

# Almost output synchronization for heterogeneous time-varying networks for a class of non-introspective, nonlinear agents without exchange of controller states

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## SUMMARY

We consider almost output synchronization for directed heterogeneous time-varying networks where agents are non-introspective (i.e., agents have no access to their own states or outputs) in the presence of external disturbances. The nonlinear agents have a triangular structure and are globally Lipschitz continuous. The network can be time-varying with network switches occurring at arbitrary moments. A purely decentralized time-invariant protocol based on a low-gain and high-gain method is designed for each agent to achieve almost output synchronization while reducing the impact of disturbances on the output synchronization error. Copyright © 2016 John Wiley & Sons, Ltd.

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## 1. INTRODUCTION

In the last decade, the topic of synchronization in a multi-agent system has received considerable attention. Its potential applications can be seen in cooperative control of autonomous vehicles, distributed sensor networks, swarming and flocking, and others. The objective of synchronization is to guarantee an asymptotic agreement on a common state or output trajectory through decentralized control protocols [1–4]. Most work has focused on state synchronization based on full-state/partial-state coupling in a homogenous network (i.e., agents have identical dynamics), where the agent dynamics progress from single-integrator and double-integrator dynamics to more general dynamics (e.g., [5–13]). The counterpart of state synchronization is output synchronization, which is mostly performed in heterogeneous networks (i.e., agents are non-identical). When the agent has access to part of its own states, it is frequently referred to as introspective, and otherwise, it is referred to as non-introspective. Quite a few of the recent work has assumed agents are introspective (e.g., [14–17]), while others have considered non-introspective agents [18].

In [5], for homogeneous networks, a controller structure was introduced, which included not only sharing the relative outputs over the network but also sharing the relative states of the protocol over the network. This type of additional communication is not always natural. Some papers such as [8] (homogeneous network) and [15] (heterogeneous network but introspective) already avoided this additional communication of controller states.

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Almost synchronization is a notion that was studied in [19] (introspective), [20] (homogeneous, non-introspective), and [21] (heterogeneous, introspective, without exchange of controller states) where they deal with agents that are affected by external disturbances. The goal of this work is to reduce the impact of disturbances on the synchronization error to an arbitrary degree of accuracy (expressed in the  $\mathcal{H}_\infty$  norm).

The majority of the work assumes the topology associated with the network is fixed. Extensions to time-varying topologies are performed in the framework of switching topologies. Synchronization with switching topologies is studied utilizing concepts of dwell time and average dwell time (e.g., [22–24]). It is assumed that time-varying topologies switch among a finite set of topologies. In our previous work [25], we allow the network topology switches in an infinite set of topologies with certain properties. In [26], switching laws are designed to achieve synchronization.

Some authors have also studied synchronization in networks with nonlinear agent dynamics [14, 26–32], sometimes in combination with network heterogeneity. Explicit controller design for nonlinear networks has to a large degree centered on the relatively strict assumption of passivity. Passivity can in some cases be ensured by first applying local prefeedbacks to the system; however, this requires the system to be introspective.

In our earlier work for nonlinear systems [18], we addressed the issue of synchronization for a homogeneous network consisting of SISO, non-introspective agents. The main focus of this paper is to solve the almost output synchronization problem for heterogeneous network again consisting of SISO, non-introspective, nonlinear agents. The nonlinear agents have a diagonal structure and are globally Lipschitz as explained later. Our paper has three main contributions over our earlier work for nonlinear systems:

- We allow for heterogeneous networks.
- We consider almost synchronization in the presence of external disturbances.
- We allow for time-varying graphs.

### 1.1. Notations and definitions

Given a matrix  $A$ ,  $A'$  denotes its conjugate transpose, and  $\|A\|$  is the induced 2-norm. For square matrices,  $\lambda_i(A)$  denotes its  $i$ -th eigenvalue, and it is said to be Hurwitz stable if all eigenvalues are in the open left-half complex plane. We denote by  $\text{blkdiag}\{A_i\}$  a block-diagonal matrix with  $A_1, \dots, A_N$  as the diagonal elements and by  $\text{col}\{x_i\}$  or  $[x_1; \dots; x_N]$  a column vector with  $x_1, \dots, x_N$  stacked together, where the range of index  $i$  can be identified from the context.  $A \otimes B$  denotes the Kronecker product between  $A$  and  $B$ .  $I_n$  denotes the  $n$ -dimensional identity matrix, and  $0_n$  denotes the  $n \times 1$  zero column vector; sometimes, we drop the subscript if the dimension is clear from the context. Finally, the  $\mathcal{H}_\infty$  norm of a transfer function  $T$  is indicated by  $\|T\|_\infty$ .

A *weighted directed graph*  $\mathcal{G}$  is defined by a triple  $(\mathcal{V}, \mathcal{E}, \mathcal{A})$  where  $\mathcal{V} = \{1, \dots, N\}$  is a node set,  $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$  is a set of pairs of nodes indicating connections among nodes, and  $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N \times N} : \mathcal{E} \rightarrow \mathbb{R}^+$  is the weighting matrix, and  $a_{ij} > 0$  iff  $(i, j) \in \mathcal{E}$ . Each pair in  $\mathcal{E}$  is called an *edge*. A *path* from node  $i_1$  to  $i_k$  is a sequence of nodes  $\{i_1, \dots, i_k\}$  such that  $(i_j, i_{j+1}) \in \mathcal{E}$  for  $j = 1, \dots, k-1$ . A *directed tree* with *root*  $r$  is a subset of nodes of the graph  $\mathcal{G}$  such that a path exists between  $r$  and every other node in this subset. A *directed spanning tree* is a directed tree containing all the nodes of the graph. For a weighted graph  $\mathcal{G}$ , the matrix  $L = [\ell_{ij}]$  with

$$\ell_{ij} = \begin{cases} \sum_{k=1}^N a_{ik}, & i = j, \\ -a_{ij}, & i \neq j, \end{cases}$$

is called the *Laplacian matrix* associated with the graph  $\mathcal{G}$ . Because our graph  $\mathcal{G}$  has non-negative weights, we know that  $L$  has all its eigenvalues in the closed right-half plane and at least one eigenvalue at zero associated with right eigenvector  $\mathbf{1}$ .

