlo uncertainty analysis. Sensitivity analysis is relatively fast and cheap but it correctly describes only the local behavior of the model outcome uncertainty as a result of the model parameter uncertainties. When the uncertain parameters are dependent, sensitivity analysis is difficult. We extend the classical sensitivity analysis so that the results conform better to those of the Monte Carlo uncertainty analysis. Monte Carlo uncertainty analysis provides a global view. Since it can include parameter dependencies, it is more accurate than sensitivity analysis. By two examples we demonstrate both approaches and illustrate the effects uncertainty and dependence can have.

1. INTRODUCTION

There is a considerable & growing interest in the use of Markov-reward models for dependability & performability modeling of computer systems [27]. A topic that is receiving little attention is the uncertainty in model parameters and the propagation of this uncertainty to the model outcome. This paper focuses on that topic.

Acronyms[1]

CTMC continuous-time Markov chain
MRM Markov-reward model
SIFT software implemented fault tolerance
SPN stochastic Petri net
SPNP SPN package.

[1]The singular & plural of an acronym are always spelled the same.

Notation

bold implies a matrix
___ implies a vector
X $(X_t,\ t \geq 0)$: continuous-time Markov chain
S $\{1,\ldots,K\}$: state space of X
$\mathbf{Q}$ generator of X, a matrix
$\underline{\pi}^0$ $(\pi_1^0,\ldots,\pi_K^0)$: initial probability vector of X
$\underline{\pi}$ $(\pi_1,\ldots,\pi_K)$: steady-state probability vector of X
$\underline{r}$ $(r_1,\ldots,r_K)$: reward vector
Y $\underline{r}\cdot\underline{\pi}$: steady-state performability of X
$\underline{\Lambda}$ $(\Lambda_1,\ldots,\Lambda_n)$: vector of uncertain model-parameters
$\underline{\lambda}$ $(\lambda_1,\ldots,\lambda_n)$: a realization of $\underline{\Lambda}$
$\underline{\epsilon}$ $(\epsilon_1,\ldots,\epsilon_n)$: vector of very small quantities, $\epsilon_i << 1$
$F_{\underline{\Lambda}}(\underline{\lambda})$ uncertainty distribution of $\underline{\Lambda}$
$\mathbf{Q}_i$ derivative of $\mathbf{Q}$ with respect to Λ_i, a matrix
$F_Y(y)$ induced distribution of Y given $F_{\underline{\Lambda}}$
$\hat{}$ implies a sample value
$\mathcal{I}(\cdot)$ indicator function: $\mathcal{I}(\text{True})=1$, $\mathcal{I}(\text{False})=0$
$\rho(\Lambda_1,\Lambda_2)$ s-correlation between r.v. Λ_1 & Λ_2
$\rho_r(\Lambda_1,\Lambda_2)$ rank-correlation between r.v. Λ_1 & Λ_2.

Other, standard notation is given in "Information for Readers & Authors" at the rear of each issue.

1.1 Markov-Reward Models

Markov-reward models can combine performance & reliability aspects of a system in the following way. A CTMC $X = (X_t,\ t \geq 0)$ is defined on a finite state space $S = \{1,2,\ldots,K\}$. The elements $s \in S$ are *structure states* since every s describes the system structure in terms of the number of operational components. Given $s \in S$, the system performance is described by the reward rate r_s. The rewards for all possible states are denoted by $\underline{r}$.

The CTMC X is completely described by $\mathbf{Q}$ & $\underline{\pi}^0$. Whenever X is aperiodic and irreducible, the *stationary* (steady-state) distribution $\underline{\pi}$ is unique, does not depend on $\underline{\pi}^0$, and is obtained by solving the system of linear equations:

$$\underline{\pi}\mathbf{Q} = \underline{0}, \text{ and } \underline{\pi}\cdot\underline{1}=1. \tag{1}$$

Also,

$$Y = \underline{\pi}\cdot\underline{r}.$$

In this paper,

- the normalization of the probabilities is integrated with $\underline{\pi}Q = \underline{0}$.
- we deal only with uncertainty in the non-zero rates of Q.
- we restrict our attention to steady-state measures; they are also useful for bounding transient measures [23; 25: chapter 6].

From an abstract point of view, we can interpret a Markov-reward model as a function $\mathfrak{F}$ of the model structure M and the model parameters $\underline{\Lambda}$: $Y(\underline{\Lambda}) = \mathfrak{F}_M(\underline{\Lambda})$. When there is uncertainty about the values of $\underline{\Lambda}$ they are treated as r.v. and their (joint) probability distribution quantifies the uncertainty about their values. In general the 'average model outcome' does not equal the 'outcome of the model, evaluated at the average value of the parameters'. The equality holds only when $\mathfrak{F}_M$ is linear in $\underline{\Lambda}$. A complicating factor is the usual s-dependence of the uncertainties.

