



# A Hamiltonian vorticity–dilatation formulation of the compressible Euler equations



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## ABSTRACT

Using the Hodge decomposition on bounded domains the compressible Euler equations of gas dynamics are reformulated using a density weighted vorticity and dilatation as primary variables, together with the entropy and density. This formulation is an extension to compressible flows of the well-known vorticity–stream function formulation of the incompressible Euler equations. The Hamiltonian and associated Poisson bracket for this new formulation of the compressible Euler equations are derived and extensive use is made of differential forms to highlight the mathematical structure of the equations. In order to deal with domains with boundaries also the Stokes–Dirac structure and the port-Hamiltonian formulation of the Euler equations in density weighted vorticity and dilatation variables are obtained.

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## 1. Introduction

The dynamics of an inviscid compressible gas is described by the compressible Euler equations and an equation of state. The compressible Euler equations have been extensively used to model many different types of compressible flows, since in many applications the effects of viscosity are small or can be neglected. This has motivated over the years extensive theoretical and numerical studies of the compressible Euler equations. The Euler equations for a compressible, inviscid and non-isentropic gas in a domain  $\Omega \subseteq \mathbb{R}^3$  are defined as

$$\rho_t = -\nabla \cdot (\rho u), \quad (1)$$

$$u_t = -u \cdot \nabla u - \frac{1}{\rho} \nabla p, \quad (2)$$

$$s_t = -u \cdot \nabla s, \quad (3)$$

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with  $u = u(x, t) \in \mathbb{R}^3$  the fluid velocity,  $\rho = \rho(x, t) \in \mathbb{R}^+$  the mass density and  $s(x, t) \in \mathbb{R}$  the entropy of the fluid, which is conserved along streamlines. The spatial coordinates are  $x \in \Omega$  and time  $t$  and the subscript means differentiation with respect to time. The pressure  $p(x, t)$  is given by an equation of state

$$p = \rho^2 \frac{\partial U}{\partial \rho}(\rho, s), \quad (4)$$

where  $U(\rho, s)$  is the internal energy function that depends on the density  $\rho$  and the entropy  $s$  of the fluid. The compressible Euler equations have a rich mathematical structure [1] and can be represented as an infinite dimensional Hamiltonian system [2,3]. Depending on the field of interest, various types of variables have been used to define the Euler equations, e.g. conservative, primitive and entropy variables [1]. The conservative variable formulation is for instance a good starting point for numerical discretizations that can capture flow discontinuities [4], such as shocks and contact waves, whereas the primitive and entropy variables are frequently used in theoretical studies.

In many flows vorticity is, however, the primary variable of interest. Historically, the Kelvin circulation theorem and Helmholtz theorems on vortex filaments have played an important role in describing incompressible flows, in particular the importance of vortical structures. This has motivated the use of vorticity as primary variable in theoretical studies of incompressible flows, see e.g. [5,2], and the development of vortex methods to compute incompressible vortex dominated flows [6].

The use of vorticity as primary variable is, however, not very common for compressible flows. This is partly due to the fact that the equations describing the evolution of vorticity in a compressible flow are considerably more complicated than those for incompressible flows. Nevertheless, vorticity is also very important in many compressible flows. A better insight into the role of vorticity, and also dilatation to account for compressibility effects, is not only of theoretical importance, but also relevant for the development of numerical discretizations that can compute these quantities with high accuracy.

In this article we will present a vorticity–dilatation formulation of the compressible Euler equations. Special attention will be given to the Hamiltonian formulation of the compressible Euler equations in terms of the density weighted vorticity and dilatation variables on domains with boundaries. This formulation is an extension to compressible flows of the well-known vorticity–stream function formulation of the incompressible Euler equations [5,2]. An important theoretical tool in this analysis is the Hodge decomposition on bounded domains [7]. Since bounded domains are crucial in many applications we also consider the Stokes–Dirac structure of the compressible Euler equations. This results in a port-Hamiltonian formulation [8] of the compressible Euler equations in terms of the vorticity–dilatation variables, which clearly identifies the flows and efforts entering and leaving the domain. An important feature of our presentation is that we extensively use the language of differential forms. Apart from being a natural way to describe the underlying mathematical structure it is also important for our long term objective, viz. the derivation of finite element discretizations that preserve the mathematical structure as much as possible. A nice way to achieve this is by using discrete differential forms and exterior calculus, as highlighted in [9–11].

The outline of this article is as follows. In the introductory Section 2 we summarize the main techniques that we will use in our analysis. A crucial element is the use of the Hodge decomposition on bounded domains, which we briefly discuss in Section 2.2. This analysis is based on the concept of Hilbert complexes, which we summarize in Section 2.1. The Hodge Laplacian problem is discussed in Section 2.3. Here we show how to deal with inhomogeneous boundary conditions, which is of great importance for our applications. These results will be used in Section 3 to define via the Hodge decomposition a new set of variables, viz., the density weighted vorticity and dilatation, and to formulate the Euler equations in terms of these new variables. Section 4 deals with the Hamiltonian formulation of the Euler equations using the density weighted vorticity and dilatation, together with the density and entropy, as primary variables. The Poisson bracket for the Euler equations in these variables is derived in Section 5. In order to account for bounded domains we extend the results obtained for the Hamiltonian formulation in Sections 4 and 5 to the port-Hamiltonian framework in Section 6. First, we extend in Section 6.1 the Stokes–Dirac structure for the isentropic compressible Euler equations presented in [12] to the non-isentropic Euler equations. Next, we derive the Stokes–Dirac structure for the compressible Euler equations in the vorticity–dilatation formulation in Section 6.3 and use this in Section 6.5 to obtain a port-Hamiltonian formulation of the compressible Euler equations in vorticity–dilatation variables. Finally, in Section 7 we finish with some conclusions.

## 2. Preliminaries

This preliminary section is devoted to summarize the main concepts and techniques that we use throughout this paper in our analysis.

### 2.1. Review of Hilbert complexes

In this section we discuss the abstract framework of Hilbert complexes, which is the basis of the exterior calculus in Arnold, Falk and Winther [10] and to which we refer for a detailed presentation. We also refer to Brüning and Lesch [13] for a functional analytic treatment of Hilbert complexes.

**Definition 1.** A Hilbert complex  $(W, d)$  consists of a sequence of Hilbert spaces  $W^k$ , along with closed, densely-defined linear operators  $d^k : W^k \rightarrow W^{k+1}$ , possibly unbounded, such that the range of  $d^k$  is contained in the domain of  $d^{k+1}$  and  $d^{k+1} \circ d^k = 0$  for each  $k$ .

A Hilbert complex is *bounded* if, for each  $k$ ,  $d^k$  is a bounded linear operator from  $W^k$  to  $W^{k+1}$  and it is *closed* if for each  $k$ , the range of  $d^k$  is closed in  $W^{k+1}$ .

**Definition 2.** Given a Hilbert complex  $(W, d)$ , a domain complex  $(V, d)$  consists of domains  $\mathcal{D}(d^k) = V^k \subset W^k$ , endowed with the graph inner product

$$\langle u, v \rangle_{V^k} = \langle u, v \rangle_{W^k} + \langle d^k u, d^k v \rangle_{W^{k+1}}.$$

**Remark 1.** Since  $d^k$  is a closed map, each  $V^k$  is closed with respect to the norm induced by the graph inner product. From the Closed Graph Theorem, it follows that  $d^k$  is a bounded operator from  $V^k$  to  $V^{k+1}$ . Hence,  $(V, d)$  is a bounded Hilbert complex. The domain complex is closed if and only if the original complex  $(W, d)$  is.

**Definition 3.** Given a Hilbert complex  $(W, d)$ , the space of  $k$ -cocycles is the null space  $\mathfrak{Z}^k = \ker d^k$ , the space of  $k$ -coboundaries is the image  $\mathfrak{B}^k = d^{k-1}V^{k-1}$ , the  $k$ th harmonic space is the intersection  $\mathfrak{H}^k = \mathfrak{Z}^k \cap \mathfrak{B}^{k \perp W}$ , and the  $k$ th reduced cohomology space is the quotient  $\mathfrak{Z}^k / \mathfrak{B}^k$ . When  $\mathfrak{B}^k$  is closed,  $\mathfrak{Z}^k / \mathfrak{B}^k$  is called the  $k$ th cohomology space.

**Remark 2.** The harmonic space  $\mathfrak{H}^k$  is isomorphic to the reduced cohomology space  $\mathfrak{Z}^k / \overline{\mathfrak{B}^k}$ . For a closed complex, this is identical to the homology space  $\mathfrak{Z}^k / \mathfrak{B}^k$ , since  $\mathfrak{B}^k$  is closed for each  $k$ .

**Definition 4.** Given a Hilbert complex  $(W, d)$ , the dual complex  $(W^*, d^*)$  consists of the spaces  $W_k^* = W^k$ , and adjoint operators  $d_k^* = (d^{k-1})^* : V_k^* \subset W_k^* \rightarrow V_{k-1}^* \subset W_{k-1}^*$ . The domain of  $d_k^*$  is denoted by  $V_k^*$ , which is dense in  $W^k$ .

**Definition 5.** We can define the  $k$ -cycles  $\mathfrak{Z}_k^* = \ker d_k^* = \mathfrak{B}^{k \perp W}$  and  $k$ -boundaries  $\mathfrak{B}_k^* = d_{k+1}^* V_k^*$ .

2.2. The  $L^2$ -de Rham complex and Hodge decomposition

The basic example of a Hilbert complex is the  $L^2$ -de Rham complex of differential forms. Let  $\Omega \subseteq \mathbb{R}^n$  be an  $n$ -dimensional oriented manifold with Lipschitz boundary  $\partial\Omega$ , representing the space of spatial variables. Assume that there is a Riemannian metric  $\langle \cdot, \cdot \rangle$  on  $\Omega$ . We denote by  $L^2 \Lambda^k(\Omega)$  the space of smooth differential  $k$ -forms on  $\Omega$ ,  $d$  is the exterior derivative operator, taking differential  $k$ -forms on the domain  $\Omega$  to differential  $(k + 1)$ -forms,  $\delta$  represents the coderivative operator and  $\star$  the Hodge star operator associated to the Riemannian metric  $\langle \cdot, \cdot \rangle$ . The operations grad, curl, div,  $\times$ ,  $\cdot$  from vector analysis can be identified with operations on differential forms, see e.g. [14].

We can define the  $L^2$ -inner product of any two differential  $k$ -forms on  $\Omega$  as

$$\langle \omega, \eta \rangle_{L^2 \Lambda^k} = \int_{\Omega} \omega \wedge \star \eta = \int_{\Omega} \langle \omega, \eta \rangle v_{\Omega} = \int_{\Omega} \star(\omega \wedge \star \eta) v_{\Omega}, \tag{5}$$

where  $v_{\Omega}$  is the Riemannian volume form. The Hilbert space  $L^2 \Lambda^k(\Omega)$  is the space of differential  $k$ -forms for which  $\|\omega\|_{L^2 \Lambda^k} = \sqrt{\langle \omega, \omega \rangle_{L^2 \Lambda^k}} < \infty$ . When  $\Omega$  is omitted from  $L^2 \Lambda^k$  in the inner product (5), then the integral is always over  $\Omega$ . The exterior derivative  $d = d^k$  may be viewed as an unbounded operator from  $L^2 \Lambda^k$  to  $L^2 \Lambda^{k+1}$ . Its domain, denoted by  $H \Lambda^k(\Omega)$ , is the space of differential forms in  $L^2 \Lambda^k(\Omega)$  with the weak derivative in  $L^2 \Lambda^{k+1}(\Omega)$ , that is

$$\mathcal{D}(d) = H \Lambda^k(\Omega) = \{ \omega \in L^2 \Lambda^k(\Omega) \mid d\omega \in L^2 \Lambda^{k+1}(\Omega) \},$$

which is a Hilbert space with the inner product

$$\langle \omega, \eta \rangle_{H \Lambda^k} = \langle \omega, \eta \rangle_{L^2 \Lambda^k} + \langle d\omega, d\eta \rangle_{L^2 \Lambda^{k+1}}.$$

For an oriented Riemannian manifold  $\Omega \subseteq \mathbb{R}^3$ , the  $L^2$  de Rham complex is

$$0 \rightarrow L^2 \Lambda^0(\Omega) \xrightarrow{d} L^2 \Lambda^1(\Omega) \xrightarrow{d} L^2 \Lambda^2(\Omega) \xrightarrow{d} L^2 \Lambda^3(\Omega) \rightarrow 0. \tag{6}$$

Note that  $d$  is a bounded map from  $H \Lambda^k(\Omega)$  to  $L^2 \Lambda^{k+1}(\Omega)$  and  $\mathcal{D}(d) = H \Lambda^k(\Omega)$  is densely-defined in  $L^2 \Lambda^k(\Omega)$ . Since  $H \Lambda^k(\Omega)$  is complete with the graph norm,  $d$  is a closed operator (equivalent statement to the Closed Graph Theorem). Thus, the  $L^2$  de Rham domain complex for  $\Omega \subseteq \mathbb{R}^3$  is

$$0 \rightarrow H \Lambda^0(\Omega) \xrightarrow{d} H \Lambda^1(\Omega) \xrightarrow{d} H \Lambda^2(\Omega) \xrightarrow{d} H \Lambda^3(\Omega) \rightarrow 0. \tag{7}$$

The coderivative operator  $\delta : L^2 \Lambda^k(\Omega) \mapsto L^2 \Lambda^{k-1}(\Omega)$  is defined as

$$\delta \omega = (-1)^{n(k+1)+1} \star d \star \omega, \quad \omega \in L^2 \Lambda^k(\Omega). \tag{8}$$

Since we assumed that  $\Omega$  has Lipschitz boundary, the trace theorem holds and the trace operator  $\text{tr} = \text{tr}_{\partial\Omega}$  maps  $H\Lambda^k(\Omega)$  boundedly into the Sobolev space  $H^{-1/2}\Lambda^k(\partial\Omega)$ . Moreover, the trace operator extends to a bounded surjection from  $H^1\Lambda^k(\Omega)$  onto  $H^{1/2}\Lambda^k(\partial\Omega)$ , see [9]. We denote the space  $H\Lambda^k(\Omega)$  with vanishing trace as

$$\mathring{H}\Lambda^k(\Omega) = \{\omega \in H\Lambda^k(\Omega) \mid \text{tr } \omega = 0\}. \tag{9}$$

In analogy with  $H\Lambda^k(\Omega)$ , we can define the space

$$H^*\Lambda^k(\Omega) = \{\omega \in L^2\Lambda^k(\Omega) \mid \delta\omega \in L^2\Lambda^{k-1}(\Omega)\}. \tag{10}$$

Since  $H^*\Lambda^k(\Omega) = \star H\Lambda^{n-k}(\Omega)$ , for  $\omega \in H^*\Lambda^k(\Omega)$ , the quantity  $\text{tr}(\star\omega)$  is well defined, and we can define

$$\mathring{H}^*\Lambda^k(\Omega) = \star\mathring{H}\Lambda^{n-k}(\Omega) = \{\omega \in H^*\Lambda^k(\Omega) \mid \text{tr}(\star\omega) = 0\}. \tag{11}$$

The adjoint  $d^* = d_k^*$  of  $d^{k-1}$  has domain  $\mathcal{D}(d^*) = \mathring{H}^*\Lambda^k(\Omega)$  and coincides with the operator  $\delta$  defined in (8), (see [10]). Hence, the dual complex of (7) is

$$0 \leftarrow \mathring{H}^*\Lambda^0(\Omega) \xleftarrow{\delta} \mathring{H}^*\Lambda^1(\Omega) \xleftarrow{\delta} \mathring{H}^*\Lambda^2(\Omega) \xleftarrow{\delta} \mathring{H}^*\Lambda^3(\Omega) \leftarrow 0. \tag{12}$$

The integration by parts formula also holds

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle + \int_{\partial\Omega} \text{tr } \omega \wedge \text{tr}(\star\eta), \quad \omega \in \Lambda^{k-1}(\Omega), \eta \in \Lambda^k(\Omega), \tag{13}$$

and we have

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle, \quad \omega \in H\Lambda^{k-1}(\Omega), \eta \in \mathring{H}^*\Lambda^k(\Omega). \tag{14}$$