#### Definition 1

A matrix pair  $(A, C)$  is said to contain the matrix pair  $(S, R)$  if there exists a matrix  $\Pi$  such that  $\Pi S = A\Pi$  and  $C\Pi = R$ .

*Remark 1*

Definition 1 implies that for any initial condition  $\omega(0)$  of the system  $\dot{\omega} = S\omega$ ,  $y_r = R\omega$ , there exists an initial condition  $x(0)$  of the system  $\dot{x} = Ax$ ,  $y = Cx$ , such that  $y(t) = y_r(t)$  for all  $t \geq 0$  ([33]).

*Definition 2*

Let  $\mathcal{L}_N \subset \mathbb{R}^{N \times N}$  be the family of all possible Laplacian matrices associated to a graph with  $N$  agents. We denote by  $\mathcal{G}_L$  the graph associated with a Laplacian matrix  $L \in \mathcal{L}_N$ . Then, a time-varying graph  $\mathcal{G}(t)$  with  $N$  agents is assumed to be of the form

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)},$$

where  $\sigma : \mathbb{R} \rightarrow \mathcal{L}_N$  is a piecewise constant, right-continuous function with minimal dwell-time  $\tau$  [34]; that is,  $\sigma(t)$  remains fixed for  $t \in [t_k, t_{k+1})$ ,  $k \in \mathbb{N}$  and switches at  $t = t_k$ ,  $k = 1, 2, \dots$ , where  $t_{k+1} - t_k \geq \tau$  for  $k = 0, 1, \dots$ . For ease of presentation, we assume  $t_0 = 0$ .

## 2. NONLINEAR MULTI-AGENT SYSTEMS

We consider a multi-agent system/network consisting of  $N$  non-identical non-introspective agents  $\tilde{\Sigma}_i$  with  $i \in \{1, 2, \dots, N\}$  described by

$$\tilde{\Sigma}_i : \begin{cases} \dot{\tilde{x}}_i = \tilde{A}_i \tilde{x}_i + \tilde{B}_i \tilde{u}_i + \tilde{\phi}_i(t, \tilde{x}_i) + \tilde{G}_i w_i, \\ y_i = \tilde{C}_i \tilde{x}_i, \end{cases} \quad (1)$$

where  $\tilde{x}_i \in \mathbb{R}^{\tilde{n}_i}$ ,  $\tilde{u}_i \in \mathbb{R}$ , and  $y_i \in \mathbb{R}$  are the state, input, and output of agent  $i$ . Moreover,  $\tilde{\phi}_i(t, \tilde{x}_i)$  is continuously differentiable and globally Lipschitz continuous with respect to  $\tilde{x}_i$ , uniformly in  $t$ , and piecewise continuous with respect to  $t$ . The relative degree of agent  $i$  is denoted by  $\tilde{\rho}_i$ . In the preceding text,  $w_i \in \mathbb{R}^{w_i}$  is the external disturbance, which is either in the set  $\Gamma_{\kappa}^{\text{rms}}$  or in the set  $\Gamma_{\kappa}^{\infty}$  for given  $\kappa$  as defined in the following.

*Definition 3*

The set of disturbances with power less than  $\kappa$  is defined as

$$\Gamma_{\kappa}^{\text{rms}} = \{ w \in L_{2,\text{loc}} : \|w\|_{\text{rms}} := \limsup_{T \rightarrow \infty} \frac{1}{T} \int_0^T w(t)' w(t) dt < \kappa \}.$$

The set of disturbances that are bounded by  $\kappa$  is defined as

$$\Gamma_{\kappa}^{\infty} = \{ w \in L_{\infty} : \|w\|_{\infty} < \kappa \}.$$

We make the following assumption on the agent dynamics.

*Assumption 1*

For each agent  $i = 1, \dots, N$ , we assume that

- $(\tilde{A}_i, \tilde{B}_i, \tilde{C}_i)$  is minimum-phase and
- $(\tilde{A}_i, \tilde{B}_i)$  is stabilizable and  $(\tilde{A}_i, \tilde{C}_i)$  is detectable.

The general agent model (1) can always be transformed to the *special coordinate basis* (SCB) form [35]. In other words, there exist nonsingular state and input transformations  $\Gamma_{x_i}$  and  $\Gamma_{u_i}$  such that

$$\left( \hat{A}_i, \hat{B}_i, \hat{C}_i, \hat{G}_i \right) = \left( \Gamma_{x_i}^{-1} \tilde{A}_i \Gamma_{x_i}, \Gamma_{x_i}^{-1} \tilde{B}_i \Gamma_{u_i}, \tilde{C}_i \Gamma_{x_i}, \Gamma_{x_i}^{-1} \tilde{G}_i \right),$$

and moreover,  $(\hat{A}_i, \hat{B}_i, \hat{C}_i)$  is in the SCB form. Define  $\tilde{x}_i = \Gamma_{x_i} \hat{x}_i$  and  $\tilde{u}_i = \Gamma_{u_i} \hat{u}_i$  and partition  $\hat{x}_i$  as  $[\hat{x}_{ia}; \hat{x}_{id}]$ ; then we obtain