Assumptions (Markov-reward models)

1. The number of states in the Markov-reward models is finite.
2. The Markov chain is aperiodic and irreducible.
3. We address uncertainty only in the non-zero rates of the Q-matrix.
4. $Q(\underline{\Lambda})$ is linear in the components $\Lambda_1,\ldots, \Lambda_n$ of $\underline{\Lambda}$.[2] ◄

1.2 Propagation of Uncertainties

The effect of uncertainty about the model parameters on the model predictions can be analyzed in three ways: sensitivity analysis, uncertainty analysis, perturbation analysis.

Sensitivity Analysis

$\mathfrak{F}_M$ is evaluated for a specific value $\underline{\lambda}$ (most likely scenario), yielding $Y(\underline{\lambda})$. Then, $\partial\mathfrak{F}_M/\partial\underline{\Lambda}$ is evaluated at $\underline{\Lambda} = \underline{\lambda}$, yielding insight into how $Y(\underline{\lambda})$ changes when $\underline{\lambda}$ is changed by $\underline{\epsilon}$. The interpretation of the derivatives alone is difficult if there is not indication what values for $\underline{\epsilon}$ are reasonable. Furthermore this approach is accurate only when ϵ is small enough. Still, it is often seen in the Markov-reward modeling field [6, 10, 14]; these 3 papers emphasize calculation of the derivatives (sensitivities). To a certain extent the sensitivities are combined with knowledge about the uncertainty of model parameters in [4].

Uncertainty Analysis

An $F_{\underline{\Lambda}}$ is associated with $\underline{\Lambda}$: $F_{\underline{\Lambda}}(\underline{\lambda}) = \Pr\{\underline{\Lambda} \le \underline{\lambda}\} = \Pr\{(\Lambda_1 \le \lambda_1)\cap\ldots\cap(\Lambda_n \le \lambda_n)\}$ where the $\Lambda_1,\ldots,\Lambda_n$ can be s-dependent. Because $\underline{\Lambda}$ is a r.v., $Y(\underline{\Lambda})$ is a r.v. as well. The aim now is to derive $F_Y(y) = \Pr\{Y \le y\}$. In general, analytic methods cannot be used for usual Markov-reward models. One therefore resorts to Monte Carlo simulation [9, 15, 16, 20, 24].

Perturbation Analysis

One tries to obtain insight into the deviation $|Y(\underline{\Lambda}) - Y(\tilde{\underline{\Lambda}})|$ given a bound on the perturbation $\|\tilde{\underline{\Lambda}} - \underline{\Lambda}\|$[12]. In the context of Markov-reward models, Van Dijk [8] recently obtained some interesting results. However, it seems difficult to apply them to general models, and to include s-dependencies between the uncertainties. We do not treat perturbation analysis in this paper. ◄

We extend our earlier analytic work [13] in which we addressed only a 1-dimensional case. Now we address sensitivities with respect to more than 1 model parameter, and extend the use of sensitivities to propagate uncertainties (section 2). Section 3 discusses s-dependence between uncertainties, and section 4 provides a concise overview of Monte Carlo uncertainty analysis. Section 5 applies the methods to the simple Markov-reward model treated before [13]. Section 6 applies these methods to a model of a fault-tolerant flight control computer, in order to investigate the impact of coverage uncertainty.

2. SENSITIVITY ANALYSIS

This section evaluates the sensitivity of $Y(\underline{\Lambda})$ with respect to $\underline{\Lambda}$. A Taylor-series expansion for $Y = \underline{r}\cdot\underline{\pi}$ as a function of $\underline{\Lambda}$ is derived. As a special case we address the 1-dimensional sensitivity of Y with respect to Λ. Then the sensitivity results are combined with knowledge about the uncertainty of the input parameters.

2.1 Derivation of Sensitivities

Assumption #4 greatly simplifies the analysis. We can write,

$$Q(\underline{\Lambda}) = Q_0 + \sum_{i=1}^{n} \Lambda_i Q_i,$$

the $Q_0, Q_1,\ldots,Q_n$ are constants.

This implies that the first-order derivative of $Q(\underline{\Lambda})$ with respect to Λ_i equals Q_i and that all second- and higher-order derivatives equal $\mathbf{0}$. Differentiate (1) with respect to Λ_i:

$$\frac{\partial\underline{\pi}(\Lambda)}{\partial\Lambda_i} Q(\underline{\Lambda}) + \underline{\pi}(\underline{\Lambda})Q_i = 0, \text{ or,}$$

$$\frac{\partial\underline{\pi}(\underline{\Lambda})}{\partial\Lambda_i} = -\underline{\pi}(\underline{\Lambda})Q_i Q^{-1}(\underline{\Lambda}), \quad (i=1,2,\ldots,n). \tag{2}$$