Since the  $L^2$ -de Rham complexes (6) and (7) are closed Hilbert complexes, the Hodge decomposition of  $L^2\Lambda^k$  and  $H\Lambda^k$  are:

$$L^2\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{B}_k^*, \tag{15}$$

$$H\Lambda^k = \mathfrak{B}^k \oplus \mathfrak{H}^k \oplus \mathfrak{Z}^{k\perp}, \tag{16}$$

where  $\mathfrak{B}_k^* = \{\delta\omega \mid \omega \in \mathring{H}^*\Lambda^{k+1}(\Omega)\}$ , and  $\mathfrak{Z}^{k\perp} = H\Lambda^k \cap \mathfrak{B}_k^*$  denotes the orthogonal complement of  $\mathfrak{Z}^k$  in  $H\Lambda^k$ . The space of harmonic forms, both for the original complex and the dual complex, is

$$\mathfrak{H}^k = \{\omega \in H\Lambda^k(\Omega) \cap \mathring{H}^*\Lambda^k(\Omega) \mid d\omega = 0, \delta\omega = 0\}. \tag{17}$$

Problems with essential boundary conditions are important for applications. This is why we briefly review the de Rham complex with essential boundary conditions. Take as domain of the exterior derivative  $d^k$  the space  $\mathring{H}\Lambda^k(\Omega)$ . The de Rham complex with homogeneous boundary conditions on  $\Omega \subset \mathbb{R}^3$  is

$$0 \rightarrow \mathring{H}\Lambda^0(\Omega) \xrightarrow{d} \mathring{H}\Lambda^1(\Omega) \xrightarrow{d} \mathring{H}\Lambda^2(\Omega) \xrightarrow{d} \mathring{H}\Lambda^3(\Omega) \rightarrow 0. \tag{18}$$

From (13), we obtain that

$$\langle d\omega, \eta \rangle = \langle \omega, \delta\eta \rangle, \quad \omega \in \mathring{H}\Lambda^{k-1}(\Omega), \eta \in H^*\Lambda^k(\Omega). \tag{19}$$

Hence, the adjoint  $d^*$  of the exterior derivative with domain  $\mathring{H}\Lambda^k(\Omega)$  has domain  $H^*\Lambda^k(\Omega)$  and coincides with the operator  $\delta$ . Finally, the second Hodge decomposition of  $L^2\Lambda^k$  and of  $\mathring{H}\Lambda^k$  follow immediately:

$$L^2\Lambda^k = \mathring{\mathfrak{B}}^k \oplus \mathring{\mathfrak{H}}^k \oplus \mathfrak{B}_k^*, \tag{20}$$

$$\mathring{H}\Lambda^k = \mathring{\mathfrak{B}}^k \oplus \mathring{\mathfrak{H}}^k \oplus \mathfrak{Z}^{k\perp}, \tag{21}$$

where  $\mathring{\mathfrak{B}}^k = d\mathring{H}\Lambda^{k-1}(\Omega)$ ,  $\mathfrak{Z}^{k\perp} = \mathring{H}\Lambda^k \cap \mathfrak{B}_k^*$  and the space of harmonic forms is

$$\mathring{\mathfrak{H}}^k = \{\omega \in \mathring{H}\Lambda^k(\Omega) \cap H^*\Lambda^k(\Omega) \mid d\omega = 0, \delta\omega = 0\}. \tag{22}$$

### 2.3. The Hodge Laplacian problem

In this section we first review the Hodge Laplacian problem with homogeneous natural and essential boundary conditions. The main result of this section is to show how to deal with inhomogeneous boundary conditions, which are crucial for applications.

#### 2.3.1. The Hodge Laplacian problem with homogeneous natural boundary conditions

Given the Hilbert complex (7) and its dual complex (12), the Hodge Laplacian operator applied to a  $k$ -form is an unbounded operator  $L_k = d^{k-1}d_k^* + d_{k+1}^*d^k : \mathcal{D}(L_k) \subset L^2\Lambda^k \rightarrow L^2\Lambda^k$ , with domain (see [10])

$$\mathcal{D}(L_k) = \{u \in H\Lambda^k \cap \overset{\circ}{H}^*\Lambda^k \mid d^k u \in \overset{\circ}{H}^*\Lambda^{k+1}, d_k^* u \in H\Lambda^{k-1}\}. \tag{23}$$

In the following, we will not use the subscripts and superscripts  $k$  of the operators when they are clear from the context and use  $d^* = \delta$ .

For any  $f \in L^2\Lambda^k$ , there exists a unique solution  $u = K_k f \in \mathcal{D}(L_k)$  satisfying

$$L_k u = f \pmod{\mathfrak{H}}, \quad u \perp \mathfrak{H}, \tag{24}$$

with  $K_k$  the solution operator, see [10]. The solution  $u$  satisfies the Hodge Laplacian (homogeneous) boundary value problem

$$(d\delta + \delta d)u = f - P_{\mathfrak{H}}f \text{ in } \Omega, \quad \text{tr}(\star u) = 0, \quad \text{tr}(\star du) = 0 \text{ on } \partial\Omega, \tag{25}$$

where  $P_{\mathfrak{H}}f$  is the orthogonal projection of  $f$  into  $\mathfrak{H}$ , with the condition  $u \perp \mathfrak{H}$  required for uniqueness of the solution. The boundary conditions in (25) are both natural in the mixed variational formulation of the Hodge Laplacian problem, as discussed in [10].

The Hodge Laplacian problem is closely related to the Hodge decomposition in the following way. Since  $d\delta u \in \mathfrak{B}^k$  and  $\delta du \in \mathfrak{B}_k^*$ , the differential equation in (25), or equivalently  $f = d\delta u + \delta du + \alpha$ ,  $\alpha \in \mathfrak{H}^k$ , is exactly the Hodge decomposition of  $f \in L^2\Lambda^k(\Omega)$ . If we restrict  $f$  to an element of  $\mathfrak{B}^k$  or  $\mathfrak{B}_k^*$ , we obtain two problems that are important for our applications (see also [10]).

**The  $\mathfrak{B}$  problem.** If  $f \in \mathfrak{B}^k$ , then  $u = K_k f$  satisfies  $d\delta u = f$ ,  $u \perp \overset{\circ}{\mathfrak{Z}}_k^*$ , where  $\overset{\circ}{\mathfrak{Z}}_k^* = \{\omega \in \overset{\circ}{H}^*\Lambda^k(\Omega) \mid \delta\omega = 0\}$ . It also follows that the solution  $u \in \mathfrak{B}^k$ . To see this, consider  $u \in \mathcal{D}(L_k)$  and the Hodge decomposition  $u = u_{\mathfrak{B}} + u_{\mathfrak{H}} + u_{\perp}$ , where  $u_{\mathfrak{B}} \in \mathfrak{B}^k$ ,  $u_{\mathfrak{H}} \in \mathfrak{H}^k$  and  $u_{\perp} \in \mathfrak{B}_k^* \cap H\Lambda^k$ . Then,

$$L_k u = d\delta u_{\mathfrak{B}} + \delta du_{\perp} = f.$$

If  $f \in \mathfrak{B}^k$ , then  $u = u_{\mathfrak{B}}$ , hence  $u \in \mathfrak{B}^k$ . Then,  $du_{\mathfrak{B}} = 0$  since  $d^2 = 0$ , and since  $\mathfrak{B}^k = \overset{\circ}{\mathfrak{Z}}_k^{\perp}$ , it follows that  $u \perp \overset{\circ}{\mathfrak{Z}}_k^*$  is also satisfied and  $u$  solves uniquely the Hodge Laplacian boundary value problem, see [10],

$$d\delta u = f, \quad du = 0, \quad u \perp \overset{\circ}{\mathfrak{Z}}_k^* \text{ in } \Omega \tag{26}$$

$$\text{tr}(\star u) = 0, \quad \text{on } \partial\Omega. \tag{27}$$

**The  $\mathfrak{B}_k^*$  problem.** If  $f \in \mathfrak{B}_k^*$ , then  $u = K_k f$  satisfies  $\delta du = f$ , with  $u \perp \overset{\circ}{\mathfrak{Z}}_k$ . Similarly as for the  $\mathfrak{B}$  problem, it can be shown that the solution  $u \in \mathfrak{B}_k^*$ . Consider  $u \in \mathcal{D}(L_k)$  and the Hodge decomposition  $u = u_{\mathfrak{B}} + u_{\mathfrak{H}} + u_{\perp}$ . Then,

$$L_k u = d\delta u_{\mathfrak{B}} + \delta du_{\perp} = f.$$

If  $f \in \mathfrak{B}_k^*$ , then  $u = u_{\perp}$ , hence  $u \in \mathfrak{B}_k^*$ . Therefore,  $\delta u_{\perp} = 0$  and  $u \perp \overset{\circ}{\mathfrak{Z}}_k$ . Consequently,  $u$  solves uniquely the Hodge Laplacian boundary value problem

$$\delta du = f, \quad \delta u = 0, \quad u \perp \overset{\circ}{\mathfrak{Z}}_k \text{ in } \Omega \tag{28}$$

$$\text{tr}(\star du) = 0, \quad \text{on } \partial\Omega. \tag{29}$$

#### 2.3.2. The Hodge Laplacian problem with homogeneous essential boundary conditions

Considering now the Hilbert complex with (homogeneous) boundary conditions (18) and its dual complex, the Hodge Laplacian problem with essential boundary conditions is

$$L_k u = d\delta u + \delta du = f \pmod{\mathfrak{H}}, \quad \text{in } \Omega \tag{30}$$

$$\text{tr}(\delta u) = 0, \quad \text{tr } u = 0 \text{ on } \partial\Omega, \tag{31}$$

with the condition  $u \perp \overset{\circ}{\mathfrak{H}}$ , which has a unique solution,  $u = K_k f$ . Both boundary conditions in (31) are essential in the mixed variational formulation of the Hodge Laplacian problem (see [10]). Here the domain of the Laplacian is

$$\mathcal{D}(L_k) = \{u \in \overset{\circ}{H}\Lambda^k \cap H^*\Lambda^k \mid du \in H^*\Lambda^{k+1}, \delta u \in \overset{\circ}{H}\Lambda^{k-1}\}. \tag{32}$$

We can briefly formulate the  $\mathfrak{B}$  and  $\mathfrak{B}^*$  problems as follows.

The  $\mathfrak{B}$  problem. If  $f \in \mathring{\mathfrak{B}}^k$ , then  $u = K_k f \in \mathring{\mathfrak{B}}^k$  satisfies  $d\delta u = f$ ,  $u \perp \mathfrak{Z}_k^*$ . Then  $u$  solves uniquely the  $\mathfrak{B}$  problem

$$d\delta u = f, \quad du = 0, \quad u \perp \mathfrak{Z}_k^* \text{ in } \Omega \tag{33}$$

$$\text{tr}(\delta u) = 0, \quad \text{on } \partial\Omega. \tag{34}$$

The  $\mathfrak{B}^*$  problem. If  $f \in \mathfrak{B}_k^*$ , then  $u = K_k f$  satisfies  $\delta du = f$ , with  $u \perp \mathfrak{Z}^k$ . Similarly,  $u$  solves uniquely

$$\delta du = f, \quad \delta u = 0, \quad u \perp \mathfrak{Z}^k \text{ in } \Omega \tag{35}$$

$$\text{tr}(u) = 0, \quad \text{on } \partial\Omega. \tag{36}$$

The next section shows how to transform the inhomogeneous Hodge Laplacian boundary value problem into a homogeneous one.

### 2.3.3. The Hodge Laplacian with inhomogeneous essential boundary conditions

Consider the case when the essential boundary conditions are inhomogeneous, that is, the boundary value problem

$$L_k u = d\delta u + \delta du = f \text{ in } \Omega \tag{37}$$

$$\text{tr } u = r_b, \quad \text{tr}(\delta u) = r_N \text{ on } \partial\Omega, \tag{38}$$

with the condition

$$\langle f, \alpha \rangle = \int_{\partial\Omega} r_N \wedge \text{tr}(\star\alpha), \quad \forall \alpha \in \mathring{\mathfrak{H}}^k. \tag{39}$$

Here the domain of the Hodge Laplacian operator is

$$\mathcal{D}_k^D = \left\{ u \in H\Lambda^k \cap H^*\Lambda^k \mid du \in H^*\Lambda^{k+1}, \delta u \in H\Lambda^{k-1}, \right. \\ \left. \text{tr } u = r_b \in H^{1/2}\Lambda^k(\partial\Omega), \text{tr}(\delta u) = r_N \in H^{1/2}\Lambda^{k-1}(\partial\Omega) \right\}. \tag{40}$$

This problem has a solution, which is unique up to a harmonic form  $\alpha \in \mathring{\mathfrak{H}}^k$ . Following the idea of Schwarz in [7], the inhomogeneous boundary value problem can be transformed to a homogeneous problem in the following way. For a given  $r_b \in H^{1/2}\Lambda^k(\partial\Omega)$ , using a bounded, linear trace lifting operator (see [15,9,7]), we can find  $\eta \in H^1\Lambda^k(\Omega)$ , such that  $\text{tr } \eta = r_b$ . The Hodge decomposition of  $\eta$

$$\eta = d\phi_\eta + \delta\beta_\eta + \alpha_\eta, \quad d\phi_\eta \in \mathring{\mathfrak{B}}^k, \delta\beta_\eta \in \mathfrak{B}_k^*, \alpha_\eta \in \mathring{\mathfrak{H}}^k$$

means for the trace that

$$\text{tr } \eta = d(\text{tr } \phi_\eta) + \text{tr}(\delta\beta_\eta) + \text{tr } \alpha_\eta = \text{tr}(\delta\beta_\eta),$$

viz. the components  $d\phi_\eta$  and  $\alpha_\eta$  of the extension  $\eta$  do not contribute to the  $\text{tr } \eta$ . Hence, we can construct  $\eta = \delta\beta_\eta$ . Then,  $L_k \eta = \delta d\delta\beta_\eta$  and it can be easily shown that  $\langle L_k \eta, \alpha \rangle = 0$  for all  $\alpha \in \mathring{\mathfrak{H}}^k$ .

On the other hand, given  $r_N \in H^{1/2}\Lambda^{k-1}(\partial\Omega)$ , the extension result of Lemma 3.3.2 in [7], guarantees the existence of  $\bar{\eta} \in H^1\Lambda^k\Omega$ , such that

$$\text{tr } \bar{\eta} = 0 \text{ and } \text{tr}(\delta\bar{\eta}) = r_N.$$

Take  $\bar{u} = u - \eta - \bar{\eta}$ . Then,  $L_k \bar{u} = f - L_k \eta - L_k \bar{\eta} =: \bar{f}$ ,  $\text{tr } \bar{u} = 0$ ,  $\text{tr}(\delta\bar{u}) = 0$  and using the condition (39), we can show that  $\bar{f} \perp \mathring{\mathfrak{H}}^k$ . Hence,  $\bar{u}$  is the solution of the homogeneous boundary value problem (30)–(31) with the right hand side  $\bar{f}$ .

### 2.3.4. The Hodge Laplacian with inhomogeneous natural boundary conditions

Consider the Hodge Laplacian operator  $L_k = d\delta + \delta d : \mathcal{D}(L_k) \subset L^2\Lambda^k \rightarrow L^2\Lambda^k$ , with domain

$$\mathcal{D}_k^N = \left\{ u \in H\Lambda^k \cap H^*\Lambda^k \mid du \in H^*\Lambda^{k+1}, \delta u \in H\Lambda^{k-1}, \right. \\ \left. \text{tr}(\star u) = g_b \in H^{-1/2}\Lambda^{n-k}(\partial\Omega), \text{tr}(\star du) = g_N \in H^{-1/2}\Lambda^{n-k-1}(\partial\Omega) \right\}. \tag{41}$$

Our next step is to transform the inhomogeneous boundary value problem

$$(d\delta + \delta d)u = f \text{ in } \Omega \tag{42}$$

$$\text{tr}(\star u) = g_b, \quad \text{tr}(\star du) = g_N \text{ on } \partial\Omega, \tag{43}$$

with the condition

$$\langle f, \alpha \rangle = - \int_{\partial\Omega} \text{tr } \alpha \wedge g_N \quad \forall \alpha \in \mathring{\mathfrak{H}}^k, \tag{44}$$

and the side condition for uniqueness  $u \perp \mathfrak{H}^k$ , into the Hodge Laplacian homogeneous boundary value problem (25). This can be considered as the dual of the problem with inhomogeneous essential boundary conditions, treated in Section 2.3.3. For completeness, we briefly summarize the steps of the construction.