$$\begin{cases} \dot{\hat{x}}_{ia} = \hat{A}_{ia} \hat{x}_{ia} + \hat{\phi}_{ia}(t, \hat{x}_{ia}, \hat{x}_{id}) + \hat{L}_{ia} y_i + \hat{G}_{ia} w_i, \\ \dot{\hat{x}}_{id} = \hat{A}_{id} \hat{x}_{id} + \hat{\phi}_{id}(t, \hat{x}_{ia}, \hat{x}_{id}) \\ \quad + \hat{B}_{id}(\hat{u}_i + \hat{E}_{ida} \hat{x}_{ia} + \hat{E}_{idd} \hat{x}_{id}) + \hat{G}_{id} w_i, \\ y_i = \hat{C}_{id} \hat{x}_{id}, \end{cases} \quad (2)$$

for  $i \in \{1, 2, \dots, N\}$ , where  $\hat{\phi}_{ia}$  and  $\hat{\phi}_{id}$  are (possibly time-varying) nonlinearities and

$$\hat{A}_{id} = \begin{pmatrix} 0 & I_{\hat{\rho}_i-1} \\ 0 & 0 \end{pmatrix}, \quad \hat{B}_{id} = \begin{pmatrix} 0_{\hat{\rho}_i-1} \\ 1 \end{pmatrix}, \quad \hat{C}_{id} = (1 \ 0'_{\hat{\rho}_i-1}). \tag{3}$$

*Remark 2*

The transformation matrices  $\Gamma_{x_i}$  and  $\Gamma_{u_i}$  can be calculated using available software, either numerically [36] or symbolically [37].

Given the assumptions on  $\tilde{\phi}_i$ , we know that for each  $i \in \{1, 2, \dots, N\}$ , the functions  $\hat{\phi}_{ia}$  and  $\hat{\phi}_{id}$  are continuously differentiable and globally Lipschitz continuous with respect to  $(\hat{x}_{ia}, \hat{x}_{id})$ , uniformly in  $t$ , and piecewise continuous with respect to  $t$ .

We make the following assumption about the nonlinearities:

*Assumption 2*

We have  $\hat{\phi}_{ia} = 0$ , while the nonlinearity  $\hat{\phi}_{id}$  satisfies the following lower-triangular structure:

$$\frac{\partial \hat{\phi}_{idj}(t, \hat{x}_{ia}, \hat{x}_{id})}{\partial \hat{x}_{idk}} = 0, \quad \forall k > j, \tag{4}$$

where  $\hat{\phi}_{idj}(t, \hat{x}_{ia}, \hat{x}_{id})$  denotes the  $j$ '-th element of  $\hat{\phi}_{id}(t, \hat{x}_{ia}, \hat{x}_{id})$  and  $\hat{x}_{idk}$  denotes the  $k$ '-th element of  $\hat{x}_{id}$ . Finally,  $\hat{\phi}_{id}(t, 0, 0)$  is a bounded signal.

*Remark 3*

The aforementioned assumption is independent of the specific transformation into the canonical form (2). For details, we refer to [18, Theorem 3]. Note that the aforementioned lower-triangular structure has been widely considered in nonlinear control in areas such as high-gain control and backstepping. The results of this paper can be extended to the more general strict feedback form [38], but this requires a different state feedback design (based on [39]) instead of the Riccati-based approach from this paper.

The topology of time-varying networks can be described by a time-varying graph  $\mathcal{G}(t)$ , which is defined by a triple  $(\mathcal{V}, \mathcal{E}(t), \mathcal{A}(t))$ , where  $\mathcal{V} = \{1, \dots, N\}$  is a node set (each node denotes an agent in the network),  $\mathcal{E}(t)$  is a time-varying set of a pair of nodes, and  $\mathcal{A}(t) = [a_{ij}(t)]$  is the weighted time-varying adjacency matrix. The Laplacian matrix of  $\mathcal{G}(t)$  is defined as  $L(t) = [\ell_{ij}(t)]$ . Note that we assume that  $\mathcal{A}(t)$  is a piecewise constant matrix and right continuous in time, which implies that  $L(t)$  has the same properties.

The time-varying network provides each agent with a linear combination of its own output relative to those of other neighboring agents; that is, agent  $i \in \{1, 2, \dots, N\}$  has access to the quantity

$$\zeta_i(t) = \sum_{j=1}^N a_{ij}(t)(y_i(t) - y_j(t)), \tag{5}$$

which is equivalent to

$$\zeta_i(t) = \sum_{j=1}^N \ell_{ij}(t)y_j(t). \tag{6}$$

### 3. ALMOST OUTPUT SYNCHRONIZATION UNDER SWITCHING TOPOLOGIES

In this section, we consider the almost output synchronization problem for time-varying, heterogeneous multi-agents systems/networks with nonlinear agents defined in Section 2, where the goal is to make the agents asymptotically converge to a reference trajectory in the presence of external disturbances. The reference trajectory in this paper is generated by an autonomous exosystem

$$\begin{cases} \dot{x}_r = Sx_r, & x_r(0) = x_{r0} \in \Omega, \\ y_r = Rx_r, \end{cases} \tag{7}$$

where  $x_r \in \mathbb{R}^{n_r}$ ,  $y_r \in \mathbb{R}$  and  $\Omega$  is a compact set. The eigenvalues of  $S$  are in the closed left-half complex plane, and the eigenvalues on the imaginary axis have equal geometric and algebraic multiplicity guaranteeing that the reference signal  $y_r$  is bounded. Moreover, we assume without loss of generality that  $(S, R)$  is observable.

Define  $e_i \triangleq y_i - y_r$  as the regulated output synchronization error for agent  $i \in \{1, 2, \dots, N\}$  and  $\mathbf{e} = \text{col}\{e_i\}$ . In order to achieve our goal, it is clear that a non-empty subset of agents must have knowledge of their outputs relative to the reference trajectory  $y_r$  generated by the reference system. More specifically, each agent has access to the quantity

$$\psi_i = \iota_i(y_i - y_r), \quad \iota_i = \begin{cases} 1, & i \in \pi, \\ 0, & i \notin \pi, \end{cases} \quad (8)$$

where  $\pi$  is a subset of  $\{1, 2, \dots, N\}$ . Then, we need the following assumption on the graph.