$\partial Y(\underline{\Lambda})/\partial\Lambda_i$ is then obtained as the inner product $[\partial\underline{\pi}(\underline{\Lambda})/\partial\Lambda_i]\cdot\underline{r}$. Further differentiation of (2) yields for general $k, k_1, k_2, \ldots, k_n \in \mathbf{N}$ with $\Sigma_{i=1}^{n} k_i = k$, that:

[2]This simplifies the analysis, and is generally true because its entries signify failure rates, repair rates *etc*, which are usually linear combinations of the per-component failure and repair rates.

$$\frac{\partial^k \underline{\pi}(\underline{\Lambda})}{\partial \Lambda_1^{k_1} \dots \partial \Lambda_n^{k_n}} = (-1)^k \underline{\pi}(\underline{\Lambda}) \sum_{\{k!\}} (\boldsymbol{Q}_1 \boldsymbol{Q}^{-1}(\underline{\Lambda}))^{k_1} \cdot \dots \cdot (\boldsymbol{Q}_n \boldsymbol{Q}^{-1}(\Lambda))^{k_n}. \quad (3)$$

The sum is over all $k!$ different orders in which one can multiply the k_1 matrices $(\boldsymbol{Q}_1\boldsymbol{Q}^{-1}(\underline{\Lambda}))$, the k_2 matrices $(\boldsymbol{Q}_2\boldsymbol{Q}^{-1}(\underline{\Lambda}))$, ..., and the k_n matrices $(\boldsymbol{Q}_n\boldsymbol{Q}^{-1}(\underline{\Lambda}))$. For matrices that are not linear in $\underline{\Lambda}$, eq (3) becomes intrinsically more complicated. Using (3) we obtain the Taylor-series expansion of $\underline{\pi}(\underline{\Lambda})$ around $\underline{\Lambda} = \underline{\lambda}$:

$$\underline{\pi}(\underline{\lambda} + \underline{\epsilon}) = \underline{\pi}(\underline{\lambda}) + \sum_{k=1}^{\infty} \frac{1}{k!} \cdot \sum_{\Omega(k)} (\epsilon_{i_1} \cdot \dots \cdot \epsilon_{i_k}) \cdot \frac{\partial^k \underline{\pi}}{\partial \Lambda_{i_1} \dots \partial \Lambda_{i_k}} (\underline{\lambda});$$

$\Omega(k)$ = the set of all n^k k-tuples $(i_1,\dots,i) \in \{1,\dots,n\}^k$.

Finally, $Y(\underline{\lambda}+\underline{\epsilon}) = \underline{r} \cdot \underline{\pi}(\underline{\lambda}+\underline{\epsilon})$.

2.2 1-Dimensional Sensitivity Analysis

Let $n=1$ (only 1 parameter, Λ); then $\boldsymbol{Q}(\Lambda) = \boldsymbol{Q}_0 + \Lambda \boldsymbol{Q}_1$. Then (3) becomes:

$$\frac{\partial^k \pi(\Lambda)}{\partial \Lambda^k} = (-1)^k \cdot k! \cdot \underline{\pi}(\Lambda) \cdot (\boldsymbol{Q}_1 \boldsymbol{Q}^{-1}(\Lambda))^k, \; k=1,2,\dots .$$

The Taylor-series expansion of $\underline{\pi}$ around $\Lambda = \lambda$, is:

$$\underline{\pi}(\lambda + \epsilon) = \sum_{k=0}^{\infty} \frac{\partial^k \underline{\pi}(\lambda)}{\partial \Lambda^k} \cdot \frac{\epsilon^k}{k!} = \underline{\pi}(\lambda) \cdot \sum_{k=0}^{\infty} (-\epsilon \boldsymbol{Q}_1 \boldsymbol{Q}^{-1}(\lambda))^k = \underline{\pi}(\lambda) (\boldsymbol{I} + \epsilon \boldsymbol{Q}_1 \boldsymbol{Q}^{-1}(\lambda))^{-1}. \quad (4)$$

This is an alternating series for which the last equality in (4) holds if, for $k=1,2,\dots$,

$$\| (\epsilon \boldsymbol{Q}_1 \boldsymbol{Q}^{-1}(\lambda)^{k+1} \| < \| (\epsilon \boldsymbol{Q}_1(\lambda) \boldsymbol{Q}^{-1}(\lambda))^k \|, \text{ and,}$$

$$\lim_{k \to \infty} (\| (\epsilon\, \boldsymbol{Q}_1(\lambda) \boldsymbol{Q}^{-1}(\lambda))^k \|) = 0.$$

These two requirements give the radius of convergence of the series expansion:

$$|\epsilon| < 1 / \|\boldsymbol{Q}_1\, \boldsymbol{Q}^{-1}(\lambda)\|.$$

For small models we can often derive $\underline{\pi}(\underline{\Lambda})$ symbolically, and the Taylor-series expansion is not needed at all. For moderate-size models with one uncertain parameter we can use the exact expression (4). For larger models the first few terms of this alternating Taylor-series expansion can be taken. The error made in summing only the first k elements is smaller than the absolute value of element $(k+1)$.