For  $g_b \in H^{-1/2} \Lambda^{n-k}(\partial\Omega)$  we can find  $\tau \in H^* \Lambda^k(\Omega)$  with  $\text{tr}(\star\tau) = g_b$ . Note here that since  $\tau \in H^* \Lambda^k(\Omega)$ , then  $\star\tau \in H \Lambda^{n-k}(\Omega)$ , so  $\text{tr}(\star\tau)$  is well-defined. Moreover, using the Hodge decomposition

$$\tau = d\phi_\tau + \delta\beta_\tau + \alpha_\tau, \quad d\phi_\tau \in \mathfrak{B}^k, \quad \delta\beta_\tau \in \mathfrak{B}_k^{\circ}, \quad \alpha_\tau \in \mathfrak{H}^k,$$

we have  $\text{tr}(\star\tau) = \text{tr}(\star d\phi_\tau)$ , viz. the terms  $\delta\beta_\tau$  and  $\alpha_\tau$  do not contribute to the trace, hence we can take  $\tau = d\phi_\tau$ . Then,  $L_k\tau = d\delta d\phi_\tau$  and  $\langle L_k\tau, \alpha \rangle = 0$  for all  $\alpha \in \mathfrak{H}^k$ . Similarly, for  $g_N \in H^{-1/2} \Lambda^{n-k-1}(\partial\Omega)$ , we can find  $\bar{\tau} \in H^* \Lambda^k(\Omega)$ , with

$$\star\bar{\tau}|_{\partial\Omega} = 0, \quad \text{tr}(\delta\star\bar{\tau}) = g_N.$$

Taking  $\bar{u} = u - \tau - \bar{\tau}$ , we obtain  $L_k\bar{u} = f - L_k\tau - L_k\bar{\tau} =: \bar{f}$ . Moreover,  $\text{tr}(\star\bar{u}) = 0$ ,  $\text{tr}(\star d\bar{u}) = 0$  and by using (44), we obtain  $\bar{f} \perp \mathfrak{H}^k$ .

Consequently, solving the inhomogeneous boundary value problem (42)–(44) for given  $\bar{f} \in L^2 \Lambda^k$  is equivalent with solving the Hodge Laplacian problem with homogeneous boundary conditions (25), for given  $\bar{f}$ .

Note that, since the  $\mathfrak{B}$  and  $\mathfrak{B}^*$  problems are special cases of the Hodge Laplacian problem, they can be solved also for inhomogeneous boundary conditions.

### 3. The Euler equations in density weighted vorticity and dilatation formulation

In this section we will derive, via the Hodge decomposition, a Hamiltonian formulation of the compressible Euler equations using a density weighted vorticity and dilatation as primary variables. This will provide an extension of the well known vorticity–stream function formulation of the incompressible Euler equations to compressible flows, see e.g. [2,3]. Special attention will be given to the proper boundary conditions for the potential  $\phi$  and the vector stream function  $\beta$ .

The analysis of the Hamiltonian formulation and Stokes–Dirac structure of the compressible Euler equations is most easily performed using techniques from differential geometry. For this purpose we first reformulate (1)–(3) in terms of differential forms. We identify the mass density  $\rho$  and the entropy  $s$  with a 3-form on  $\Omega$ , that is, with an element of  $\Lambda^3(\Omega)$ , the pressure  $p \in \Lambda^0(\Omega)$ , and the velocity field  $u$  with a 1-form on  $\Omega$ , viz., with an element of  $\Lambda^1(\Omega)$ . Let  $u^\sharp$  be the vector field corresponding to the 1-form  $u$  (using index raising, or sharp mapping),  $i_{u^\sharp}$  denotes the interior product by  $u^\sharp$ .

For an arbitrary vector field  $X$  and  $\alpha \in \Lambda^k(\Omega)$ , the following relation between the interior product and Hodge star operator is valid, (see e.g. [16]),

$$i_X \alpha = \star(X^\flat \wedge \star\alpha), \tag{45}$$

where  $X^\flat$  is the 1-form related to  $X$  by the flat mapping. Following the identifications of the variables suggested in [8] for the isentropic fluid using differential forms, and completing them with the entropy balance equation, the Euler equations of gas dynamics can then be formulated in differential forms as

$$\rho_t = -d(i_{u^\sharp} \rho), \tag{46}$$

$$u_t = -d\left(\frac{1}{2} \langle u^\sharp, u^\sharp \rangle_v\right) - i_{u^\sharp} du - \frac{1}{\star\rho} dp \tag{47}$$

$$s_t = -u \wedge \star d(\star s) = u \wedge \delta s, \tag{48}$$

where  $\langle \cdot, \cdot \rangle_v$  is the inner product of two vectors.

Using the  $L^2$ -de Rham complex described in Section 2.2, let us start with the Hodge decomposition of the differential 1-form  $\sqrt{\tilde{\rho}} \wedge u \in L^2 \Lambda^1(\Omega)$ , denoted by  $\zeta$ ,

$$\zeta = \sqrt{\tilde{\rho}} \wedge u = d\phi + \delta\beta + \alpha, \tag{49}$$

where  $\tilde{\rho} = \star\rho$ . The choice of  $\zeta$  will be motivated in the next section.

**Definition 6.** Using the Hodge decomposition (49), define the density weighted vorticity as  $\omega = d\zeta$  and the density weighted dilatation as  $\theta = -\delta\zeta$ .

There are two Hodge decompositions, (15) and (20), hence two sets of boundary conditions on the Hodge components

- (a)  $d\phi \in \mathfrak{B}^1, \delta\beta \in \mathfrak{B}_1^{\circ}, \alpha \in \mathfrak{H}^1,$
- (b)  $d\phi \in \mathring{\mathfrak{B}}^1, \delta\beta \in \mathfrak{B}_1^*, \alpha \in \mathring{\mathfrak{H}}^1.$

**Remark 3.** If the classical vorticity is  $\xi = du$  then, the density weighted vorticity is

$$\omega = d\zeta = d(\sqrt{\rho} \wedge u) = (d\sqrt{\rho}) \wedge u + \sqrt{\rho} \wedge \xi. \tag{50}$$

When the flow is incompressible, then  $\omega = \sqrt{\rho} \wedge \xi$ , i.e., the density weighted vorticity. Using the equation for the velocity (47), the vorticity evolution equation for constant density is

$$\xi_t = -d(i_{u^\sharp} du) = -d(i_{u^\sharp} \xi). \tag{51}$$

On the other hand, the evolution equation for density weighted vorticity for incompressible flows is

$$\omega_t = \sqrt{\rho} \wedge \xi_t = -d(i_{u^\sharp} \omega). \tag{52}$$

Hence,  $\omega$  and  $\xi$  satisfy the same evolution equations.

The density weighted dilatation  $\theta$  for an incompressible flow is

$$\theta = -\delta(\sqrt{\rho} \wedge u) = -\sqrt{\rho} \wedge \delta u. \tag{53}$$

The incompressibility constraint in differential forms is  $\delta u = 0$ , which implies  $\theta = 0$ .

**Lemma 1.** The potential function  $\phi$  and vector stream function  $\beta$  in the Hodge decomposition (49) solve the following boundary value problems

$$\mathfrak{B}\text{-problem} \begin{cases} d\delta\beta = \omega, & d\beta = 0 & \text{in } \Omega \\ \text{tr}(\star\beta) = 0 & & \text{on } \partial\Omega, \end{cases} \quad \mathfrak{B}^*\text{-problem} \begin{cases} \delta d\phi = -\theta & \text{in } \Omega \\ \text{tr}(\star d\phi) = 0 & & \text{on } \partial\Omega, \end{cases} \tag{54}$$

when  $\zeta \in H\Lambda^1(\Omega) \cap \mathring{H}^*\Lambda^1(\Omega) \cap \mathfrak{H}^{1\perp}$  and

$$\mathfrak{B}\text{-problem} \begin{cases} d\delta\beta = \omega, & d\beta = 0 & \text{in } \Omega \\ \text{tr}(\delta\beta) = 0 & & \text{on } \partial\Omega, \end{cases} \quad \mathfrak{B}^*\text{-problem} \begin{cases} \delta d\phi = -\theta & \text{in } \Omega \\ \text{tr}(\phi) = 0 & & \text{on } \partial\Omega, \end{cases} \tag{55}$$

when  $\zeta \in \mathring{H}\Lambda^1(\Omega) \cap H^*\Lambda^1(\Omega) \cap \mathring{\mathfrak{H}}^{1\perp}$ .

**Proof.** The proof of this lemma is constructive and we consider two cases.

**Case 1.** The first approach is to choose  $\zeta \in H\Lambda^1(\Omega) \cap \mathring{H}^*\Lambda^1(\Omega) \cap \mathfrak{H}^{1\perp}$ . Then, the Hodge decomposition (a) of  $\zeta$  is used to define two new variables.

Since  $\zeta \in \mathcal{D}(d) = H\Lambda^1(\Omega)$ , we can define the density weighted vorticity  $\omega \in \mathfrak{B}^2 \subset L^2\Lambda^2(\Omega)$  as

$$\omega = d\zeta = d(\sqrt{\rho} \wedge u) = d(d\phi + \delta\beta) = d\delta\beta, \tag{56}$$

where  $\phi \in H\Lambda^0(\Omega)$ ,  $\beta \in H^*\Lambda^2(\Omega)$  with  $\text{tr}(\star\beta) = 0$  and  $\delta\beta \in \mathcal{D}(d) = H\Lambda^1(\Omega)$ . Moreover, since  $\omega \in \mathfrak{B}^2$ , it follows that  $\beta \in \mathfrak{B}^2$  hence,  $d\beta = 0$ . Observe here, that (56) is the  $\mathfrak{B}$  problem (26)–(27) with homogeneous natural boundary conditions.

Consider now  $\zeta \in \mathcal{D}(\delta) = \mathring{H}^*\Lambda^1(\Omega)$ . Define the density weighted dilatation  $\theta \in \mathfrak{B}_0^* \subset L^2\Lambda^0(\Omega)$  as

$$\theta = -\delta\zeta = -\delta(\sqrt{\rho} \wedge u) = -\delta d\phi - \delta\delta\beta = -\delta d\phi, \tag{57}$$

where  $\phi \in H\Lambda^0(\Omega)$ ,  $\beta \in \mathring{H}^*\Lambda^2(\Omega)$  and  $d\phi \in H^*\Lambda^1(\Omega)$  with  $0 = \text{tr}(\star\zeta) = \text{tr}(\star d\phi)$ . Observe that (57) with this boundary condition is the  $\mathfrak{B}^*$  problem (28)–(29) for  $\phi$ , where  $\delta\phi = 0$  is satisfied since  $H\Lambda^{-1}$  is understood to be zero and  $\text{tr}(\star\phi) = 0$  since  $\star\phi$  is a 3-form.

**Case 2.** Let us choose now  $\zeta \in \mathring{H}\Lambda^1(\Omega) \cap H^*\Lambda^1(\Omega) \cap \mathring{\mathfrak{H}}^{1\perp}$ . Then, we use the Hodge decomposition (b) of  $\zeta$  to define two new variables.

Since  $\zeta \in H\Lambda^1(\Omega)$ ,  $\text{tr} \zeta = 0$  and using the decomposition in (b), we obtain the weakly imposed essential boundary condition  $0 = \text{tr} \zeta = \text{tr}(\delta\beta)$ . Define the density weighted vorticity  $\omega \in \mathring{\mathfrak{B}}^2$  as

$$\omega = d\zeta = d\delta\beta, \tag{58}$$

where  $\beta \in H^*\Lambda^2(\Omega)$ . This is the  $\mathfrak{B}$  problem (33)–(34) for  $\beta$  with homogeneous essential boundary conditions. Similarly, since  $\zeta \in H^*\Lambda^1(\Omega)$ , we can define the density weighted dilatation  $\theta \in \mathfrak{B}_0^*$ , as

$$\theta = -\delta d\phi, \tag{59}$$

where  $\phi \in \mathring{H}\Lambda^0(\Omega)$ ,  $d\phi \in H^*\Lambda^1(\Omega)$ . The Hodge decomposition in (b) implies the strongly imposed boundary condition  $\text{tr} \phi = 0$ . This is again the  $\mathfrak{B}^*$  problem (35)–(36) for  $\phi$ , with homogeneous essential boundary conditions.  $\square$



**Remark 4.** When the flow is incompressible, then  $0 = \theta = -\delta\zeta = -\delta d\phi$ . This implies that

$$0 = -\langle \delta d\phi, \phi \rangle = -\langle d\phi, d\phi \rangle + \int_{\partial\Omega} \text{tr} \phi \wedge \text{tr}(\star d\phi) = -\langle d\phi, d\phi \rangle.$$

The boundary integral is zero in either of the two Hodge decompositions. Then,  $d\phi = 0$  hence, the Hodge decomposition is  $\zeta = \delta\beta + \alpha$  and the  $\mathfrak{B}^*$ -problems in Lemma 1 cancel. The  $\mathfrak{B}$ -problems remain unchanged.

**Remark 5.** Note, in Lemma 1, we can consider inhomogeneous boundary conditions for all problems. More precisely, for Case 1, the  $\mathfrak{B}$  problem for  $\beta$  has a unique solution  $\beta \in \mathfrak{B}^2$ , with the inhomogeneous boundary condition  $\text{tr}(\star\beta) = g_b \in H^{-1/2}\Lambda^1(\partial\Omega)$ , by transforming it first to a homogeneous problem with modified right hand side  $\delta d\beta = \bar{\omega}$ , as we discussed in Section 2.3.4. The  $\mathfrak{B}^*$  problem for  $\phi$  has a unique solution with the inhomogeneous boundary condition  $\text{tr}(\star d\phi) = g_N \in H^{-1/2}\Lambda^2(\partial\Omega)$ . Hence, solving the  $\mathfrak{B}^*$  problem (57) with inhomogeneous boundary conditions, is equivalent with solving the  $\mathfrak{B}^*$  homogeneous problem with modified right hand side  $\delta d\phi = -\bar{\theta}$ , with  $\bar{\theta} = \theta + \bar{\tau}$ , as seen in Section 2.3.4.

In Case 2, the inhomogeneous essential boundary conditions for  $\phi$  and  $\beta$  are  $\text{tr} \phi = r_b \in H^{1/2}\Lambda^0(\partial\Omega)$  and  $\text{tr}(\delta\beta) = r_N \in H^{1/2}\Lambda^1(\partial\Omega)$ , respectively. We can transform the equations into a homogeneous problem as in Section 2.3.3.

**Corollary 1.** *The non-isentropic compressible Euler equations can be formulated in the variables  $\rho, \omega, \theta$  and  $s$  as*

$$\rho_t = -d(\sqrt{\rho} \wedge \star \zeta), \tag{60}$$

$$\omega_t = d \zeta_t, \tag{61}$$

$$\theta_t = -\delta \zeta_t, \tag{62}$$

$$s_t = \frac{1}{\sqrt{\rho}} \wedge \zeta \wedge \delta s, \tag{63}$$

with  $\zeta$  given by (49).