#### Assumption 3

Every node of the network graph  $\mathcal{G}$  is a member of a directed tree with the root contained in  $\pi$ .

In the following, we will refer to the node set  $\pi$  as the *root set* in view of Assumption 3 (a special case is when  $\pi$  consists of a single element, which is the root of a directed spanning tree of  $\mathcal{G}$ ).

Based on the Laplacian matrix  $L(t)$  of our time-varying network graph  $\mathcal{G}(t)$ , we define the expanded Laplacian matrix as

$$\bar{L}(t) = L(t) + \text{blkdiag}\{\iota_i\} = [\bar{\ell}_{ij}(t)].$$

Note that  $\bar{L}(t)$  is also written as  $\bar{L}_t$ , and it is clearly not a Laplacian matrix associated to some graph because it does not have a zero row sum for any fixed  $t$ . From [40, Lemma 7], all eigenvalues of  $\bar{L}(t)$  are in the open right-half complex plane for any  $t \in \mathbb{R}$ .

It should be noted that, in practice, perfect information of the communication topology is usually not available for controller design and only some rough characterization of the network can be obtained. Next, we will define a set of time-varying graphs based on some rough information of the graph. Before doing so, we first define a set of fixed graphs, based on which the set of time-varying graphs is defined.

#### Definition 4

For a given root set  $\pi$ , and  $\alpha, \beta, \varphi > 0$  and positive integer  $N$ , the set  $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$  is the set of directed graphs composed of  $N$  nodes satisfying the following properties:

- The eigenvalues of the associated expanded Laplacian matrix  $\bar{L}$ , denoted by  $\lambda_1, \dots, \lambda_N$ , satisfy  $\text{Re}\{\lambda_i\} > \beta$  and  $|\lambda_i| < \alpha$ .
- The condition number<sup>‡</sup> of the expanded Laplacian matrix  $\bar{L}$  is less than  $\varphi$ .

#### Remark 4

Note that for undirected graphs, the condition number of the Laplacian matrix is always bounded. Moreover, if we have a *finite* set of possible graphs, each of which is such that every node of the network graph  $\mathcal{G}$  is a member of a directed tree, which has a root contained in the root set  $\pi$ , then there always exists a set of the graph  $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$  for suitable  $\alpha, \beta, \varphi > 0$  and  $N$  containing these graphs. The only limitation is that we cannot find one protocol for a (infinite) sequence of graphs whose Laplacian either diverges or becomes more and more ill-conditioned.

#### Definition 5

Given a root set  $\pi$ , and  $\alpha, \beta, \varphi, \tau > 0$  and positive integer  $N$ , we define the set of time-varying network graphs  $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$  as the set of all time-varying graphs  $\mathcal{G}$  for which

$$\mathcal{G}(t) = \mathcal{G}_{\sigma(t)} \in \mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$$

<sup>‡</sup>In this context, we mean by condition number the minimum of  $\|U\| \|U^{-1}\|$  over all possible matrices  $U$  whose columns are the (generalized) eigenvectors of the expanded Laplacian matrix  $\bar{L}$ .

for all  $t \in \mathbb{R}$  where  $\sigma : \mathbb{R} \rightarrow \mathcal{L}_N$  is a piecewise constant, right-continuous function with minimal dwell-time  $\tau$ ; that is, for any two neighboring discontinuities  $t_k$  and  $t_{k-1}$ , we have  $t_k - t_{k-1} > \tau$ .

*Remark 5*

Note that the minimal dwell time is assumed to avoid chattering problems. However, it can be arbitrarily small.

We will define the almost regulated output synchronization problem as follows.

*Problem 1*

Consider multi-agent systems (1) and (5) and reference systems (7) and (8). For any given root set  $\pi$ , and  $\alpha, \beta, \varphi, \tau > 0$  and positive integer  $N$  defining a set of time-varying network graphs  $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$ , the *almost regulated output synchronization* problem is to find, if possible, for any  $\gamma > 0$ , and for any disturbance bound  $\kappa$ , a linear time-invariant dynamic protocol such that, for any time-varying graph  $\mathcal{G} \in \tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$ , for all initial conditions of agents and the reference system, the almost regulated output synchronization error satisfies

- for all  $w_i \in \Gamma_{\kappa}^{\infty}, i = 1, \dots, N$ ,

$$\limsup_{t \rightarrow \infty} \|\mathbf{e}(t)\| < \gamma; \tag{9}$$

- for all  $w_i \in \Gamma_{\kappa}^{\text{rms}}, i = 1, \dots, N$ ,

$$\|\mathbf{e}\|_{\text{rms}} < \gamma. \tag{10}$$

The main result in this section is presented in the following theorem.

*Theorem 1*

Consider multi-agent systems (1) and (5), and reference systems (7) and (8). Let a root set  $\pi$ , and  $\alpha, \beta, \varphi, \tau > 0$  and positive integer  $N$  be given, and hence, a set of time-varying network graphs  $\tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$  be defined.

Under Assumptions 1 and 2, the almost regulated output synchronization problem is solvable; that is, for any given  $\gamma > 0$ , for any disturbance bound  $\kappa$ , there exists a family of distributed dynamic protocols, parametrized in terms of low-gain and high-gain parameters  $\delta, \varepsilon$ , of the form

$$\begin{cases} \dot{x}_{i,c} = \mathcal{A}_i(\delta, \varepsilon)x_{i,c} + \mathcal{B}_i(\delta, \varepsilon) \begin{pmatrix} \zeta_i \\ \psi_i \end{pmatrix}, \\ \tilde{u}_i = \mathcal{C}_i(\delta, \varepsilon)x_{i,c} + \mathcal{D}_i(\delta, \varepsilon) \begin{pmatrix} \zeta_i \\ \psi_i \end{pmatrix}, \end{cases} \quad i \in \{1, 2, \dots, N\}, \tag{11}$$

where  $x_{i,c} \in \mathbb{R}^{q_i}$ , such that for any time-varying graph  $\mathcal{G} \in \tilde{\mathbb{G}}_{\alpha, \beta, \pi}^{\varphi, \tau, N}$ , for all initial conditions of agents and the reference system, the almost regulated output synchronization error satisfies (9) and (10).