2.3 Using Sensitivities to Propagate Uncertainties

In classical sensitivity analysis, the propagation of the uncertainties is based on the first-order Taylor-series expansion of the model. Based on this approximation the mean & variance of the model predictions $\underline{Y}$ are calculated. Two directions of generalization can be considered:

- improve the approximation of the model by taking more terms of its Taylor-series expansion.
- calculate more moments of $\underline{Y}$ than the mean & variance.

The classical sensitivity analysis proceeds by approximating the real model, $Y(\underline{\Lambda}) = \underline{r} \cdot \underline{\pi}(\underline{\Lambda})$, by its first-order Taylor-series approximation around $\underline{\Lambda} = \mathrm{E}\{\underline{\Lambda}\}$, *ie*, for all $\underline{\lambda} \in \mathbf{R}^n$ one approximates:

$$\underline{\pi}(\underline{\lambda}) \approx \underline{\pi}(\mathrm{E}\{\underline{\Lambda}\}) + \sum_{i=1}^{n} \frac{\partial \underline{\pi}(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i} \cdot (\underline{\lambda} - \mathrm{E}\{\underline{\Lambda}\}).$$

Let $\mathrm{E}\{\underline{\pi}\} = (\mathrm{E}\{\pi_1\},\dots,\mathrm{E}\{\pi_K\})$.

$$\mathrm{E}\{\underline{\pi}(\underline{\Lambda})\} \approx \underline{\pi}(\mathrm{E}\{\underline{\Lambda}\}), \; \mathrm{E}\{Y(\underline{\Lambda})\} \approx \underline{r} \cdot \underline{\pi}(\mathrm{E}\{\underline{\Lambda}\}).$$

The linearity of the original model around $\underline{\Lambda} = \mathrm{E}\{\underline{\Lambda}\}$ determines how good this approximation is. For the variance of Y, use:

$$\mathrm{Var}\{Y\} = \mathrm{E}\{(Y(\underline{\Lambda}) - Y(E\{\underline{\Lambda}\}))^2\}$$
$$\approx \sum_{i=1}^{n} \left(\frac{\partial Y(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i} \right)^2 \cdot \mathrm{Var}\{\Lambda_i\}$$
$$+2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \frac{\partial Y(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i} \cdot \frac{\partial Y(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_j} \cdot \mathrm{Cov}\{\Lambda_i, \Lambda_j\}. \quad (5)$$

In (5), the sensitivity of performability measures with respect to model parameters is combined with the uncertainty of these parameters to show the effect of parameter uncertainties. Eq (5) shows that s-dependence between the input parameters influences the variance in the output measure. If the uncertain input parameters are s-independent, the second sum vanishes.

When one tries to approximate higher-order moments of the model prediction by the first-order Taylor-series expansion, many higher-order cross-product moments appear. These terms disappear only when the r.v. are s-independent. In that case, the skewness of the distribution of $\underline{\pi}(\underline{\Lambda})$ is:

$$\mathrm{E}\{(\pi_s(\underline{\Lambda}) - \mathrm{E}\{\pi_s(\underline{\Lambda})\})^3\} \approx \sum_{i=1}^{n} \left(\frac{\partial \pi_s(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i} \right)^3 \cdot \mathrm{E}\{(\Lambda_i - E\{\Lambda_i\})^3\}.$$

Now consider a second-order Taylor-series approximation for the model $Y(\underline{\Lambda}) = \underline{\pi}(\underline{\Lambda}) \cdot \underline{r}$. The mean value of the approximation for $Y(\underline{\Lambda})$ depends on the covariances between the parameters:

$$\mathrm{E}\{Y(\underline{\Lambda})\} \approx Y(\mathrm{E}\{\underline{\Lambda}\}) + \tfrac{1}{2}\sum_{i=1}^{n} \frac{\partial^2 Y(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i^2}\cdot \mathrm{Var}\{\Lambda_i\}$$

$$+ \sum_{i=1}^{n}\sum_{j=i+1}^{n} \frac{\partial^2 Y(\mathrm{E}\{\underline{\Lambda}\})}{\partial \Lambda_i \partial \Lambda_j}\cdot \mathrm{Cov}\{\Lambda_i,\Lambda_j\}. \quad (6)$$

Using this quadratic approximation, the variance and all higher-order moments of Y depend on many higher-order cross-moment terms. It is impractical to consider these expressions.

Summarizing, sensitivity analysis can be used only for analyzing local behavior. It is difficult to account for *s*-dependencies between parameters. For many Markov-reward models, sensitivities are difficult or very costly to obtain.