**Proof.** The statement of this corollary can easily be verified by introducing the Hodge decomposition (49) into the Euler equations (46)–(48).  $\square$

Summarizing, we use the Hodge decomposition (49) to define the density weighted vorticity  $\omega$  and dilatation  $\theta$ . In order to have them well defined, we choose the proper spaces for  $\zeta$ . The potential function  $\phi$  and vector stream function  $\beta$  in the Hodge components are the solutions of the  $\mathfrak{B}^*$  and  $\mathfrak{B}$  problems, respectively, with natural or essential boundary conditions.

#### 4. Functional derivatives of the Hamiltonian in density weighted vorticity–dilatation formulation

In this section we transform the Hamiltonian functional for the non-isentropic compressible Euler equations, into the new set of variables  $\rho, \theta, \omega, s$ , and calculate the variational derivatives with respect to these new variables.

In order to simplify the discussion, we assume from here on that the domain  $\Omega$  is simply connected. Since the dimension of  $\mathfrak{H}^1$  is equal to the first Betti number, i.e., the number of handles, of the domain, it follows that  $\mathfrak{H}^1 = 0$  if the domain is simply connected, see [10]. Then  $\alpha = 0$  in the Hodge decomposition (49).

Let us recall from van der Schaft and Maschke [8] the definition of the variational derivatives of the Hamiltonian functional when it depends on, for example, two energy variables. Consider a Hamiltonian density, i.e. energy per volume element,

$$H : \Lambda^p(\Omega) \times \Lambda^q(\Omega) \rightarrow \Lambda^n(\Omega), \tag{64}$$

where  $\Omega$  is an  $n$ -dimensional manifold, resulting in the total energy

$$\mathcal{H}[\alpha_p, \alpha_q] = \int_{\Omega} H(\alpha_p, \alpha_q) \in \mathbb{R}, \tag{65}$$

where square brackets are used to indicate that  $\mathcal{H}$  is a functional of the enclosed functions. Let  $\alpha_p, \partial\alpha_p \in \Lambda^p(\Omega)$ , and  $\alpha_q, \partial\alpha_q \in \Lambda^q(\Omega)$ . Then under weak smoothness assumptions on  $H$ ,

$$\begin{aligned} \mathcal{H}[\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q] &= \int_{\Omega} H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) \\ &= \int_{\Omega} H(\alpha_p, \alpha_q) + \int_{\Omega} (\delta_p \mathcal{H} \wedge \partial\alpha_p + \delta_q \mathcal{H} \wedge \partial\alpha_q) \\ &\quad + \text{higher order terms in } \partial\alpha_p, \partial\alpha_q, \end{aligned} \tag{66}$$

for certain uniquely defined differential forms  $\delta_p \mathcal{H} \in (\Lambda^p(\Omega))^*$  and  $\delta_q \mathcal{H} \in (\Lambda^q(\Omega))^*$ , which can be regarded as the variational derivatives of  $\mathcal{H}$  with respect to  $\alpha_p$  and  $\alpha_q$ , respectively. The dual linear space  $(\Lambda^p(\Omega))^*$  can be naturally identified with  $\Lambda^{n-p}(\Omega)$ , and similarly the dual space  $(\Lambda^q(\Omega))^*$  with  $\Lambda^{n-q}(\Omega)$ .

For the non-isentropic compressible Euler equations, the energy density is given as the sum of the kinetic energy and internal energy densities. The Hamiltonian functional for the compressible Euler equations in differential forms is (see [8])

$$\mathcal{H}[\rho, u, s] = \int_{\Omega} \left( \frac{1}{2} \langle u^\sharp, u^\sharp \rangle_v \rho + U(\tilde{\rho}, \tilde{s}) \rho \right), \tag{67}$$

where  $\tilde{s} = \star s$ . The Hamiltonian functional can further be written as

$$\begin{aligned} \mathcal{H}[\rho, u, s] &= \int_{\Omega} \left( \frac{1}{2} \langle (\sqrt{\tilde{\rho}} \wedge u)^\sharp, (\sqrt{\tilde{\rho}} \wedge u)^\sharp \rangle_v v_\Omega + U(\tilde{\rho}, \tilde{s}) \rho \right) \\ &= \int_{\Omega} \left( \frac{1}{2} \langle \langle \sqrt{\tilde{\rho}} \wedge u, \sqrt{\tilde{\rho}} \wedge u \rangle \rangle v_\Omega + U(\tilde{\rho}, \tilde{s}) \rho \right). \end{aligned} \tag{68}$$

The Hamiltonian written in form (68) motivates the choice of the variable  $\zeta$ , defined in (49), and its Hodge decomposition. In the next lemma we compute the Hamiltonian functional and its variational derivatives with respect to the Hodge decomposition of  $\zeta$ .

**Remark 6.** We have seen that the inhomogeneous  $\mathfrak{B}^*$  problem for  $\phi$  and the inhomogeneous  $\mathfrak{B}$  problem for  $\beta$  can be transformed into homogeneous boundary value problems with modified right hand side. Therefore, from here on we just use the standard de Rham theory for the bar variables  $\bar{\theta}$  and  $\bar{\omega}$ , with the corresponding homogeneous boundary conditions for the  $\phi$  and  $\beta$  variables.

**Lemma 2.** *The Hamiltonian density, when the variables  $\rho, \phi, \beta, s$  are introduced, is a mapping*

$$\begin{aligned} H : H\Lambda^3 \times \mathcal{D}_0 \times \mathcal{D}_2 \times H\Lambda^3 &\rightarrow L^2\Lambda^3, \\ (\rho, \phi, \beta, s) &\mapsto H(\rho, \phi, \beta, s), \end{aligned}$$

where  $\mathcal{D}_0$  and  $\mathcal{D}_2$  are the domains of the Hodge Laplacian for 0-forms and 2-forms, respectively, with either essential or natural inhomogeneous boundary conditions, as defined in (40) and (41). This results in the total energy

$$\mathcal{H}[\rho, \phi, \beta, s] = \int_{\Omega} \left( \frac{1}{2} (\delta d\phi \wedge \star\phi + d\delta\beta \wedge \star\beta) + U(\tilde{\rho}, \tilde{s}) \rho \right). \tag{69}$$

The variational derivatives of the Hamiltonian are

$$\frac{\delta \mathcal{H}}{\delta \rho} = \frac{\partial}{\partial \tilde{\rho}} (\tilde{\rho} U(\tilde{\rho}, \tilde{s})), \quad \frac{\delta \mathcal{H}}{\delta \phi} = -\star \bar{\theta}, \quad \frac{\delta \mathcal{H}}{\delta \beta} = \star \bar{\omega}, \quad \frac{\delta \mathcal{H}}{\delta s} = \frac{\partial U(\tilde{\rho}, \tilde{s})}{\partial \tilde{s}} \tilde{\rho}. \tag{70}$$

**Proof.** Introducing the Hodge decomposition (49) into the Hamiltonian in (68) and using the inner product (5), we obtain the following Hamiltonian when the variables  $\rho, \phi, \beta, s$  are introduced

$$\begin{aligned} \mathcal{H}[\rho, \phi, \beta, s] &= \frac{1}{2} \int_{\Omega} \langle \langle d\phi + \delta\beta, d\phi + \delta\beta \rangle \rangle v_\Omega + \int_{\Omega} U(\tilde{\rho}, \tilde{s}) \rho \\ &= \frac{1}{2} \langle \langle d\phi + \delta\beta, d\phi + \delta\beta \rangle \rangle + \int_{\Omega} U(\tilde{\rho}, \tilde{s}) \rho. \end{aligned} \tag{71}$$

Using the definition of the density weighted vorticity and dilatation, after partial integration, the inner product in the Hamiltonian in (71) reduces to

$$\begin{aligned} \langle \langle d\phi + \delta\beta, d\phi + \delta\beta \rangle \rangle &= \langle \langle d\phi, d\phi \rangle \rangle + \langle \langle d\phi, \delta\beta \rangle \rangle + \langle \langle \delta\beta, \delta\beta \rangle \rangle + \langle \langle \delta\beta, d\phi \rangle \rangle \\ &= \langle \langle \phi, \delta d\phi \rangle \rangle + \langle \langle \beta, d\delta\beta \rangle \rangle = \langle \langle \phi, -\bar{\theta} \rangle \rangle + \langle \langle \beta, \bar{\omega} \rangle \rangle, \end{aligned}$$

where  $\langle \langle d\phi, \delta\beta \rangle \rangle = 0$  and  $\langle \langle \delta\beta, d\phi \rangle \rangle = 0$  since (49) is an orthogonal decomposition. Note that this is valid for both types of boundary conditions, since in either case the boundary integrals are zero. Introducing this inner product into (71) we obtain (69).

Consider (71) in the form

$$\mathcal{H}[\rho, \phi, \beta, s] = \frac{1}{2} (\langle \langle d\phi, d\phi \rangle \rangle + \langle \langle \delta\beta, \delta\beta \rangle \rangle) + \int_{\Omega} U(\tilde{\rho}, \tilde{s}) \rho, \tag{72}$$

where the inner products are in the space  $L^2\Lambda^1(\Omega)$ . Let us calculate  $\delta_\phi \mathcal{H}$  first. For  $\phi, \partial\phi \in \mathcal{D}(L_0)$  and  $\beta \in \mathcal{D}(L_2)$ , with either essential or natural homogeneous boundary conditions on  $\phi$  and  $\partial\phi$ , we have

$$\begin{aligned} \mathcal{H}[\rho, \phi + \partial\phi, \beta, s] &= \int_{\Omega} H(\rho, \phi, \beta, s) + \langle \langle d\phi, d(\partial\phi) \rangle \rangle + \{\text{h.o.t. in } \partial\phi\} \\ &= \int_{\Omega} H(\rho, \phi, \beta, s) + \langle \langle \delta d\phi, \partial\phi \rangle \rangle + \{\text{h.o.t. in } \partial\phi\}. \end{aligned}$$

Therefore,

$$\frac{\delta \mathcal{H}}{\delta \phi} = \star \delta d\phi = -d(\star d\phi) = -\star \bar{\theta}. \tag{73}$$

Similarly, let us calculate the variational derivative  $\delta_\beta \mathcal{H}$ . For  $\beta, \partial\beta \in \mathcal{D}(L_2)$ , we have for either boundary conditions,

$$\mathcal{H}[\rho, \phi, \beta + \partial\beta, s] = \int_\Omega H(\rho, \phi, \beta, s) + \langle \partial\beta, d\delta\beta \rangle + \{\text{h. o. t. in } \partial\beta\}. \tag{74}$$

Hence, we obtain that

$$\frac{\delta \mathcal{H}}{\delta \beta} = \star d\delta\beta = \star \bar{\omega}, \tag{75}$$

which completes the proof of this lemma. The variational derivatives of the Hamiltonian with respect to the variables  $\rho$  and  $s$ , given in (70), can easily be calculated, see [8].  $\square$

**Remark 7.** We have defined the problem in the bar variables to account for inhomogeneous boundary conditions, see Section 2.3. From here on we drop the bar to make the notation simpler.

Our aim is now to formulate the Hamiltonian as a functional of  $\rho, \theta, \omega, s$  and calculate the variational derivatives with respect to these new variables.

**Lemma 3.** *The Hamiltonian density in the variables  $\rho, \theta, \omega, s$  is a mapping*

$$\begin{aligned} H : H\Lambda^3 \times \mathfrak{B}_0^* \times \mathfrak{B}^2 \times H\Lambda^3 &\rightarrow L^2\Lambda^3, \\ (\rho, \theta, \omega, s) &\mapsto H(\rho, \theta, \omega, s), \end{aligned} \tag{76}$$

which results in the total energy

$$\mathcal{H}[\rho, \theta, \omega, s] = \int_\Omega \left( \frac{1}{2} (\theta \wedge \star K_0 \theta + \omega \wedge \star K_2 \omega) + U(\tilde{\rho}, \tilde{s}) \rho \right), \tag{77}$$

where  $K_k$  is the solution operator of the Hodge Laplacian operator  $L_k$ ,  $k = 0, 2$ . The variational derivatives of the Hamiltonian functional are:

$$\frac{\delta \mathcal{H}}{\delta \rho} = \frac{\partial}{\partial \tilde{\rho}} (\tilde{\rho} U(\tilde{\rho}, \tilde{s})), \quad \frac{\delta \mathcal{H}}{\delta \omega} = \star K_2 \omega = \star \beta, \tag{78}$$

$$\frac{\delta \mathcal{H}}{\delta \theta} = \star K_0 \theta = -\star \phi, \quad \frac{\delta \mathcal{H}}{\delta s} = \frac{\partial U(\tilde{\rho}, \tilde{s})}{\partial \tilde{s}} \tilde{\rho}. \tag{79}$$

**Proof.** Using Lemma 2, and that  $\theta$  and  $\omega$  are in the domain of the solution operators  $K_0$  and  $K_2$  of the Hodge Laplacian problems for  $\phi$  and  $\beta$ , respectively, the Hamiltonian can be written as

$$\mathcal{H}[\rho, \theta, \omega, s] = \int_\Omega \left( \frac{1}{2} (\theta \wedge \star K_0 \theta + \omega \wedge \star K_2 \omega) + U(\tilde{\rho}, \tilde{s}) \rho \right). \tag{80}$$

Let  $\theta, \partial\theta \in \mathfrak{B}_0^*$  and  $\omega, \partial\omega \in \mathfrak{B}^2$ . The variational derivatives of the Hamiltonian with respect to  $\theta$  and  $\omega$  can be obtained from

$$\begin{aligned} \mathcal{H}[\rho, \theta + \partial\theta, \omega + \partial\omega, s] &= \frac{1}{2} \int_\Omega (\theta + \partial\theta) \wedge \star K_0 (\theta + \partial\theta) + \int_\Omega U(\tilde{\rho}, \tilde{s}) \rho + \frac{1}{2} \int_\Omega (\omega + \partial\omega) \wedge \star K_2 (\omega + \partial\omega) \\ &= \int_\Omega H(\rho, \theta, \omega, s) + \frac{1}{2} \int_\Omega (\theta \wedge \star K_0 (\partial\theta) + \partial\theta \wedge \star K_0 \theta) \\ &\quad + \frac{1}{2} \int_\Omega (\omega \wedge \star K_2 (\partial\omega) + \partial\omega \wedge \star K_2 \omega) + \{\text{h.o.t. in } \partial\theta, \partial\omega\}. \end{aligned} \tag{81}$$

Here  $\partial\theta$  and  $\partial\omega$  denote the variation of  $\theta$  and  $\omega$ , respectively, to avoid confusion with the coderivative operator  $\delta$ . In order to further investigate the last two integrals in (81), we use Lemma 2, where the variational derivatives of the Hamiltonian with respect to  $\phi$  and  $\beta$  are given, then apply the variational chain rule to obtain  $\delta_\theta \mathcal{H}$  and  $\delta_\omega \mathcal{H}$ .

To obtain the variational derivative of the Hamiltonian with respect to  $\theta$ , apply the variational chain rule as follows

$$\begin{aligned} \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \phi} \wedge \partial \phi &= \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \theta} \wedge \partial \theta = - \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \theta} \wedge \delta d(\partial \phi) = - \left\langle \star \frac{\delta \mathcal{H}}{\delta \theta}, \delta d(\partial \phi) \right\rangle \\ &= - \int_{\Omega} d \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) \wedge \partial \phi + \int_{\partial \Omega} \text{tr}(\partial \phi) \wedge \text{tr} \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) + \int_{\partial \Omega} \text{tr}(\star d(\partial \phi)) \wedge \text{tr} \left( \star \frac{\delta \mathcal{H}}{\delta \theta} \right), \end{aligned} \tag{82}$$

where  $\partial \phi, \partial \theta = -\delta d(\partial \phi)$  denote the variations of  $\phi$  and  $\theta$ , to avoid confusion with the coderivative operator  $\delta$ . Analogously, to obtain the variational derivative of the Hamiltonian with respect to  $\omega$ , consider the variational chain rule

$$\begin{aligned} \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \beta} \wedge \partial \beta &= \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \omega} \wedge \partial \omega = \int_{\Omega} \frac{\delta \mathcal{H}}{\delta \omega} \wedge d\delta(\partial \beta) = \left\langle \star \frac{\delta \mathcal{H}}{\delta \omega}, d\delta(\partial \beta) \right\rangle \\ &= \int_{\Omega} \delta d \left( \frac{\delta \mathcal{H}}{\delta \omega} \right) \wedge \partial \beta + \int_{\partial \Omega} \text{tr}(\delta(\partial \beta)) \wedge \text{tr} \frac{\delta \mathcal{H}}{\delta \omega} - \int_{\partial \Omega} \text{tr}(\star \partial \beta) \wedge \text{tr} \left( \star d \frac{\delta \mathcal{H}}{\delta \omega} \right). \end{aligned} \tag{83}$$

We now have two choices.