To be more specific, there exists for any  $\gamma, \kappa$  a  $\delta^* \in (0, 1]$  such that, for each  $\delta \in (0, \delta^*]$ , there exists an  $\varepsilon^* \in (0, 1]$  such that for any  $\varepsilon \in (0, \varepsilon^*]$ , the protocol (11) yields (9) and (10).

The proof will be presented in the following subsection in a constructive way.

*3.1. Proof of Theorem 1*

In this section, we will present the constructive proof in three steps.

*Step 1:* In this step, we augment agent (1) with a pre-compensator in such a way that the inter-connection of agent (1) and the pre-compensator is of relative degree  $\rho$  and contains the reference system (7).

With the method presented in the Appendix A, we can find a linear pre-compensator without finite zeros given by

$$\begin{cases} \dot{z}_i = A_{ip}z_i + B_{ip}u_i, \\ \hat{u}_i = C_{ip}z_i, \end{cases} \tag{12}$$

for  $i = 1, \dots, N$ , such that the interconnection of agent (2) and pre-compensator (12) can be represented in the form

$$\begin{cases} \dot{x}_i = A_i x_i + B_i u_i + \phi_i(t, x_i) + G_i w_i, \\ y_i = C_i x_i, \end{cases} \quad (13)$$

where  $x_i \in \mathbb{R}^{n_i}$ ,  $u_i \in \mathbb{R}$ ,  $y_i \in \mathbb{R}$  are states, inputs, and outputs of the interconnection system of agent (2) and pre-compensator (12). Moreover,  $(A_i, C_i)$  contains  $(S, R)$ , and  $(A_i, B_i, C_i)$  has relative degree  $\rho$ .

As mentioned in Appendix A, we can guarantee that  $(A_i, B_i, C_i)$  are in the SCB form, where  $x_i = [x_{ia}; x_{id}]$ , with  $x_{ia} \in \mathbb{R}^{n_i - \rho}$  representing the finite zero structure and  $x_{id} \in \mathbb{R}^\rho$  the infinite zero structure, and

$$\begin{cases} \dot{x}_{ia} = A_{ia} x_{ia} + L_{ia} y_i + G_{ia} w_i, \\ \dot{x}_{id} = A_d x_{id} + \phi_{id}(t, x_{ia}, x_{id}) \\ \quad + B_d (u_i + E_{ida} x_{ia} + E_{idd} x_{id}) + G_{id} w_i, \\ y_i = C_d x_{id}, \end{cases} \quad (14)$$

for  $i = 1, \dots, N$ , where  $\phi_{id}(t, x_{ia}, x_{id})$  is a (possibly time-varying) nonlinearity and satisfies Assumption 2 regarding the lower-triangular structure, and

$$A_d = \begin{pmatrix} 0 & I_{\rho-1} \\ 0 & 0 \end{pmatrix}, \quad B_d = \begin{pmatrix} 0_{\rho-1} \\ 1 \end{pmatrix}, \quad C_d = (1 \ 0'_{\rho-1}). \quad (15)$$

*Step 2:* For each interconnection system (13), we will design a purely decentralized controller based on a low-gain and high-gain method [18]. Let  $\delta \in (0, 1]$  be the low-gain parameter and  $\varepsilon \in (0, 1]$  be the high-gain parameter. First, select  $K$  such that  $A_d - KC_d$  is Hurwitz stable. Next, choose  $F_\delta = -B'_d P_d$ , where  $P_d$  is a unique positive definite solution of the following algebraic Riccati equation:

$$P_d A_d + A'_d P_d - \beta P_d B_d B'_d P_d + \delta^2 I = 0, \quad (16)$$

where  $\beta > 0$  is the given lower bound on the real parts of the eigenvalues of all possible expanded Laplacian matrices  $\bar{L}$ . Such a solution exists and is unique because  $(A_d, B_d)$  is controllable [41]. Next, define

$$S_\varepsilon = \text{blkdiag}\{1, \varepsilon, \dots, \varepsilon^{\rho-1}\},$$

$$K_\varepsilon = \varepsilon^{-1} S_\varepsilon^{-1} K, \quad \text{and} \quad F_{\delta\varepsilon} = \varepsilon^{-\rho} F_\delta S_\varepsilon.$$

Then, we define the dynamic controller for each agent  $i \in \{1, 2, \dots, N\}$ :

$$\begin{cases} \dot{\hat{x}}_{id} = A_d \hat{x}_{id} + K_\varepsilon (\zeta_i + \psi_i - C_d \hat{x}_{id}), \\ u_i = F_{\delta\varepsilon} \hat{x}_{id}, \end{cases} \quad (17)$$

where  $\psi_i$  is defined in (8).

The following part proves that (12) together with (17) solves the almost regulated output synchronization problem.