3. UNCERTAINTY DEPENDENCIES

In (5) & (6) the covariances between the parameters are important in calculating the propagation of the uncertainties on the parameter values to the model predictions. *s*-Dependencies between uncertainties in the model parameters can exist because they —

- are physically induced (assumed to be captured by the model itself), or
- stem from the fact that parameter choices are based on common knowledge or information sources (this should be reflected in the choice of $F_{\underline{\Lambda}}$).

These *s*-dependencies are important when parameters are not estimated from measurements but merely guessed-at by experts [22].

s-Dependencies between r.v. can be measured by *s*-correlations or rank-correlations. The $\rho(X,Y)$ measures the degree of linear relationship between X & Y:

$$\rho(X,Y) = \mathrm{Cov}\{X,Y\}/\sqrt{\mathrm{Var}\{X\}\cdot\mathrm{Var}\{Y\}}. \quad (7)$$

Let F_X & F_Y be the Cdf's of X & Y respectively. The rank-correlation,

$$\rho_r(X,Y) = \rho(F_X(X),F_Y(Y)),$$

measures the degree of monotone relationship between X & Y [19].

Two model parameters guessed by the same expert should be assumed to be *s*-correlated because the expert tends to be either on the high side, or on the low side for both guesses. The idea is to model the information sources by so-called latent variables $\underline{\mathcal{L}}$, and to couple the uncertain model parameters $\underline{\Lambda}$ to these latent variables $\underline{\mathcal{L}}$. The degree of coupling depends on the degree-of-subjectivity of the information source on which the uncertainty distribution of the parameters is assessed; *ie*, for parameters Λ_i whose distribution is obtained from information source $\mathcal{L}_j$, more subjectivity in $\mathcal{L}_j$ means that $\rho_r(\Lambda_i,\mathcal{L}_j)$ is higher. Two variables each having ρ_r with $\mathcal{L}_j$ have an approximate rank-correlation ρ_r^2 between each other (the exact value depends on the bivariate distribution) [22].

From a computational point of view it is important that, conditional upon the value of $\underline{\mathcal{L}}$, the uncertainty distributions of the parameters are *s*-independent. Consequently only 2-dimensional probability distributions may be used for sampling *s*-correlated r.v.

4. MONTE CARLO UNCERTAINTY ANALYSIS

For most practical problems, the Cdf $F_Y(y) = \Pr\{Y \le y\}$, induced by $F_{\underline{\Lambda}}(\underline{\lambda})$, cannot be obtained analytically. But this distribution can be estimated by Monte Carlo simulation. Take m samples $\underline{\lambda}_1,\ldots,\underline{\lambda}_m$ from $F_{\underline{\Lambda}}(\underline{\lambda})$; evaluate the model for these samples. The result is, for $i=1,\ldots,m$: $y_i = Y(\underline{\Lambda}:=\underline{\lambda}_i)$. The sample mean $\hat{\mu}\{y\}$, standard deviation $\hat{\sigma}\{y\}$, and quantiles of Y, can be estimated in a standard way [18: chapter II.16; 28].

4.1 Probability Distributions and Entropy

An important question in uncertainty analysis is how to choose a particular uncertainty distribution from the incomplete knowledge that one has about the uncertain variable. We follow the maximum-entropy approach [11,21,22]. If one only knows, *eg*, that the uncertain parameter takes values in $[a,b]$, then the maximum-entropy distribution is uniform on $[a,b]$, denoted by $U[a,b]$. Similarly, if one knows only the order of magnitude of an uncertain parameter, then just take the logs of the boundaries; *eg*, it is known that $\log_{10}$(parameter) is in [-4, -2], then the maximum-entropy distribution is the log-uniform distribution on $[10^{-4}, 10^{-2}]$ denoted $\mathrm{LU}[10^{-4}, 10^{-2}]$.

4.2 Diagonal-Band Distribution

Drawing a realization λ for a 1-dimensional r.v. Λ with Cdf F_Λ can be done by the inverse-transform technique. For the generation of *s*-dependent r.v., the basic idea is to generate *s*-dependent $U[0,1]$ numbers $u_1,\ldots,u_n$ and then apply the inverse transforms for each of the variables individually. This can be done in many ways, because a joint distribution is not entirely characterized by its marginals and *s*-correlations. For instance one could use *s*-dependent *s*-normal r.v. for the generation of *s*-dependent uniform r.v. [17, 20]. Alternatively, interesting from a theoretical view, use that joint distribution for the generation of *s*-dependent uniforms which has maximum-entropy, given the specified *s*-correlations [22: chapter 2]. For our calculations we used the computationally very attractive diagonal-band distributions [7; 22: chapter 3].