First, when Case 1 applies in Lemma 1, then  $\text{tr}(\star d(\partial \phi)) = 0$ , hence the variational equation (82) becomes

$$\int_{\Omega} \frac{\delta \mathcal{H}}{\delta \phi} \wedge \partial \phi = - \int_{\Omega} d \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) \wedge \partial \phi + \int_{\partial \Omega} \text{tr}(\partial \phi) \wedge \text{tr} \left( \frac{\delta \mathcal{H}}{\delta \theta} \right), \tag{84}$$

$\forall \partial \phi \in H\Lambda^0(\Omega)$ , with  $\text{tr}(\star d(\partial \phi)) = 0$ . Choose  $\partial \phi$  such that the boundary integral in (84) is zero. We thus obtain that  $\frac{\delta \mathcal{H}}{\delta \theta}$  solves the differential equation

$$d \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) = - \frac{\delta \mathcal{H}}{\delta \phi} = \star \theta \quad \text{in } \Omega. \tag{85}$$

Consider now the variation  $\partial \phi \in \mathcal{D}(L_0)$  arbitrary. Inserting (85) into the variational equation (84), we obtain that

$$\text{tr} \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) = 0, \tag{86}$$

which together with (85) is precisely the  $\mathfrak{B}$  problem with essential boundary conditions (33)–(34) for  $\frac{\delta \mathcal{H}}{\delta \theta}$ , with weakly imposed boundary condition (86). On the other hand, combining (85) with (73) leads to

$$\frac{\delta \mathcal{H}}{\delta \theta} = - \star \phi + h, \tag{87}$$

with  $h \in \mathfrak{Z}_3^*$ , the null space of  $\delta$ . The  $\mathfrak{B}$  problem for  $\frac{\delta \mathcal{H}}{\delta \theta}$  has however, a unique solution  $\frac{\delta \mathcal{H}}{\delta \theta} \in \overset{\circ}{\mathfrak{B}}^3$ , hence the side condition  $\frac{\delta \mathcal{H}}{\delta \theta} \perp \mathfrak{Z}_3^*$  is satisfied. Consequently,

$$\frac{\delta \mathcal{H}}{\delta \theta} = - \star \phi, \tag{88}$$

where  $\phi$  is the unique solution of the  $\mathfrak{B}^*$  problem (28)–(29).

We still need to calculate  $\delta_{\omega} \mathcal{H}$ , when Case 1 applies. Since  $\text{tr}(\star \partial \beta) = 0$ , the last integral in (83) cancels. Using the same arguments as before, we obtain the  $\mathfrak{B}^*$  problem with essential boundary condition

$$\delta d \left( \frac{\delta \mathcal{H}}{\delta \omega} \right) = \frac{\delta \mathcal{H}}{\delta \beta} \quad \text{in } \Omega, \quad \text{tr} \left( \frac{\delta \mathcal{H}}{\delta \omega} \right) = 0 \quad \text{on } \partial \Omega. \tag{89}$$

Combined with (75), we obtain the following equation

$$\delta d \left( \frac{\delta \mathcal{H}}{\delta \omega} \right) = \star d \delta \beta \quad \text{in } \Omega, \tag{90}$$

which leads to  $\frac{\delta \mathcal{H}}{\delta \omega} = \star \beta + h$ , with  $h \in \mathfrak{Z}_1^1$ , the null space of  $d$ . On the other hand, the  $\mathfrak{B}^*$  problem (89) has a unique solution  $\frac{\delta \mathcal{H}}{\delta \omega} \in \mathfrak{B}_1^* = \mathfrak{Z}_1^{1\perp}$ . Therefore,

$$\frac{\delta \mathcal{H}}{\delta \omega} = \star \beta, \tag{91}$$

where  $\beta$  is the unique solution of the  $\mathfrak{B}$  problem (26)–(27).

When Case 2 applies, the first boundary integral in both variational equations (82) and (83) will be zero. In a completely analogous way as in Case 1, we obtain the following  $\mathfrak{B}$  problem for  $\frac{\delta \mathcal{H}}{\delta \theta}$  when natural boundary conditions hold

$$d\delta \left( \frac{\delta \mathcal{H}}{\delta \theta} \right) = \star \theta \quad \text{in } \Omega, \quad \text{tr} \left( \star \frac{\delta \mathcal{H}}{\delta \theta} \right) = 0 \quad \text{on } \partial \Omega, \tag{92}$$

and the  $\mathfrak{B}^*$  problem with natural boundary condition for  $\frac{\delta \mathcal{H}}{\delta \omega}$

$$\delta d \left( \frac{\delta \mathcal{H}}{\delta \omega} \right) = \star \omega \quad \text{in } \Omega, \quad \text{tr} \left( \star d \frac{\delta \mathcal{H}}{\delta \omega} \right) = 0 \quad \text{on } \partial \Omega. \tag{93}$$

Analyzing the solution of these problems leads to the same variational derivatives (88) and (91).

The variational derivatives of the Hamiltonian with respect to the variables  $\rho$  and  $s$ , given in (78) and (79), respectively, can easily be calculated, see [8].  $\square$

Summarizing, when  $\phi$  and  $\beta$  solve a  $\mathfrak{B}^*$  and  $\mathfrak{B}$  problem, respectively, with (in)homogeneous natural or essential boundary conditions, the variational derivatives  $\frac{\delta \mathcal{H}}{\delta \theta}$  and  $\frac{\delta \mathcal{H}}{\delta \omega}$  will solve a dual problem, viz. a  $\mathfrak{B}$  and  $\mathfrak{B}^*$  problem, respectively, with the corresponding (dual) boundary conditions.

### 5. Poisson bracket

The nonlinear system (46)–(48) has a Hamiltonian formulation with the Poisson bracket of Morrison and Green [3,2] with the Hamiltonian given by (67) in the  $\rho, u, s$  variables, (see also [8]). The Poisson bracket has the form

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & - \int_{\Omega} \left[ \underbrace{\left( \frac{\delta \mathcal{F}}{\delta \rho} \wedge d \frac{\delta \mathcal{G}}{\delta u} - \frac{\delta \mathcal{G}}{\delta \rho} \wedge d \frac{\delta \mathcal{F}}{\delta u} \right)}_{T_1} + \star i_X \star \underbrace{\left( \star \frac{\delta \mathcal{G}}{\delta u} \wedge \star \frac{\delta \mathcal{F}}{\delta u} \right)}_{T_2} \right. \\ & \left. + \underbrace{\frac{1}{\rho} \wedge d\tilde{s} \wedge \left( \frac{\delta \mathcal{F}}{\delta s} \wedge \frac{\delta \mathcal{G}}{\delta u} - \frac{\delta \mathcal{G}}{\delta s} \wedge \frac{\delta \mathcal{F}}{\delta u} \right)}_{T_3} \right], \end{aligned} \tag{94}$$

with the vector field  $X = \left( \frac{\star du}{\rho} \right)^\sharp$ . The aim of this section is to transform the Poisson bracket into the new set of variables  $\rho, \theta, \omega, s$  and to properly account for the boundary conditions. We derive the bracket in a well-chosen functional space using the functional chain rule.

**Lemma 4.** Consider the Hodge decomposition of  $\zeta$  in (49) and let  $\mathcal{F}[\rho, \theta, \omega, s]$  and  $\mathcal{G}[\rho, \theta, \omega, s]$  be arbitrary functionals. Assume that

$$\zeta \in \mathring{H}\Lambda^1(\Omega) \cap H^* \Lambda^1(\Omega) \cap \mathring{\mathfrak{H}}^{1\perp}, \quad \text{and} \quad \text{tr} \left( \star \frac{\delta \mathcal{F}}{\delta \theta} \right) = 0, \quad \text{tr} \left( \star \frac{\delta \mathcal{G}}{\delta \theta} \right) = 0 \tag{95}$$

or

$$\zeta \in H\Lambda^1(\Omega) \cap H^* \Lambda^1(\Omega) \cap \mathring{\mathfrak{H}}^{1\perp}, \quad \text{and} \quad \text{tr} \left( \frac{\delta \mathcal{F}}{\delta \omega} \right) = 0, \quad \text{tr} \left( \frac{\delta \mathcal{G}}{\delta \omega} \right) = 0. \tag{96}$$

Then, the bracket (94) in terms of the variables  $\rho, \theta, \omega, s$  has the form

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & - \int_{\Omega} \left[ \left( \frac{\delta \mathcal{F}}{\delta \rho} \wedge d \left( \sqrt{\rho} \wedge \alpha(\mathcal{G}) \right) - \frac{\delta \mathcal{G}}{\delta \rho} \wedge d \left( \sqrt{\rho} \wedge \alpha(\mathcal{F}) \right) \right) + \star \left( \frac{\zeta}{2\rho} \wedge \alpha(\mathcal{F}) \right) \wedge d \left( \sqrt{\rho} \wedge \alpha(\mathcal{G}) \right) \right. \\ & \left. - \star \left( \frac{\zeta}{2\rho} \wedge \alpha(\mathcal{G}) \right) \wedge d \left( \sqrt{\rho} \wedge \alpha(\mathcal{F}) \right) + \frac{d\tilde{s}}{\sqrt{\rho}} \wedge \left( \frac{\delta \mathcal{F}}{\delta s} \wedge \alpha(\mathcal{G}) - \frac{\delta \mathcal{G}}{\delta s} \wedge \alpha(\mathcal{F}) \right) \right], \end{aligned} \tag{97}$$

where  $\alpha(\cdot)$  is the following operator on functionals

$$\alpha(\cdot) = d \frac{\delta \cdot}{\delta \omega} + \delta \frac{\delta \cdot}{\delta \theta}. \tag{98}$$

**Proof.** The transformation of the bracket requires the use of the chain rule for functional derivatives. First, we would like to know how the value of  $\zeta[\rho, u]$ , given by (49), changes as  $\rho$  and  $u$  are slightly perturbed, say  $\rho \rightarrow \rho + \epsilon \partial\rho$  and  $u \rightarrow u + \epsilon \partial u$ . The first variation  $\partial\zeta$  of  $\zeta$  induced by  $\partial\rho$  is given by (see e.g. [2])

$$\partial\zeta[\rho; \partial\rho, u] = \left. \frac{d}{d\epsilon} \zeta[\rho + \epsilon \partial\rho, u] \right|_{\epsilon=0} = \frac{u}{2\sqrt{\bar{\rho}}} \wedge \star \partial\rho. \tag{99}$$

Similarly, the variation  $\partial\zeta$  of  $\zeta$  induced by  $\partial u$  is given by

$$\partial\zeta[\rho, u; \partial u] = \left. \frac{d}{d\epsilon} \zeta[\rho, u + \epsilon \partial u] \right|_{\epsilon=0} = \sqrt{\bar{\rho}} \wedge \partial u. \tag{100}$$

Hence, the total variation  $\partial\zeta$  of  $\zeta$  is

$$\partial\zeta[\rho, u; \partial\rho, \partial u] = \frac{u}{2\sqrt{\bar{\rho}}} \wedge \star \partial\rho + \sqrt{\bar{\rho}} \wedge \partial u. \tag{101}$$

We can define a functional of  $\rho, \theta, \omega, s$  by introducing the Hodge decomposition (49) into  $\mathcal{F}[\rho, u, s]$  and obtain  $\bar{\mathcal{F}}[\rho, \theta, \omega, s] = \mathcal{F}[\rho, u, s]$ . This means that the following variational equation holds

$$\left\langle \star \frac{\delta \mathcal{F}}{\delta \rho}, \partial\rho \right\rangle + \left\langle \star \frac{\delta \mathcal{F}}{\delta u}, \partial u \right\rangle = \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \rho}, \partial\rho \right\rangle + \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \theta}, \partial\theta \right\rangle + \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \omega}, \partial\omega \right\rangle, \tag{102}$$

where  $\partial\theta = -\delta(\partial\zeta)$  and  $\partial\omega = d(\partial\zeta)$ . By partial integration, we obtain

$$\begin{aligned} \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \theta}, \partial\theta \right\rangle &= - \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \theta}, \delta(\partial\zeta) \right\rangle = - \left\langle d \star \frac{\delta \bar{\mathcal{F}}}{\delta \theta}, \partial\zeta \right\rangle + \int_{\partial\Omega} \text{tr} \left( \star \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \right) \wedge \text{tr}(\star \partial\zeta) \\ &= \int_{\Omega} \delta \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \wedge \partial\zeta = \int_{\Omega} \left( \delta \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \wedge \frac{u}{2\sqrt{\bar{\rho}}} \wedge \star \partial\rho + \delta \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \wedge \sqrt{\bar{\rho}} \wedge \partial u \right), \end{aligned}$$

where the boundary integral cancels, in case of decomposition (a) because  $\text{tr}(\star \partial\zeta) = 0$ , and in case of decomposition (b) because of the assumption  $\text{tr}(\star \frac{\delta \mathcal{F}}{\delta \theta}) = 0$ . Similarly, we obtain by partial integration that

$$\begin{aligned} \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \omega}, \partial\omega \right\rangle &= \left\langle \star \frac{\delta \bar{\mathcal{F}}}{\delta \omega}, d(\partial\zeta) \right\rangle = \left\langle d \star \frac{\delta \bar{\mathcal{F}}}{\delta \omega}, \partial\zeta \right\rangle + \int_{\partial\Omega} \text{tr}(\partial\zeta) \wedge \text{tr} \left( \frac{\delta \bar{\mathcal{F}}}{\delta \omega} \right) \\ &= \int_{\Omega} \left( d \frac{\delta \bar{\mathcal{F}}}{\delta \omega} \wedge \frac{u}{2\sqrt{\bar{\rho}}} \wedge \star \partial\rho + d \frac{\delta \bar{\mathcal{F}}}{\delta \omega} \wedge \sqrt{\bar{\rho}} \wedge \partial u \right). \end{aligned}$$

Here the boundary integral cancels, in case of decomposition (a) because of the assumption  $\text{tr}(\frac{\delta \mathcal{F}}{\delta \omega}) = 0$ , and in case of decomposition (b) because  $\text{tr}(\partial\zeta) = 0$ . Since the variational equation (102) holds for all  $\partial\rho, \partial u$ , we obtain the relations

$$\left. \frac{\delta \mathcal{F}}{\delta \rho} \right|_{\rho, u, s} = \left. \frac{\delta \bar{\mathcal{F}}}{\delta \rho} \right|_{\rho, \omega, \theta, s} + \star \left( \frac{u}{2\sqrt{\bar{\rho}}} \wedge \left( d \frac{\delta \bar{\mathcal{F}}}{\delta \omega} + \delta \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \right) \right) \tag{103}$$

$$\frac{\delta \mathcal{F}}{\delta u} = \sqrt{\bar{\rho}} \wedge \left( d \frac{\delta \bar{\mathcal{F}}}{\delta \omega} + \delta \frac{\delta \bar{\mathcal{F}}}{\delta \theta} \right). \tag{104}$$

From here on we will drop the overbar on  $\mathcal{F}$ . We can write the functional chain rules in (103) and (104) as

$$\left. \frac{\delta \mathcal{F}}{\delta \rho} \right|_{\rho, u, s} = \left. \frac{\delta \mathcal{F}}{\delta \rho} \right|_{\rho, \omega, \theta, s} + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \alpha(\mathcal{F}) \right), \tag{105}$$

$$\frac{\delta \mathcal{F}}{\delta u} = \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}). \tag{106}$$

Introducing the relations above into the bracket (94) we obtain (97), which completes the proof of the lemma.  $\square$

**Remark 8.** It is straightforward to verify that the bracket in (97) is skew-symmetric. The Jacobi identity is also satisfied, see [3].