Recall that  $x_i = [x_{ia}; x_{id}]$  and that (13) is a shorthand notation for (14). For each  $i \in \{1, 2, \dots, N\}$ , let

$$\bar{x}_i = x_i - \Pi_i x_r = \begin{pmatrix} x_{ia} - \Pi_{i,1} x_r \\ x_{id} - \Pi_{i,2} x_r \end{pmatrix},$$

where  $\Pi_i$  is defined to satisfy  $\Pi_i S = A_i \Pi_i$ ,  $C_i \Pi_i = R$ . Then, the dynamics of  $\bar{x}_i$  can be written as

$$\begin{cases} \dot{\bar{x}}_i = A_i \bar{x}_i + B_i u_i + \phi_i(t, x_i) + G_i w_i, \\ e_i = C_i \bar{x}_i. \end{cases} \quad (18)$$

Let  $\bar{x}_{ia} = x_{ia} - \Pi_{i,1}x_r$ ,  $\bar{x}_{id} = x_{id} - \Pi_{i,2}x_r$ . Then, by Taylor's theorem (e.g., [42, Theorem 11.1]), we can write

$$\phi_{id}(t, x_{ia}, x_{id}) = \phi_{id}(t, \Pi_{i,1}x_r, \Pi_{i,2}x_r) + \Phi_{ia}(t)\bar{x}_{ia} + \Phi_{id}(t)\bar{x}_{id},$$

where  $\Phi_{ia}(t)$  and  $\Phi_{id}(t)$  are given by

$$\begin{aligned} \Phi_{ia}(t) &= \int_0^1 \frac{\partial \phi_{id}}{\partial x_{ia}}(t, s\bar{x}_{ia} + \Pi_{i,1}x_r, s\bar{x}_{id} + \Pi_{i,2}x_r) ds, \\ \Phi_{id}(t) &= \int_0^1 \frac{\partial \phi_{id}}{\partial x_{id}}(t, s\bar{x}_{ia} + \Pi_{i,1}x_r, s\bar{x}_{id} + \Pi_{i,2}x_r) ds. \end{aligned}$$

Because the dynamics of the  $\bar{x}_i$  with output  $e_i$  is governed by the same dynamics as the dynamics of agent  $x_i$  in (13), we can present  $\bar{x}_i$  in the same form as (14), where

$$\begin{cases} \dot{\bar{x}}_{ia} = A_{ia}\bar{x}_{ia} + L_{ia}e_i + G_{ia}w_i, \\ \dot{\bar{x}}_{id} = A_d\bar{x}_{id} + \phi_{id}(t, \Pi_{i,1}x_r, \Pi_{i,2}x_r) + \Phi_{ia}(t)\bar{x}_{ia} \\ \quad + \Phi_{id}(t)\bar{x}_{id} + B_d(u_i + E_{ida}\bar{x}_{ia} + E_{idd}\bar{x}_{id}) + G_{id}w_i, \\ e_i = C_d\bar{x}_{id}. \end{cases}$$

Because of the global Lipschitz property of the nonlinearity, the elements of  $\Phi_{ia}(t)$  and  $\Phi_{id}(t)$  are uniformly bounded, and the lower-triangular structure of the nonlinearity implies that  $\Phi_{id}(t)$  is lower triangular. Moreover, because of the Lipschitz property and the fact that both  $x_r$  and  $\phi_{id}(t, 0, 0)$  are bounded, we know that  $\phi_{id}(t, \Pi_{i,1}x_r, \Pi_{i,2}x_r)$  is uniformly bounded. We obtain

$$\begin{cases} \dot{\bar{x}}_{ia} = A_{ia}\bar{x}_{ia} + L_{ia}e_i + \tilde{G}_{ia}\tilde{w}_i, \\ \dot{\bar{x}}_{id} = A_d\bar{x}_{id} + \Phi_{ia}(t)\bar{x}_{ia} + \Phi_{id}(t)\bar{x}_{id} \\ \quad + B_d(u_i + E_{ida}\bar{x}_{ia} + E_{idd}\bar{x}_{id}) + \tilde{G}_{id}\tilde{w}_i, \\ e_i = C_d\bar{x}_{id}, \end{cases} \tag{19}$$

for suitable definitions of  $\tilde{G}_{ia}$  and  $\tilde{G}_{id}$ , and where  $\tilde{w}_i$  is defined as

$$\tilde{w}_i = \begin{pmatrix} w_i \\ \phi_{id}(t, \Pi_{i,1}x_r, \Pi_{i,2}x_r) \end{pmatrix}.$$

Using similar arguments as in [25], which is presented in Appendix B for the completeness of this paper, we can assure that the complete network system can be brought in the form

$$\begin{aligned} \dot{\eta}_a &= A_a\eta_a + \tilde{W}_{ad,t}\eta_d + \tilde{G}_a\tilde{w}, \\ \varepsilon\dot{\eta}_d &= \tilde{A}_{\delta,t}\eta_d + \tilde{W}_{da,t}^\varepsilon\eta_a + \tilde{W}_{dd,t}^\varepsilon\eta_d + \varepsilon\tilde{G}_{d,t}^\varepsilon\tilde{w}, \end{aligned} \tag{20}$$

where  $A_a = \text{blkdiag}\{A_{ia}\}$ , and

$$\tilde{A}_{\delta,t} = I_N \otimes \begin{pmatrix} A_d & 0 \\ 0 & A_d - KC_d \end{pmatrix} + J_t \otimes \begin{pmatrix} B_d F_\delta - B_d F_\delta & \\ B_d F_\delta & -B_d F_\delta \end{pmatrix},$$

and

$$\begin{aligned} \tilde{W}_{ad,t} &= (W_{ad,t} \ 0) N_d^{-1}, & \tilde{G}_{d,t}^\varepsilon &= N_d \begin{pmatrix} \tilde{G}_{d,t}^\varepsilon \\ \tilde{G}_{d,t}^\varepsilon \end{pmatrix}, \\ \tilde{W}_{da,t}^\varepsilon &= N_d \begin{pmatrix} W_{da,t}^\varepsilon \\ W_{da,t}^\varepsilon \end{pmatrix}, & \tilde{W}_{dd,t}^\varepsilon &= N_d \begin{pmatrix} W_{dd,t}^\varepsilon & 0 \\ W_{dd,t}^\varepsilon & 0 \end{pmatrix} N_d^{-1}. \end{aligned}$$

Note that the function  $\eta_a$  is continuous and its behavior is completely determined by the aforementioned differential equation. However,  $\eta_d$  can have discontinuities whenever the network switches, but there exists a parameter  $m_1$  such that

$$\|\eta_d(t_k^+)\| \leq m_1 \|\eta_d(t_k^-)\|$$

for any switching time  $t_k$ , because of our bounds on  $U_t$  and  $J_t$ . Here,

$$g(t^+) = \lim_{h \downarrow 0} g(t+h), \quad g(t^-) = \lim_{h \downarrow 0} g(t-h).$$



We will use the following lemma, which is a time-varying version of [25, Lemma 2].