The diagonal-band distribution $G_d(U,V)$ is a 1-parameter bivariate distribution defined on the unit square $[0,1]\times[0,1]$. Between parameter $d \in [-1,1]$ and the *s*-correlation $\rho(U,V)$, the following relation exists:

$$\rho(U,V) = \mathrm{sign}(d)\cdot(|d|^3 - 2|d|^2 + 1).$$

A realization (u,v) for (U,V) having a diagonal-band distribution G_d is calculated as follows. Let u be a realization of the uniform(0,1) r.v. U, and v' a realization from a uniform$(u-d, u+d)$ distribution. Then select v as follows:

$v = -v'$, if $v' < 0$,

$v = 2 - v'$, if $v' > 1$,

$v = v'$, otherwise.

5. EXAMPLE: 1-UNIT, 1-STANDBY SYSTEM

This was addressed in [13]. The performability of the system was studied modeling it as a 5-state CTMC. A Monte-Carlo uncertainty analysis and a classical sensitivity analysis assessed the effect of the uncertainties in the failure & repair rates. We apply the extended sensitivity analysis derived in this paper as (6). We summarize the main data. The performability $Y = \underline{r} \cdot \underline{\pi}$ can be obtained in closed-form as:

$$Y = \Phi \cdot (0.75\Lambda_A + 0.25\Lambda_B + \Phi)/[2\Lambda_A \cdot \Lambda_B + \Phi \cdot (\Lambda_A + \Lambda_B + \Phi)];$$

Notation

Λ_A, Λ_B failure rate of unit $[A, B]$
Φ repair rate.

Consider scenarios S4 & S5 from [13]. Λ_A & Λ_B have LU(10^{-4}, 10^{-1}) and Φ has $U(0.5, 1.5)$ uncertainty distributions. In one scenario the failure rates are s-independent and in the other one completely s-dependent:

$$\rho_r(\Lambda_A, \Lambda_B) = 1 \rightarrow \mathrm{Cov}\{\Lambda_A, \Lambda_B\} = 5.147 \cdot 10^{-4}.$$

The repair rate is always s-independent of the failure rates.

TABLE 1
The Mean Performability, E{Y}, Calculated in 3 Ways

failure rate	sensitivity analysis: classical	sensitivity analysis: extended	Monte Carlo uncert. analysis
independent	0.9856	0.9849	0.9848
dependent	0.9856	0.9844	0.9839

Table 1 presents the new results and compares them to those previously calculated. The classical sensitivity analysis overestimates the mean performability. The extended sensitivity analysis, based on a quadratic approximation of the model, gives accurate results for the s-independent failure rates. It gives improved results, indicating the right direction of change, for the s-dependent failure rates.

6. COVERAGE UNCERTAINTY IN THE SIFT COMPUTER

We discuss a sensitivity analysis and a Monte Carlo uncertainty analysis of a Markov-reward model of the SIFT computer system [26, 29]. In particular, we focus on the uncertainty in the coverage factors of failures. It is not only very difficult to obtain accurate coverage factors of real systems, but these factors highly influence the system dependability & performability [1 - 3].

6.1 SIFT System

The SIFT system was designed for highly dependable inflight control [26, 29]. SIFT consists of multiple processors, interconnected by multiple busses. The processors are assigned dynamically to form triple-modular and quintuple-modular redundant groups. The processors in a group operate, in a loosely coupled way, redundantly on the same task. When all processors in a group finish their task, the processors synchronize and compare results. Processors vote on each other's results, and a processor is configured out of its group when a majority of a group determines it has failed. In a similar way the processors also vote on the correctness of busses.

6.2 Stochastic Petri-Net Model for the SIFT System

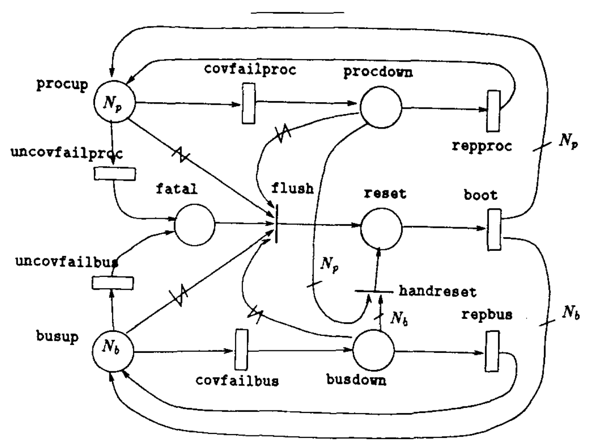

Figure 1. SPN Model of the Dependability of the SIFT Computer

The Markov-reward model of the SIFT computer uses the SPN as given in figure 1 (we used the package SPNP for our analysis [5,6]). The system consists of a configuration of N_p processors and N_b busses. Tokens in the places procup and busup represent, respectively, processors and busses that are up. These operational components can fail via the transitions covfail* and uncovfail* with component failure rates of λ_*^c & λ_*^u, respectively.[3]. Covered failures (transitions covfail*) result in the

[3] The * notation signifies either proc or bus throughout the sentence in which it is used.

movement of one token from place *up to place *down. Uncovered failures (transitions uncovfail*) result in the movement of a single token to place fatal. Once a token enters this place, all tokens are flushed into place reset via the immediate transition flush. Then the whole system is down, and after a manual repair (transition boot with rate β), the system becomes operational again.