In the next technical lemma we give the integrated-by-parts form of the bracket (97). This is only done to make it easier to verify the statement of Theorem 1. The proof is given in Appendix A.

**Lemma 5.** Under the assumptions of Lemma 4, by partial integration, the bracket in (97) can be transformed into

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} = & - \int_{\Omega} \left[ \frac{\delta \mathcal{F}}{\delta \rho} \wedge d(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G})) + \frac{\delta \mathcal{F}}{\delta s} \wedge \frac{1}{\sqrt{\bar{\rho}}} \wedge d\bar{s} \wedge \alpha(\mathcal{G}) + \frac{\delta \mathcal{F}}{\delta \omega} \wedge d\gamma(\text{grad } \mathcal{G}) - \frac{\delta \mathcal{F}}{\delta \theta} \wedge \delta\gamma(\text{grad } \mathcal{G}) \right] \\ & + \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F})) \wedge \text{tr} \left[ \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right], \end{aligned} \tag{107}$$

where

$$\text{grad } \mathcal{G} = (\delta_{\rho} \mathcal{G}, \delta_{\omega} \mathcal{G}, \delta_{\theta} \mathcal{G}, \delta_s \mathcal{G}). \tag{108}$$

Furthermore,

$$\begin{aligned} \gamma(\text{grad } \mathcal{G}) = & \frac{\zeta}{2\bar{\rho}} \wedge \star d(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G})) + \sqrt{\bar{\rho}} \wedge d \left( \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) \\ & + \star \left( \star d \left( \frac{\zeta}{\sqrt{\bar{\rho}}} \right) \wedge \star \alpha(\mathcal{G}) \right) - \frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge \frac{\delta \mathcal{G}}{\delta s}. \end{aligned} \tag{109}$$

**Remark 9.** The skew-symmetry of the bracket (107) can be directly checked by using partial integration.

**Theorem 1.** The equations of motion for the density  $\rho$ , vorticity  $\omega$ , dilatation  $\theta$  and entropy  $s$ , given by (60), (61), (62) and (63) respectively, are obtained from the bracket (107) as

$$\begin{aligned} \frac{\partial \rho}{\partial t} = \{\rho, \mathcal{H}\}, \quad \frac{\partial \omega}{\partial t} = \{\omega, \mathcal{H}\} - \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\omega)) \wedge \text{tr} \left( \frac{\delta \mathcal{H}}{\delta \rho} + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \alpha(\mathcal{H}) \right) \right), \\ \frac{\partial s}{\partial t} = \{s, \mathcal{H}\}, \quad \frac{\partial \theta}{\partial t} = \{\theta, \mathcal{H}\} - \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\theta)) \wedge \text{tr} \left( \frac{\delta \mathcal{H}}{\delta \rho} + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \alpha(\mathcal{H}) \right) \right), \end{aligned}$$

with  $\mathcal{H}$  the Hamiltonian given by (77).

**Proof.** The proof of this theorem is very straightforward if we observe that

$$\alpha(\mathcal{H}) = \star \zeta \quad \text{and} \quad \gamma(\text{grad } \mathcal{H}) = -\zeta_t. \quad \square \tag{110}$$

### 6. Stokes–Dirac structures

The treatment of infinite dimensional Hamiltonian systems in the literature seems mostly focused on systems with an infinite spatial domain, where the variables go to zero for the spatial variables tending to infinity [2], or on systems with boundary conditions such that the energy exchange through the boundary is zero. It is, however essential from an application point of view to describe a system with varying boundary conditions, including energy exchange through the boundary. In Remark 5 we already indicated how to treat inhomogeneous boundary conditions in the Hodge decomposition, which provide an essential ingredient for the Hamiltonian formulation. An alternative approach is to follow van der Schaft and Maschke [12], where a framework to overcome the difficulty of incorporating non-zero energy flow through the boundary in the Hamiltonian framework for distributed-parameter systems is presented. This is done by using the notion of a Stokes–Dirac structure [12,8]. We will demonstrate that both approaches lead to the same formulation. A general definition of a Stokes–Dirac structure is given as follows.

**Definition 7.** Let  $\mathcal{F}$  be a linear space (finite or infinite dimensional). There exists on  $\mathcal{F} \times \mathcal{E}$  the canonically defined symmetric bilinear form

$$\langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D := \langle e^1, \star f^2 \rangle + \langle e^2, \star f^1 \rangle = \int_{\Omega} (e^1 \wedge f^2 + e^2 \wedge f^1),$$

with  $f^i \in \mathcal{F}$ ,  $e^i \in \mathcal{E}$ ,  $i = 1, 2$ , and  $\langle \cdot, \cdot \rangle$  denoting the duality pairing between  $\mathcal{F}$  and its dual space  $\mathcal{E}$ . A Stokes–Dirac structure on  $\mathcal{F}$  is a linear subspace  $D \subset \mathcal{F} \times \mathcal{E}$ , such that

$$D = D^{\perp},$$

where  $\perp$  denotes the orthogonal complement with respect to the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle_D$ .

#### 6.1. Stokes–Dirac structure for the non-isentropic compressible Euler equations

The Stokes–Dirac structure for distributed-parameter systems used in [8] has a specific form by being defined on spaces of differential forms on the spatial domain of the system and its boundary. The construction of the Stokes–Dirac structure

emphasizes the geometrical content of the physical variables involved, by identifying them as appropriate differential forms. In [8] the description is given for the compressible Euler equations for an ideal isentropic fluid.

In this section we first extend the Stokes–Dirac structure given in [8] to the non-isentropic Euler equations (1)–(3). In Section 6.3 we derive the Stokes–Dirac structure for the non-isentropic Euler equations in the  $\rho, \omega, \theta, s$  variables. The linear spaces on which the Stokes–Dirac structure for the  $\rho, u, s$  variables will be defined are:

$$\mathcal{F} := \Lambda^3(\Omega) \times \Lambda^1(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\partial\Omega), \tag{111}$$

$$\mathcal{E} := \Lambda^0(\Omega) \times \Lambda^2(\Omega) \times \Lambda^0(\Omega) \times \Lambda^2(\partial\Omega). \tag{112}$$

The following theorem is an extension of Theorem 2.1 in [8] for the non-isentropic Euler equations. In the proof, we closely follow the proof of Theorem 2.1 in [8].

**Theorem 2 (Stokes–Dirac Structure).** *Let  $\Omega \subset \mathbb{R}^3$  be a three dimensional manifold with boundary  $\partial\Omega$ . Consider  $\mathcal{F}$  and  $\mathcal{E}$  as given by (111) and (112) respectively, together with the bilinear form*

$$\begin{aligned} \langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D &= \int_{\Omega} (e_{\rho}^1 \wedge f_{\rho}^2 + e_{\rho}^2 \wedge f_{\rho}^1 + e_u^1 \wedge f_u^2 + e_u^2 \wedge f_u^1 + e_s^1 \wedge f_s^2 + e_s^2 \wedge f_s^1) \\ &\quad + \int_{\partial\Omega} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1), \end{aligned} \tag{113}$$

where

$$f^i = (f_{\rho}^i, f_u^i, f_s^i, f_b^i) \in \mathcal{F}, \quad e^i = (e_{\rho}^i, e_u^i, e_s^i, e_b^i) \in \mathcal{E}, \quad i = 1, 2.$$

Then  $D \subset \mathcal{F} \times \mathcal{E}$  defined as

$$\begin{aligned} D &= \left\{ ((f_{\rho}, f_u, f_s, f_b), (e_{\rho}, e_u, e_s, e_b)) \in \mathcal{F} \times \mathcal{E} \mid f_{\rho} = de_u, f_s = \frac{1}{\rho} d\tilde{s} \wedge e_u, \right. \\ &\quad \left. f_u = de_{\rho} - \frac{1}{\rho} d\tilde{s} \wedge e_s + \frac{1}{\rho} \star((\star du) \wedge (\star e_u)), f_b = \text{tr}(e_{\rho}), e_b = -\text{tr}(e_u) \right\}, \end{aligned} \tag{114}$$

is a Stokes–Dirac structure with respect to the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle_D$  defined in (113).

**Proof.** The proof of this theorem consists of two steps.

**Step 1.** First we show that  $D \subset D^{\perp}$ . Let  $(f^1, e^1) \in D$  be fixed, and consider any  $(f^2, e^2) \in D$ . Substituting the definition of  $D$  into (113), we obtain that

$$\begin{aligned} I &:= \langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D \\ &= \int_{\Omega} \left[ e_{\rho}^1 \wedge de_u^2 + e_{\rho}^2 \wedge de_u^1 + e_u^1 \wedge (de_{\rho}^2 + T^2 + S^2) + e_u^2 \wedge (de_{\rho}^1 + T^1 + S^1) \right. \\ &\quad \left. + e_s^1 \wedge \frac{1}{\rho} d\tilde{s} \wedge e_u^2 + e_s^2 \wedge \frac{1}{\rho} d\tilde{s} \wedge e_u^1 \right] + \int_{\partial\Omega} (e_b^1 \wedge \text{tr}(e_{\rho}^2) + e_b^2 \wedge \text{tr}(e_{\rho}^1)), \end{aligned} \tag{115}$$

where  $T^i = \frac{1}{\rho} \star((\star du) \wedge (\star e_u^i))$  and  $S^i = -\frac{1}{\rho} d\tilde{s} \wedge e_s^i$  are 1-forms, for  $i = 1, 2$ . Regrouping the terms, we can write the integral (115) as

$$\begin{aligned} I &= \int_{\Omega} \underbrace{(e_{\rho}^1 \wedge de_u^2 + e_u^2 \wedge de_{\rho}^1)}_{I_1} + \underbrace{(e_{\rho}^2 \wedge de_u^1 + e_u^1 \wedge de_{\rho}^2)}_{I_2} + \underbrace{(e_u^1 \wedge T^2 + e_u^2 \wedge T^1)}_{I_3} \\ &\quad + \underbrace{\int_{\Omega} e_u^1 \wedge S^2 + e_u^2 \wedge S^1 + e_s^1 \wedge \frac{1}{\rho} d\tilde{s} \wedge e_u^2 + e_s^2 \wedge \frac{1}{\rho} d\tilde{s} \wedge e_u^1}_{I_4} + \int_{\partial\Omega} (e_b^1 \wedge \text{tr}(e_{\rho}^2) + e_b^2 \wedge \text{tr}(e_{\rho}^1)). \end{aligned}$$

Using the properties of the wedge product and the Leibniz rule for exterior differentiation, we obtain

$$I_1 = \int_{\Omega} (de_u^2 \wedge e_{\rho}^1 + e_u^2 \wedge de_{\rho}^1) = \int_{\Omega} d(e_u^2 \wedge e_{\rho}^1) = \int_{\partial\Omega} \text{tr}(e_u^2 \wedge e_{\rho}^1),$$

$$I_2 = \int_{\Omega} (de_u^1 \wedge e_{\rho}^2 + e_u^1 \wedge de_{\rho}^2) = \int_{\Omega} d(e_u^1 \wedge e_{\rho}^2) = \int_{\partial\Omega} \text{tr}(e_u^1 \wedge e_{\rho}^2).$$

Observe that

$$F(e_u^1, e_u^2) := \int_{\Omega} e_u^1 \wedge T^2 = \int_{\Omega} e_u^1 \wedge \frac{1}{\rho} \wedge \star(\star du \wedge \star e_u^2) \tag{116}$$



is skew-symmetric in  $e_u^1, e_u^2 \in \Lambda^2(\Omega)$ , that is,  $F(e_u^1, e_u^2) = -F(e_u^2, e_u^1)$ . This implies that  $I_3 = 0$ . Furthermore, introducing  $S^1$  into  $I_4$  we obtain that  $I_4 = 0$ . Therefore,

$$I = \int_{\partial\Omega} (\text{tr}(e_u^2) \wedge \text{tr}(e_\rho^1) + \text{tr}(e_u^1) \wedge \text{tr}(e_\rho^2) + e_b^1 \wedge \text{tr}(e_\rho^2) + e_b^2 \wedge \text{tr}(e_\rho^1)) = 0,$$

where the last equality is true since in  $D$  we have  $e_b^i = -\text{tr}(e_u^i)$ ,  $i = 1, 2$ . We proved that the bilinear form (113) is zero for all  $(f^2, e^2) \in D$ . Hence,  $(f^2, e^2) \in D^\perp$ .

**Step 2.** Next we show that  $D^\perp \subset D$ . Let  $(f^1, e^1) \in D^\perp$ . Then,

$$J := \langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D = 0, \quad \forall (f^2, e^2) \in D. \tag{117}$$

Since  $(f^2, e^2) \in D$ , (117) is equivalent to

$$J = \int_{\Omega} \left( \underline{e_\rho^1 \wedge de_u^2} + e_\rho^2 \wedge f_\rho^1 + \underline{e_u^1 \wedge de_\rho^2} + e_u^1 \wedge T^2 + e_u^1 \wedge S^2 + e_u^2 \wedge f_u^1 + e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^2 + e_s^2 \wedge f_s^1 \right) + \int_{\partial\Omega} (e_b^1 \wedge \text{tr}(e_\rho^2) - \text{tr}(e_u^2) \wedge f_b^1) = 0, \quad \forall e_\rho^2 \in \Lambda^0(\Omega), e_u^2 \in \Lambda^2(\Omega), e_s^2 \in \Lambda^0(\Omega). \tag{118}$$

Take  $e_\rho^2 \in \Lambda^0(\Omega)$ ,  $e_u^2 \in \Lambda^2(\Omega)$ , such that  $\text{tr}(e_\rho^2) = 0$  and  $\text{tr}(e_u^2) = 0$ . Then, the boundary integral in  $J$  vanishes. Using the Leibniz rule for the underlined terms, we obtain

$$J = \int_{\Omega} [d(e_\rho^1 \wedge e_u^2) - de_\rho^1 \wedge e_u^2 + f_\rho^1 \wedge e_\rho^2 + d(e_u^1 \wedge e_\rho^2) - de_u^1 \wedge e_\rho^2 + f_u^1 \wedge e_u^2 + e_u^1 \wedge T^2 + e_u^1 \wedge S^2 + e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^2 + e_s^2 \wedge f_s^1] = 0,$$

for all  $e_\rho^2 \in \Lambda^0(\Omega)$ ,  $e_u^2 \in \Lambda^2(\Omega)$ ,  $e_s^2 \in \Lambda^0(\Omega)$  with  $\text{tr}(e_\rho^2) = 0$  and  $\text{tr}(e_u^2) = 0$ . Using Stokes' theorem, the assumptions on  $e_\rho^2, e_u^2$ , the skew-symmetry of  $F(e_u^1, e_u^2)$  in (116) and

$$e_u^1 \wedge S^2 = e_u^1 \wedge \left( -\frac{1}{\bar{\rho}} d\tilde{s} \wedge e_s^2 \right) = -\frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^1 \wedge e_s^2,$$

we obtain

$$J = \int_{\Omega} \left[ \left( f_u^1 - de_\rho^1 - \frac{1}{\bar{\rho}} \star(\star du \wedge \star e_u^1) + e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \right) \wedge e_u^2 + (f_\rho^1 - de_u^1) \wedge e_\rho^2 + \left( f_s^1 - \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^1 \right) \wedge e_s^2 \right] = 0 \tag{119}$$

for all  $e_\rho^2 \in \Lambda^0(\Omega)$ ,  $e_u^2 \in \Lambda^2(\Omega)$ ,  $e_s^2 \in \Lambda^0(\Omega)$  for which  $\text{tr}(e_\rho^2) = 0$  and  $\text{tr}(e_u^2) = 0$ . Finally, (119) can only be satisfied if

$$f_\rho^1 = de_u^1, \tag{120}$$

$$f_u^1 = de_\rho^1 + \frac{1}{\bar{\rho}} \star(\star du \wedge \star e_u^1) - e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s}, \tag{121}$$

$$f_s^1 = \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^1, \tag{122}$$

which are the conditions stated in the definition of  $D$  in (114). We still need to verify the boundary conditions. Insert (120)–(122) into (118) and obtain

$$J = \int_{\Omega} \left( \underline{e_\rho^1 \wedge de_u^2} + e_\rho^2 \wedge de_u^1 + \underline{e_u^1 \wedge de_\rho^2} + e_u^1 \wedge T^2 + e_u^1 \wedge S^2 + \underline{e_u^2 \wedge de_\rho^1} + e_u^2 \wedge T^1 + e_u^2 \wedge \left( -e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \right) + e_s^1 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^2 + e_s^2 \wedge \frac{1}{\bar{\rho}} d\tilde{s} \wedge e_u^1 \right) + \int_{\partial\Omega} (e_b^1 \wedge \text{tr}(e_\rho^2) - \text{tr}(e_u^2) \wedge f_b^1) = 0, \quad \forall e_\rho^2 \in \Lambda^0(\Omega), e_u^2 \in \Lambda^2(\Omega), e_s^2 \in \Lambda^0(\Omega).$$