*Lemma 1*

For any  $\delta$  small enough, the matrix  $\tilde{A}_{\delta,t}$  is asymptotically stable for any Jordan matrix  $J_t$  whose eigenvalues satisfy  $\operatorname{Re}\{\lambda_{t,i}\} > \beta$  and  $|\lambda_{t,i}| < \alpha$ . Moreover, there exists  $P_\delta > 0$  and  $\nu > 0$  such that

$$P_\delta \tilde{A}_{\delta,t} + \tilde{A}'_{\delta,t} P_\delta \leq -\nu P_\delta - 4I \quad (21)$$

is satisfied for all possible Jordan matrices  $J_t$ , and there exists  $P_a > 0$  for which

$$P_a A_a + A'_a P_a = -\nu P_a - I. \quad (22)$$

*Proof*

For each fixed  $t$ , eigenvalues of  $J_t$  satisfy  $\operatorname{Re}\{\lambda_{t,i}\} > \beta$  and  $|\lambda_{t,i}| < \alpha$ . Hence, the arguments from [25] apply.  $\square$

Define  $V_a = \varepsilon^2 \eta'_a P_a \eta_a$  as a Lyapunov function for the dynamics of  $\eta_a$  in (20). Similarly, we define  $V_d = \varepsilon \eta'_d P_\delta \eta_d$  as a Lyapunov function for the dynamics of  $\eta_d$  in (20). Note that  $V_a$  is continuous while  $V_d$  experiences discontinuity during switches. From Appendix C, we find that the derivative of  $V_a$  and  $V_d$  is bounded by

$$\begin{pmatrix} \dot{V}_a \\ \dot{V}_d \end{pmatrix} \leq A_e \begin{pmatrix} V_a \\ V_d \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 2r_2^2 \\ r_3^2 \end{pmatrix} \|\tilde{w}\|^2, \quad (23)$$

componentwise where

$$A_e = \begin{pmatrix} -\nu & \varepsilon c_1 \\ c_2 & -\varepsilon^{-1}\nu - \nu + \varepsilon^2 \frac{c_1 c_2}{\nu} \end{pmatrix},$$

with eigenvalues  $\hat{\lambda}_1 = -\nu + \varepsilon^2 \frac{c_1 c_2}{\nu}$  and  $\hat{\lambda}_2 = -\varepsilon^{-1}\nu - \nu$ . Moreover, we have

$$e^{A_e t} = \frac{1}{1 + \varepsilon^3 \frac{c_2 c_1}{\nu^2}} \begin{pmatrix} e^{\hat{\lambda}_1 t} + \varepsilon^3 \frac{c_2 c_1}{\nu^2} e^{\hat{\lambda}_2 t} & \varepsilon^2 \frac{c_1}{\nu} (e^{\hat{\lambda}_1 t} - e^{\hat{\lambda}_2 t}) \\ \varepsilon \frac{c_2}{\nu} (e^{\hat{\lambda}_1 t} - e^{\hat{\lambda}_2 t}) & e^{\hat{\lambda}_2 t} + \varepsilon^3 \frac{c_2 c_1}{\nu^2} e^{\hat{\lambda}_1 t} \end{pmatrix}. \quad (24)$$

We find by integration from (23) that

$$\begin{pmatrix} V_a \\ V_d \end{pmatrix} (t) \leq e^{A_e(t-t_{k-1})} \begin{pmatrix} V_a \\ V_d \end{pmatrix} (t_{k-1}^+) + \varepsilon^2 \int_{t_{k-1}}^t e^{A_e(t-s)} \begin{pmatrix} 2r_2^2 \\ r_3^2 \end{pmatrix} \|\tilde{w}(s)\|^2 ds \quad (25)$$

componentwise. We have a potential jump at time  $t_{k-1}$  in  $V_d$ . However, there also exists  $m_2$  such that  $V_d(t_{k-1}^+) \leq m_2 V_d(t_{k-1}^-)$ , while  $V_a$  is continuous. Using (24) and the fact that  $t_k - t_{k-1} > \tau$ , we note that there exists small enough  $\varepsilon$  such that

$$(1 \ 1) e^{A_e(t_k - t_{k-1})} \begin{pmatrix} V_a \\ V_d \end{pmatrix} (t_{k-1}^+) \leq e^{-\frac{\nu}{2}(t_k - t_{k-1})} V(t_{k-1}^-), \quad (26)$$

where  $V = V_a + V_d$ . When  $w_i \in \Gamma_\kappa^\infty$ , it can be easily verified that

$$(1 \ 1) \int_{t_{k-1}}^t e^{A_e(t-s)} \begin{pmatrix} 2\varepsilon^2 r_2^2 \\ \varepsilon^2 r_3^2 \end{pmatrix} \|\tilde{w}(s)\|^2 ds \leq r \varepsilon^2 \|\tilde{w}\|_\infty^2, \quad (27)$$

where  $r$  is a sufficiently large constant. By multiplying both sides of (25) with the vector  $(1 \ 1)$  and setting  $t = t_{k-1}^-$  together with (26) and (27), we find

$$V(t_k^-) \leq e^{-\frac{\nu}{2}(t_k - t_{k-1})} V(t_{k-1}^-) + r \varepsilon^2 \|\tilde{w}\|_\infty^2. \quad (28)$$