When processors or busses have failed in a covered way, they are repaired via transitions rep* with rate $\mu_\bullet$. When all processors & busses have failed in this way and the special situation occurs that all tokens are in places *down, the system is fully down. It requires manual repair established via the immediate transition handreset. This flushes all tokens from places *down to place reset. The coverage-factor is:

$c^* = \lambda^c_*/(\lambda^c_* + \lambda^u_*)$ = Pr{failure is covered|the failure occurs}.

Numerical Values

$N_p = 10,\ N_b = 5,$

$\mu_p = 1.0,\ \mu_b = 1.0,$

$\beta = 0.1.$

Assumptions

1. The SIFT computer is operational whenever at least 2 processors and 2 busses are operational.
2. $r^d_{(n_p,n_b)} = \mathcal{I}$(system is operational).
3. $r^d_{(n_p,n_b)} = \min(n_p,\ n_b)$.
4. Failure rates, Λ^c_* & Λ^u_*, are r.v.

Notation

s $(n_p,n_b) \in S = \{0,\ldots,N_p\} \times \{0,\ldots,N_b\}$

Y_d, Y_p [dependability, performability] measure

Y_d is obtained by using $\underline{r}^d$; Y_p is obtained by using $\underline{r}^p$.

TABLE 2
Uncertainty Assumptions on the Marginal Distributions

	distribution	mean	std.dev.	coeff. of var.
Λ^c_p, Λ^c_b	LU$[10^{-4}, 10^{-2}]$	$2.150\cdot 10^{-3}$	$2.497\cdot 10^{-3}$	1.162
Λ^u_p, Λ^u_b	LU$[10^{-6}, 10^{-4}]$	$2.150\cdot 10^{-5}$	$2.497\cdot 10^{-5}$	1.162

Table 2 shows that Λ^c_* has a distribution with support on $[l_\bullet, r_\bullet]$, and that Λ^u_* has a distribution with support on $[l_\bullet/\alpha_\bullet,\ r_\bullet/\alpha_\bullet]$ where $\alpha_\bullet = 100$. This implies, for $c_\bullet$: If $\rho_r(\Lambda^c_*,\Lambda^u_*) = 1$, then $\Lambda^c_* \equiv \alpha_\bullet \Lambda^u_*$, $c_\bullet$ is fixed, and equals:

$c_\bullet = \alpha_\bullet/(\alpha_\bullet + 1) = 100/101 = 0.990099.$

Whenever $\rho_r(\Lambda^c_*,\Lambda^u_*) = 0$, the $c_\bullet$ varies between the maximum value $\alpha_\bullet/(\alpha_\bullet + 1)$ and the minimum value $\alpha_\bullet/(\alpha_\bullet + (r_\bullet/l_\bullet))$, which in this case equals 0.5. The precise distribution of $c_\bullet$ depends on the s-correlation between the failure rates and can be calculated by numerical integration or by Monte Carlo simulation.

We consider 3 different scenarios, T1 - T3, that differ only in the degree of coupling between the failure rates. We introduce 3 latent variables: $\mathcal{L}^p_{c/u}$ and $\mathcal{L}^b_{c/u}$, which couple the covered & uncovered failure rates for processors & busses respectively, and $\mathcal{L}_{p,b}$ which couples $\mathcal{L}^p_{c/u}$ & $\mathcal{L}^b_{c/u}$. Table 3 presents the amount of coupling.

TABLE 3
Rank-Correlations Between the s-Dependent r.v.

	T1	T2	T3
$\rho_r(\mathcal{L}_{p,b},\mathcal{L}^p_{c/u}), \rho_r(\mathcal{L}_{p,b},\mathcal{L}^b_{c/u})$	0.00	0.70	0.70
$\rho_r(\mathcal{L}^p_{c/u},\Lambda^c_p), \rho_r(\mathcal{L}^p_{c/u},\Lambda^{uu}_p), \rho_r(\mathcal{L}^b_{c/u},\Lambda^c_b), \rho_r(\mathcal{L}^b_{c/u},\Lambda^u_b)$	0.00	0.70	1.00