Using again the Leibniz rule for the underlined terms, we obtain

$$J = \int_{\Omega} [d(e_{\rho}^1 \wedge e_u^2) + d(e_u^1 \wedge e_{\rho}^2) + e_u^1 \wedge T^2 + e_u^2 \wedge T^1] + \int_{\partial\Omega} (e_b^1 \wedge \text{tr}(e_{\rho}^2) - \text{tr}(e_u^2) \wedge f_b^1) = 0.$$

Finally, applying Stokes' theorem and using the skew-symmetric property of  $F(e_u^1, e_u^2)$  yields

$$J = \int_{\partial\Omega} \text{tr}(e_u^2) \wedge (\text{tr}(e_{\rho}^1) - f_b^1) + (\text{tr}(e_u^1) + e_b^1) \wedge \text{tr}(e_{\rho}^2) = 0, \quad \forall e_{\rho}^2 \in \Lambda^0(\Omega), e_u^2 \in \Lambda^2(\Omega).$$

The integral  $J$  can only be zero if

$$\begin{aligned} f_b^1 &= \text{tr}(e_{\rho}^1) \\ e_b^1 &= -\text{tr}(e_u^1), \end{aligned}$$

which together with (120)–(122) proves that  $(f^1, e^1) \in D$ .  $\square$

### 6.2. Pseudo-Poisson bracket for the primitive variables derived from the Stokes–Dirac structure

Using the approach outlined in [8], we can associate a pseudo-Poisson bracket to the Stokes–Dirac structure defined in (114). The resulting skew-symmetric bracket has the form

$$\begin{aligned} \{k^1, k^1\}_D &= - \int_{\Omega} \left[ \delta_{\rho} k^1 \wedge d(\delta_u k^2) + \delta_u k^1 \wedge d(\delta_{\rho} k^2) + \frac{\star du}{\bar{\rho}} \wedge \star \delta_u k^2 \wedge \star \delta_u k^1 \right. \\ &\quad \left. + \frac{d\tilde{s}}{\bar{\rho}} \wedge (\delta_s k^1 \wedge \delta_u k^2 - \delta_s k^2 \wedge \delta_u k^1) \right] + \int_{\partial\Omega} \text{tr}(\delta_u k^1) \wedge \text{tr}(\delta_{\rho} k^2), \end{aligned} \tag{123}$$

where  $k^1, k^2$  belong to the set of functions

$$k : \Lambda^3(\Omega) \times \Lambda^1(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\partial\Omega) \rightarrow \mathbb{R}$$

whose derivatives

$$\delta k = (\delta_{\rho} k, \delta_u k, \delta_s k, \delta_b k) \in \Lambda^0(\Omega) \times \Lambda^0(\Omega) \times \Lambda^0(\Omega) \times \Lambda^2(\partial\Omega)$$

satisfy

$$\delta_b k = -\text{tr}(\delta_u k).$$

**Remark 10.** Note that if we integrate by parts the second term in the bracket of Morrison and Green (94), we obtain the bracket (123) derived from the Stokes–Dirac structure. Moreover, this bracket is skew-symmetric. The Jacobi identity for  $\{\cdot, \cdot\}_D$  in (123) is not automatically satisfied, and we call therefore  $\{\cdot, \cdot\}_D$  a pseudo-Poisson bracket as in [8].

### 6.3. Stokes–Dirac structure for vorticity–dilatation formulation of the compressible Euler equations

In this section we determine the Stokes–Dirac structure for the non-isentropic compressible Euler equations when written in the density weighted vorticity and dilatation variables. The linear spaces on which the Stokes–Dirac structure for the  $\rho, \omega, \theta, s$  variables will be defined are:

$$\mathcal{F} := \Lambda^3(\Omega) \times \Lambda^2(\Omega) \times \Lambda^0(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\partial\Omega) \tag{124}$$

$$\mathcal{E} := \Lambda^0(\Omega) \times \Lambda^1(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\Omega) \times \Lambda^2(\partial\Omega). \tag{125}$$

The Stokes–Dirac structure for the new variables is defined in the following theorem. Since the steps of the proof are analogous to the ones in the proof of Theorem 2, we only give the main steps in Appendix B.

**Theorem 3.** Let  $\Omega \subset \mathbb{R}^3$  be a three dimensional manifold with boundary  $\partial\Omega$ . Consider  $\mathcal{F}$  and  $\mathcal{E}$  as given in (124) and (125) respectively, together with the bilinear form

$$\begin{aligned} \langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D &:= \int_{\Omega} (e_{\rho}^1 \wedge f_{\rho}^2 + e_{\rho}^2 \wedge f_{\rho}^1 + e_{\omega}^1 \wedge f_{\omega}^2 + e_{\omega}^2 \wedge f_{\omega}^1 + e_{\theta}^1 \wedge f_{\theta}^2 \\ &\quad + e_{\theta}^2 \wedge f_{\theta}^1 + e_s^1 \wedge f_s^2 + e_s^2 \wedge f_s^1) + \int_{\partial\Omega} (e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1), \end{aligned} \tag{126}$$

where

$$f^i = (f_{\rho}^i, f_{\omega}^i, f_{\theta}^i, f_s^i, f_b^i) \in \mathcal{F}, \quad e^i = (e_{\rho}^i, e_{\omega}^i, e_{\theta}^i, e_s^i, e_b^i) \in \mathcal{E}, \quad i = 1, 2.$$

Then  $D \subset \mathcal{F} \times \mathcal{E}$  defined as

$$D = \left\{ (f_\rho, f_\omega, f_\theta, f_s, f_b), (e_\rho, e_\omega, e_\theta, e_s, e_b) \in \mathcal{F} \times \mathcal{E} \mid f_\rho = d(\sqrt{\bar{\rho}} \wedge d\sigma(e)), \right. \\ f_\omega = d\gamma(\bar{e}), f_\theta = -\delta\gamma(\bar{e}), f_s = \frac{1}{\sqrt{\bar{\rho}}} \wedge d\tilde{s} \wedge \sigma(e), \\ \left. f_b = \text{tr} \left( e_\rho + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \sigma(e) \right) \right), e_b = -\text{tr} \left( \sqrt{\bar{\rho}} \wedge \sigma(e) \right), \text{tr}(e_\omega) = 0, \text{tr}(\star e_\theta) = 0 \right\}, \quad (127)$$

with  $\sigma(e) = de_\omega + \delta e_\theta$ ,  $\bar{e} = (e_\rho, e_\omega, e_\theta, e_s)$ , is a Stokes–Dirac structure with respect to the bilinear form  $\langle\langle \cdot, \cdot \rangle\rangle_D$  defined in (126).

#### 6.4. Pseudo-Poisson bracket for the vorticity–dilatation variable derived from the Stokes–Dirac structure

Using the approach outlined in [8], we can define a pseudo-Poisson bracket from the Stokes–Dirac structure (127) as follows

$$\{k^1, k^1\}_D = - \int_\Omega \left[ \delta_\rho k^1 \wedge d \left( \sqrt{\bar{\rho}} \wedge d\alpha(k^2) \right) + \frac{d\tilde{s}}{\sqrt{\bar{\rho}}} \wedge (\delta_s k^1 \wedge \alpha(k^2)) \right. \\ \left. + \delta_\omega k^1 \wedge d(\gamma(\text{grad } k^2)) - \delta_\theta k^1 \wedge \delta(\gamma(\text{grad } k^2)) \right] \\ + \int_{\partial\Omega} \text{tr} \left( \sqrt{\bar{\rho}} \wedge \alpha(k^1) \right) \wedge \text{tr} \left( \delta_\rho k^2 + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \alpha(k^2) \right) \right), \quad (128)$$

where  $\alpha(k) = d(\delta_\omega k) + \delta(\delta_\theta k)$  and the operator  $\gamma(\cdot)$  is defined in (109). Here  $k^1, k^2$  belong to the set of functions

$$k : \Lambda^3(\Omega) \times \Lambda^2(\Omega) \times \Lambda^0(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\partial\Omega) \rightarrow \mathbb{R}$$

whose derivatives

$$\delta k = (\delta_\rho k, \delta_\omega k, \delta_\theta k, \delta_s k, \delta_b k) \in \Lambda^0(\Omega) \times \Lambda^1(\Omega) \times \Lambda^3(\Omega) \times \Lambda^0(\Omega) \times \Lambda^2(\partial\Omega)$$

satisfy (see last two lines of (128))

$$\delta_b k = -\text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(k)), \quad \text{tr}(\delta_\omega k) = 0, \quad \text{tr}(\star \delta_\theta k) = 0.$$

**Remark 11.** Note that the bracket defined in (107) is identical to the one obtained from the Stokes–Dirac structure (128). Hence, this bracket is skew-symmetric.

#### 6.5. Distributed-parameter port-Hamiltonian system

In this last section we make the connection between the Hamiltonian system and Stokes–Dirac structure for the non-isentropic compressible Euler equations in vorticity–dilatation formulation. Consider the Hamiltonian density  $H$  in (76) and total energy  $\mathcal{H}$  in (77), with the gradient vector denoted as

$$\text{grad } \mathcal{H} = (\delta_\rho \mathcal{H}, \delta_\omega \mathcal{H}, \delta_\theta \mathcal{H}, \delta_s \mathcal{H}) \in H^* \Lambda^0 \times \mathfrak{B}_1^* \times \mathfrak{B}^3 \times H^* \Lambda^0.$$

Consider now the time function

$$F : \mathbb{R} \rightarrow H\Lambda^3 \times \mathfrak{B}^2 \times \mathfrak{B}_0^* \times H\Lambda^3,$$

$$t \mapsto (\rho(t), \omega(t), \theta(t), s(t))$$

and the Hamiltonian  $\mathcal{H}[\rho(t), \omega(t), \theta(t), s(t)]$  along this trajectory. Then,

$$\frac{d\mathcal{H}}{dt} = \int_\Omega \delta_\rho \mathcal{H} \wedge \frac{\partial \rho}{\partial t} + \delta_\omega \mathcal{H} \wedge \frac{\partial \omega}{\partial t} + \delta_\theta \mathcal{H} \wedge \frac{\partial \theta}{\partial t} + \delta_s \mathcal{H} \wedge \frac{\partial s}{\partial t}.$$

The variables  $\rho_t, \omega_t, \theta_t$  and  $s_t$  represent generalized velocities of the energy variables  $\rho, \omega, \theta, s$ . They are connected to the Stokes–Dirac structure  $D$  in (127) by setting

$$f_\rho = -\frac{\partial \rho}{\partial t}, \quad f_\omega = -\frac{\partial \omega}{\partial t}, \quad f_\theta = -\frac{\partial \theta}{\partial t}, \quad f_s = -\frac{\partial s}{\partial t}.$$

Finally, setting  $e_\rho = \delta_\rho \mathcal{H}$ ,  $e_\omega = \delta_\omega \mathcal{H}$ ,  $e_\theta = \delta_\theta \mathcal{H}$ ,  $e_s = \delta_s \mathcal{H}$ , in the Stokes–Dirac structure, we obtain the distributed-parameter port-Hamiltonian system for the non-isentropic compressible Euler equations.

In order to simplify notations, let us split the operator  $\gamma(\text{grad } \mathcal{G})$  in (109) acting on a functional  $\mathcal{G}$ , as follows

$$\gamma(\text{grad } \mathcal{G}) = \gamma_\rho(\text{grad } \mathcal{G}) + \gamma_{\omega,\theta}(\text{grad } \mathcal{G}) + \gamma_s(\text{grad } \mathcal{G}), \tag{129}$$

where

$$\gamma_{\omega,\theta}(\text{grad } \mathcal{G}) = \frac{\zeta}{2\bar{\rho}} \wedge \star d(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G})) + \sqrt{\bar{\rho}} \wedge d\left(\star\left(\alpha(\mathcal{G}) \wedge \frac{\zeta}{2\bar{\rho}}\right)\right) + \star\left(\star d\left(\frac{\zeta}{\sqrt{\bar{\rho}}}\right) \wedge \star\alpha(\mathcal{G})\right), \tag{130}$$

with  $\alpha(\mathcal{G})$  defined in (98), and

$$\gamma_\rho(\text{grad } \mathcal{G}) = \sqrt{\bar{\rho}} \wedge d \frac{\delta \mathcal{G}}{\delta \rho}, \quad \gamma_s(\text{grad } \mathcal{G}) = -\frac{1}{\sqrt{\bar{\rho}}} \wedge d\tilde{s} \wedge \frac{\delta \mathcal{G}}{\delta s}.$$

**Corollary 2.** *The distributed-parameter port-Hamiltonian system for the three dimensional manifold  $\Omega$ , state space  $H\Lambda^3 \times \mathfrak{B}^2 \times \mathfrak{B}_0^* \times H\Lambda^3$ , Stokes–Dirac structure  $D$ , (127), and Hamiltonian  $\mathcal{H}$ , (76), is given as*

$$\begin{aligned} \begin{bmatrix} -\frac{\partial \rho}{\partial t} \\ \frac{\partial \omega}{\partial t} \\ \frac{\partial \theta}{\partial t} \\ \frac{\partial t}{\partial s} \\ -\frac{\partial t}{\partial t} \end{bmatrix} &= \begin{bmatrix} 0 & d(\sqrt{\bar{\rho}} \wedge d\cdot) & d(\sqrt{\bar{\rho}} \wedge \delta\cdot) & 0 \\ d\gamma_\rho(\cdot) & d\gamma_\omega(\cdot) & d\gamma_\theta(\cdot) & d\gamma_s(\cdot) \\ -\delta\gamma_\rho(\cdot) & -\delta\gamma_\omega(\cdot) & -\delta\gamma_\theta(\cdot) & -\delta\gamma_s(\cdot) \\ 0 & \frac{d\tilde{s}}{\sqrt{\bar{\rho}}} \wedge d\cdot & \frac{d\tilde{s}}{\sqrt{\bar{\rho}}} \wedge \delta\cdot & 0 \end{bmatrix} \begin{bmatrix} \delta_\rho \mathcal{H} \\ \delta_\omega \mathcal{H} \\ \delta_\theta \mathcal{H} \\ \delta_s \mathcal{H} \end{bmatrix} \\ \begin{bmatrix} f_b \\ e_b \end{bmatrix} &= \text{tr} \left( \begin{bmatrix} 1 & \star\left(\frac{\zeta}{2\bar{\rho}} \wedge d\cdot\right) & \star\left(\frac{\zeta}{2\bar{\rho}} \wedge \delta\cdot\right) \\ 0 & -\sqrt{\bar{\rho}} \wedge d\cdot & -\sqrt{\bar{\rho}} \wedge \delta\cdot \end{bmatrix} \begin{bmatrix} \delta_\rho \mathcal{H} \\ \delta_\omega \mathcal{H} \\ \delta_\theta \mathcal{H} \end{bmatrix} \right). \end{aligned} \tag{131}$$

Note that (131) might be a good starting point for nonlinear boundary control systems. By the power-conserving property of any Stokes–Dirac structure, i.e.,

$$\langle\langle f, e \rangle, (f, e) \rangle\rangle_D = 0, \quad \forall (f, e) \in D,$$

it follows that any distributed-parameter port-Hamiltonian system satisfies along its trajectories the energy balance

$$\frac{d\mathcal{H}}{dt} = \int_{\partial\Omega} e_b \wedge f_b. \tag{132}$$

This expresses that the increase in internally stored energy in the domain  $\Omega$  is equal to the power supplied to the system through the boundary  $\partial\Omega$ .