6.3 Sensitivity Analysis

We want to obtain the sensitivities $\partial Y/\partial c_p$ and $\partial Y/\partial c_b$. A difficulty is that Y is calculated as a function of the failure rates λ^c_* and λ^u_*, and not of the $c_\bullet$. From $c_\bullet = \lambda^c_*/(\lambda^c_* + \lambda^u_*)$ one can use the chain rule, and obtain:

$$\partial Y/\partial c_\bullet = (\partial Y/\partial \lambda^c_*) \cdot [(\lambda^c_* + \lambda^u_*)^2/\lambda^u_*] - (\partial Y/\partial \lambda^u_*) \cdot [(\lambda^c_* + \lambda^u_*)^2/\lambda^c_\bullet]. \quad (8)$$

The model is evaluated for the average-parameter scenario from table 2 using the SPNP package. It also calculates the first order derivatives:

$\partial Y_d/\partial c_p = 0.217925,\ \partial Y_p/\partial c_p = 1.087027;$

$\partial Y_d/\partial c_b = 0.108959,\ \partial Y_p/\partial c_b = -0.563338.$

An increase in processor coverage increases the system dependability & performability. An increase in the bus coverage yields a higher system dependability but, surprisingly, a lower system performability. An increase of the bus coverage implies that failures of busses more often proceeds along the line of individual failures, than along the line of a single, non-covered catastrophic failure. The $\underline{r}^p$ is such that the latter, however, is better from a performability point of view.

The uncertainties in the failure rates are propagated to Y_d & Y_p by combining the assumptions on the uncertainties of the failure rates with the sensitivities, *ie*, by using a linear approximation for $\underline{\pi}(\underline{\lambda})$. Table 4 shows the results for T1, T2, T3. The approximations for the mean values are s-independent of the degree of s-dependence between the r.v. The standard deviations increase appreciably as a function of the s-dependence. It might seem difficult to calculate the standard deviations because the covariances in (7) have to be calculated. This can be done by numerical integration or Monte Carlo simulation. A separate calculation is not necessary because these values are delivered almost automatically by the following Monte-Carlo uncertainty analysis.

TABLE 4
Uncertainty Propagation by Classical Sensitivity Analysis

	$E\{Y_d\}$	$E\{Y_p\}$	$\sigma\{Y_d\}$	$\sigma\{Y_p\}$
T1	0.996792	4.973180	0.002768	0.018684
T2	0.996792	4.973180	0.002952	0.022211
T3	0.996792	4.973180	0.003137	0.025917

6.4 Uncertainty Analysis

For T1 - T3 we generated $m=10^4$ samples each, using the Unicorn package [9]. Table 5 presents the results for Y_d & Y_p. The distributions of Y_d & Y_p are skewed to the left: the 50% quantile is much closer to the 95% quantile than to the 5% quantile. Going from T1 via T2 to T3, we observe 2 trends in the samples:

- Although the 50% quantile of Y_d & Y_p tends to increase slightly as a function of the *s*-correlation between the failure rates, *viz*, as a function of $c_\bullet$, the mean of Y_d & Y_p is almost the same for the 3 scenarios.
- The standard deviations of Y_d & Y_d increase 13% - 37% as a function of the *s*-correlation between the failure rates. This increase agrees with the fact that the 5% quantile decreases and the 95% quantile increases if the *s*-correlations increase.

TABLE 5
Uncertainty Propagation by Uncertainty Analysis

	Y_d			Y_p		
quantile	T1	T2	T3	T1	T2	T3
5%	0.99090	0.99056	0.99032	4.93733	4.92978	4.92115
50%	0.99765	0.99779	0.99808	4.97690	4.97936	4.98346
95%	0.99968	0.99971	0.99975	4.99582	4.99717	4.99786
mean	0.99679	0.99678	0.99681	4.97296	4.97300	4.97336
std.dev.	0.00277	0.00296	0.00313	0.01881	0.02245	0.02581

A final remark is devoted to the rank-correlations between the coverage factors and the dependability measure for the three scenarios. In the non *s*-correlated and intermediately *s*-correlated scenarios there exists a positive correlation (of about 0.20, ..., 0.57) between Y_d and C_p & C_b. In T3 with the high *s*-correlations (where also the coverage factors are the highest) these *s*-correlation values are negative (about −0.25). So, in T3 better coverage means a worse system: a larger fraction of the time it is in an inferior, albeit working, state. A different reward structure could alter this effect.

6.5 Evaluation of SIFT Uncertainty Study

When we compare the sensitivity-analysis results with the Monte-Carlo-uncertainty analysis results, we see that for the mean and standard deviation of the steady-state dependability & performability, the approximations of the sensitivity analysis are good. Of course, the Monte Carlo uncertainty analysis gives more results, the most surprising one that with a low *s*-dependence between the failure rates, better coverage implies a better system, whereas with a high *s*-dependence better coverage implies a worse system.

ACKNOWLEDGMENT

We thank K.S. Trivedi and G. Ciardo for making the SPNP software available, and R. van Dorp for developing the Unicorn software package. The anonymous reviewers improved the clarity & conciseness of the paper.

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Manuscript received 1994 April 20.

IEEE Log Number 94-09301

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