### 7. Conclusions

The main results of this article concern a novel Hamiltonian vorticity–dilatation formulation of the compressible Euler equations. This formulation uses the density weighted vorticity and dilatation, together with the entropy and density, as primary variables. We obtained this new formulation using the following steps. First, we defined the Hamiltonian functional with respect to the chosen primary variables and calculated its functional derivatives. Next, we derived a pseudo-Poisson bracket, (port)–Hamiltonian formulation and Stokes–Dirac structure for the vorticity–dilatation formulation of the compressible Euler equations and showed the relation between these different formulations. An essential tool in this analysis was the use of the Hodge decomposition on bounded domains. These results extend the vorticity–stream function formulation of the Euler equations for incompressible flows to compressible flows.

The long term goal of this research is the development of finite element formulations that preserve these mathematical structures also at the discrete level. A nice direction for port-Hamiltonian systems using mixed finite elements is described in [17] or using pseudo-spectral methods in [18,19]. In future research we will explore this using the concept of discrete differential forms and exterior calculus as outlined in [9,10].

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**Appendix A. Proof of Lemma 5**

In this appendix we give the proof of Lemma 5.

**Proof.** Using the notations above, we obtain for the  $T_1$ -term in (94)

$$T_1 = - \int_{\Omega} \left[ \frac{\delta \mathcal{F}}{\delta \rho} \wedge d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G}) \right) + \alpha(\mathcal{F}) \wedge \frac{\xi}{2\bar{\rho}} \wedge \star d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G}) \right) - \frac{\delta \mathcal{G}}{\delta \rho} \wedge d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}) \right) - \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \wedge \star d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}) \right) \right]. \tag{A.1}$$

Using the integration by parts formula (13), we rewrite the last two terms in (A.1) as follows

$$\begin{aligned} \left\langle \star \frac{\delta \mathcal{G}}{\delta \rho} + \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}}, d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}) \right) \right\rangle &= \left\langle \delta \left( \star \frac{\delta \mathcal{G}}{\delta \rho} + \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \right), \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}) \right\rangle \\ &+ \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F})) \wedge \text{tr} \left( \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \right) \right) \\ &= \int_{\Omega} d \left( \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \right) \right) \wedge \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F}) \\ &+ \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F})) \wedge \text{tr} \left( \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \right) \right). \end{aligned} \tag{A.2}$$

Next, we consider the  $T_2$ -term and introduce the new variables  $\omega$  and  $\theta$  into (94) to obtain

$$\star \left( \star \frac{\delta \mathcal{G}}{\delta u} \wedge \star \frac{\delta \mathcal{F}}{\delta u} \right) = \tilde{\rho} \wedge \star (\star \alpha(\mathcal{G}) \wedge \star \alpha(\mathcal{F})),$$

and using (45),

$$T_2 = - \int_{\Omega} \star i_X (\tilde{\rho} \wedge \star (\star \alpha(\mathcal{G}) \wedge \star \alpha(\mathcal{F}))) = - \int_{\Omega} \star du \wedge \star \alpha(\mathcal{G}) \wedge \star \alpha(\mathcal{F}),$$

with  $X = (\star du / \tilde{\rho})^\sharp$ . Similarly, the term  $T_3$  in (94) can be transformed into the new variables as

$$\begin{aligned} T_3 &= - \int_{\Omega} \frac{d\tilde{s}}{\sqrt{\tilde{\rho}}} \wedge \left( \frac{\delta \mathcal{F}}{\delta s} \wedge \alpha(\mathcal{G}) - \frac{\delta \mathcal{G}}{\delta s} \wedge \alpha(\mathcal{F}) \right) \\ &= - \int_{\Omega} \frac{\delta \mathcal{F}}{\delta s} \wedge \frac{d\tilde{s}}{\sqrt{\tilde{\rho}}} \wedge \alpha(\mathcal{G}) - \frac{\delta \mathcal{G}}{\delta s} \wedge \frac{d\tilde{s}}{\sqrt{\tilde{\rho}}} \wedge \alpha(\mathcal{F}). \end{aligned}$$

Adding all terms, the bracket (94) in terms of the variables  $\rho, \omega, \theta, s$  has the form

$$\begin{aligned} \{\mathcal{F}, \mathcal{G}\} &= - \int_{\Omega} \left[ \frac{\delta \mathcal{F}}{\delta \rho} \wedge d \left( \sqrt{\bar{\rho}} \wedge \alpha(\mathcal{G}) \right) + \frac{\delta \mathcal{F}}{\delta s} \wedge \frac{1}{\sqrt{\tilde{\rho}}} \wedge d\tilde{s} \wedge \alpha(\mathcal{G}) + \alpha(\mathcal{F}) \wedge \gamma(\text{grad } \mathcal{G}) \right] \\ &+ \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \alpha(\mathcal{F})) \wedge \text{tr} \left( \frac{\delta \mathcal{G}}{\delta \rho} + \star \left( \alpha(\mathcal{G}) \wedge \frac{\xi}{2\bar{\rho}} \right) \right), \end{aligned} \tag{A.3}$$

where  $\gamma(\text{grad } \mathcal{G})$  is given in (109). In the following we expand the last integral over the domain  $\Omega$  in (A.3) as

$$\begin{aligned} \langle \alpha(\mathcal{F}), \star \gamma(\text{grad } \mathcal{G}) \rangle &= \int_{\Omega} \left[ \frac{\delta \mathcal{F}}{\delta \omega} \wedge d\gamma(\text{grad } \mathcal{G}) - \frac{\delta \mathcal{F}}{\delta \theta} \wedge \delta\gamma(\text{grad } \mathcal{G}) \right] \\ &+ \int_{\partial\Omega} \left[ \text{tr} \left( \frac{\delta \mathcal{F}}{\delta \omega} \right) \wedge \text{tr}(\gamma(\text{grad } \mathcal{G})) - \text{tr} \left( \star \frac{\delta \mathcal{F}}{\delta \theta} \right) \wedge \text{tr}(\star \gamma(\text{grad } \mathcal{G})) \right]. \end{aligned}$$

If we use the boundary assumptions (95) and (96) for the variational derivatives, the last boundary integral cancels and we obtain (107).  $\square$

**Appendix B. Proof of Theorem 3**

In this appendix we show the main steps of the proof of Theorem 3.

**Proof.** The proof of Theorem 3 consists of two steps.

**Step 1.** First we show that  $D \subset D^\perp$ . Let  $(f^1, e^1) \in D$  fix, and consider any  $(f^2, e^2) \in D$ . Substituting the definition of  $D$  into (126), we obtain that

$$\begin{aligned} I &:= \langle\langle (f^1, e^1), (f^2, e^2) \rangle\rangle_D \\ &= \int_\Omega \left[ e_\rho^1 \wedge d(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) + e_\rho^2 \wedge d(\sqrt{\bar{\rho}} \wedge \sigma(e^1)) + e_s^1 \wedge \frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge \sigma(e^2) + e_s^2 \wedge \frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge \sigma(e^1) \right] \\ &\quad + \int_\Omega [e_\omega^1 \wedge d\gamma(\bar{e}^2) + e_\omega^2 \wedge d\gamma(\bar{e}^1) - e_\theta^1 \wedge \delta\gamma(\bar{e}^2) - e_\theta^2 \wedge \delta\gamma(\bar{e}^1)] \\ &\quad + \int_{\partial\Omega} e_b^1 \wedge \text{tr} \left( e_\rho^2 + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \sigma(e^2) \right) \right) + e_b^2 \wedge \text{tr} \left( e_\rho^1 + \star \left( \frac{\zeta}{2\bar{\rho}} \wedge \sigma(e^1) \right) \right). \end{aligned}$$

Consider

$$I_1 = \int_\Omega \left[ e_\rho^1 \wedge d(\sqrt{\bar{\rho}} \wedge (de_\omega^2 + \delta e_\theta^2)) + \underline{e_\omega^2 \wedge d\gamma_\rho(\bar{e}^1)} - \underline{e_\theta^2 \wedge \delta\gamma_\rho(\bar{e}^1)} \right]$$

and apply the integration by parts formula (13) for the underlined terms, to obtain

$$\langle \star e_\omega^2, d\gamma_\rho(\bar{e}^1) \rangle = \langle \delta \star e_\omega^2, \gamma_\rho(\bar{e}^1) \rangle + \int_{\partial\Omega} \text{tr}(\gamma_\rho(\bar{e}^1)) \wedge \text{tr}(e_\omega^2) = \int_\Omega de_\omega^2 \wedge \gamma_\rho(\bar{e}^1)$$

and

$$\langle \star e_\theta^2, \delta\gamma_\rho(\bar{e}^1) \rangle = \langle d \star e_\theta^2, \gamma_\rho(\bar{e}^1) \rangle - \int_{\partial\Omega} \text{tr}(\star\gamma_\rho(\bar{e}^1)) \wedge \text{tr}(\star e_\theta^2) = - \int_\Omega \delta e_\theta^2 \wedge \gamma_\rho(\bar{e}^1),$$

where we used that  $\text{tr}(e_\omega^2) = 0$  and  $\text{tr}(\star e_\theta^2) = 0$ . Inserting the definition of  $\gamma_\rho(\bar{e}^1)$ , we obtain that

$$\begin{aligned} I_1 &= \int_\Omega \left[ e_\rho^1 \wedge d(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) + de_\omega^2 \wedge \gamma_\rho(\bar{e}^1) + \delta e_\theta^2 \wedge \gamma_\rho(\bar{e}^1) \right] \\ &= \int_\Omega d \left( \sqrt{\bar{\rho}} \wedge \sigma(e^2) \wedge e_\rho^1 \right) = \int_{\partial\Omega} \text{tr} \left( \sqrt{\bar{\rho}} \wedge \sigma(e^2) \right) \wedge \text{tr}(e_\rho^1). \end{aligned}$$

Similarly, we obtain that

$$\begin{aligned} I_2 &= \int_\Omega \left[ e_\rho^2 \wedge d(\sqrt{\bar{\rho}} \wedge \sigma(e^1)) + e_\omega^1 \wedge d\gamma_\rho(\bar{e}^2) - e_\theta^1 \wedge \delta\gamma_\rho(\bar{e}^2) \right] \\ &= \int_{\partial\Omega} \text{tr} \left( \sqrt{\bar{\rho}} \wedge \sigma(e^1) \right) \wedge \text{tr}(e_\rho^2). \end{aligned}$$

Next, let

$$I_3 = \int_\Omega \left[ e_\omega^1 \wedge d(\gamma_{\omega,\theta}(\bar{e}^2)) + e_\omega^2 \wedge d(\gamma_{\omega,\theta}(\bar{e}^1)) - e_\theta^1 \wedge \delta(\gamma_{\omega,\theta}(\bar{e}^2)) - e_\theta^2 \wedge \delta(\gamma_{\omega,\theta}(\bar{e}^1)) \right],$$

with  $\gamma_{\omega,\theta}(\cdot)$  defined in (130). Note that when applied to  $\bar{e}^i$ ,  $\alpha(\mathcal{F})$  is replaced by  $\sigma(e^i)$ . Apply again partial integration and use that  $\text{tr}(e_\omega^i) = 0$  and  $\text{tr}(\star e_\theta^i) = 0$  to obtain

$$\begin{aligned} I_3 &= \int_\Omega \left[ de_\omega^1 \wedge \gamma_{\omega,\theta}(\bar{e}^2) + de_\omega^2 \wedge \gamma_{\omega,\theta}(\bar{e}^1) + \delta e_\theta^1 \wedge \gamma_{\omega,\theta}(\bar{e}^2) + \delta e_\theta^2 \wedge \gamma_{\omega,\theta}(\bar{e}^1) \right] \\ &= \int_\Omega \left[ \sigma(e^1) \wedge \gamma_{\omega,\theta}(\bar{e}^2) + \sigma(e^2) \wedge \gamma_{\omega,\theta}(\bar{e}^1) \right]. \end{aligned}$$

Inserting the definition of  $\gamma_{\omega,\theta}(\bar{e}^i)$ ,  $i = 1, 2$  and applying again partial integration, the above integral will reduce to

$$I_3 = \int_{\partial\Omega} \text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) \wedge \text{tr} \left( \star \left( \sigma(e^1) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) + \text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^1)) \wedge \text{tr} \left( \star \left( \sigma(e^2) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right).$$

Finally, observe that the term containing the entropy

$$I_4 = \int_{\Omega} \left[ e_s^1 \wedge \frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge \sigma(e^2) + e_s^2 \wedge \frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge \sigma(e^1) + \sigma(e^2) \wedge \gamma_s(\bar{e}^1) + \sigma(e^1) \wedge \gamma_s(\bar{e}^2) \right],$$

is zero when we insert  $\gamma_s(\bar{e}^i) = -\frac{d\bar{s}}{\sqrt{\bar{\rho}}} \wedge e_s^i$ ,  $i = 1, 2$ . Combining all terms, we obtain that

$$I = \int_{\partial\Omega} \text{tr} \left( e_{\rho}^1 + \star \left( \sigma(e^1) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) \wedge \left( \text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) + e_b^2 \right) + \text{tr} \left( e_{\rho}^2 + \star \left( \sigma(e^2) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) \wedge \left( \text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^1)) + e_b^1 \right) = 0,$$

where the last equality is true since in  $D$  we have  $e_b^i = -\text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^i))$ ,  $i = 1, 2$ . Hence, we proved that the bilinear form (126) is zero for all  $(f^2, e^2) \in D$ . Therefore,  $(f^2, e^2) \in D^{\perp}$ .

**Step 2.** Next we show that  $D^{\perp} \subset D$ . Let  $(f^1, e^1) \in D^{\perp}$ . Then,

$$J := \langle (f^1, e^1), (f^2, e^2) \rangle_D = 0, \quad \forall (f^2, e^2) \in D. \tag{B.1}$$

Since  $(f^2, e^2) \in D$ , the inner product above is

$$J = \int_{\Omega} \left[ e_{\rho}^1 \wedge d(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) + e_{\rho}^2 \wedge f_{\rho}^2 + e_{\omega}^1 \wedge d \star \gamma(e^2) + e_{\omega}^2 \wedge f_{\omega}^1 + e_{\theta}^1 \wedge \star d\gamma(e^2) + e_{\theta}^2 \wedge f_{\theta}^1 + e_s^1 \wedge \frac{1}{\sqrt{\bar{\rho}}} \wedge d\bar{s} \wedge \sigma(e^2) + e_s^2 \wedge f_s^1 \right] + \int_{\partial\Omega} \left[ e_b^1 \wedge \text{tr} \left( e_{\rho}^2 + \star \left( \sigma(e^2) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) - \text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) \wedge f_b^1 \right].$$

Take  $e_{\rho}^2 \in \Lambda^0(\Omega)$ ,  $e_{\omega}^2 \in \Lambda^1(\Omega)$ ,  $e_{\theta}^2 \in \Lambda^3(\Omega)$ , such that  $\text{tr} \left( \star \left( \sigma(e^2) \wedge \frac{\zeta}{2\bar{\rho}} \right) \right) = 0$ ,  $\text{tr}(e_{\rho}^2) = 0$  and  $\text{tr}(\sqrt{\bar{\rho}} \wedge \sigma(e^2)) = 0$ .

Then the boundary integral in  $J$  vanishes. After partial integration and using these vanishing traces, we obtain that  $f_{\rho}^1, f_{\omega}^1, f_{\theta}^1$  are defined as in the Stokes–Dirac structure (127). The remaining part of the proof is completely analogous to Step 2 in the proof of Theorem 2.  $\square$

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