Using these recursive inequalities, we obtain

$$V(t_k^-) \leq e^{-\frac{\nu}{2} t_k} V(0) + \frac{r \varepsilon^2}{1 - \mu} \|\tilde{w}\|_\infty^2. \quad (29)$$

where  $\mu < 1$  is such that  $e^{-\frac{\nu}{2}(t_k - t_{k-1})} \leq e^{-\frac{\nu}{2}\tau} \leq \mu$  for all  $k$ . Assume  $t_{k+1} > t > t_k$ . Because we do not necessarily have that  $t - t_k > \tau$ , we use the bound

$$(1 - \mu) e^{A_e(t-t_k)} \begin{pmatrix} V_a \\ V_d \end{pmatrix} (t_k^+) \leq m e^{-\frac{\nu}{2}(t-t_k)} V(t_k^-), \tag{30}$$

where the factor  $m$  is due to the potential discontinuous jump. Using (25) (with  $k - 1$  replaced by  $k$ ) with the bounds (29) and (30), we obtain

$$V(t) \leq m e^{-\frac{\nu}{2}t} V(0) + (m + 1) \frac{r\varepsilon^2}{1 - \mu} \|\tilde{w}\|_\infty^2.$$

Because  $w_i \in \Gamma_k^\infty$ , there exists  $\tilde{\kappa}$  independent of  $\varepsilon$  such that the norm of  $\tilde{w}(t)$  is bounded by  $\tilde{\kappa}$  for all  $t$ . Hence,

$$\limsup_{t \rightarrow \infty} \|\eta_d(t)\|^2 \leq \limsup_{t \rightarrow \infty} \frac{1}{\varepsilon \sigma_{\min}(P_\delta)} V(t) \leq (m + 1) \frac{r\varepsilon}{(1 - \mu)\sigma_{\min}(P_\delta)} \tilde{\kappa}. \tag{31}$$

On the other hand, for  $w \in \Gamma_\kappa^{\text{rms}}$ , we note that (25) implies

$$V(t_k^-) \leq \mu V(t_{k-1}^-) + r\varepsilon^2 \int_{t_{k-1}}^{t_k} \|\tilde{w}(\tau)\|^2 d\tau, \tag{32}$$

where, as noted before,  $\mu < 1$ . On the other hand, using (25) and (24) we find for  $t_{k-1} \leq t \leq t_k$ , we have

$$\int_{t_{k-1}}^t V_d(s) ds \leq r_6 V(t_{k-1}^-) + r_7 \varepsilon^2 \int_{t_{k-1}}^t \|\tilde{w}(\tau)\|^2 d\tau \tag{33}$$

for some constants  $r_6, r_7$ . We have

$$\int_0^T V_d(s) ds = \int_{t_{k-1}}^T V_d(s) ds + \sum_{i=1}^{k-1} \int_{t_{i-1}}^{t_i} V_d(s) ds$$

provided  $T > t_{k-1}$  and there are no discontinuities between  $t_{k-1}$  and  $T$ . Using (32) and (33), we then find that

$$\int_0^T V_d(s) ds \leq \frac{r_6}{1 - \mu} V(0) + \frac{r_7 \varepsilon^2}{1 - \mu} \int_0^T \|\tilde{w}(\tau)\|^2 d\tau. \tag{34}$$

From the Lyapunov function  $V_d = \varepsilon \eta_d' P_\delta \eta_d$ , we have

$$\frac{1}{T} \int_0^T \eta_d'(t) \eta_d(t) dt \leq \frac{1}{T} \frac{1}{\varepsilon \sigma_{\min}(P_\delta)} \int_0^T V_d(t) dt. \tag{35}$$

Combine (34) and (35), and taking the limit as  $T \rightarrow \infty$ , we find

$$\|\eta_d\|_{\text{rms}} \leq \varepsilon r_8 \|\tilde{w}\|_{\text{rms}} \tag{36}$$

for some constant  $r_8$ . Note that because  $w_i$  has bounded power and the additional component in  $\tilde{w}_i$  is uniformly bounded, we find that the power of  $\tilde{w}$  is bounded independent of  $\varepsilon$  and by choosing  $\varepsilon$ , we can guarantee that the power of  $\eta_d$  is arbitrarily small. Following the proof in [25], we have

$$\begin{aligned} \mathbf{e} &= (I_N \otimes C_d) (I_N \otimes S_\varepsilon^{-1}) (U J_t^{-1} \otimes I_{p\rho}) (I_{Np\rho} \ 0) N_d^{-1} \eta_d \\ &= (U J_t^{-1} \otimes C_d) (I_{Np\rho} \ 0) N_d^{-1} \eta_d \\ &= \Theta_t \eta_d, \end{aligned}$$

for suitably chosen matrix  $\Theta_t$ . Although  $\Theta_t$  is time-varying, it is uniformly bounded, because for graphs in  $\mathbb{G}_{\alpha, \beta, \pi}^{\varphi, N}$ , the matrices  $U_t$  and  $J_t$  are bounded. Then, we have

$$\|\mathbf{e}(t)\| = \|\Theta_t \eta_d(t)\| \leq \|\Theta_t\| \|\eta_d(t)\|. \tag{37}$$

Using (37) and (31), we can conclude that for  $w \in \Gamma_\kappa^\infty$ , we have (9) for any fixed  $\gamma > 0$  provided we choose  $\varepsilon$  small enough. Similarly, using (37) and (36), we can conclude that for  $w \in \Gamma_\kappa^{\text{rms}}$ , we have (10) for any fixed  $\gamma > 0$  provided we choose  $\varepsilon$  small enough.

Step 3: Combining the pre-compensator (12) in step 1 and the controller (17) in step 2, we obtain the protocol in the form of (11) as

$$\begin{aligned} A_i &= \begin{pmatrix} A_d - K_\varepsilon C_d & 0 \\ B_{ip} F_{\delta\varepsilon} & A_{ip} \end{pmatrix}, \quad B_i = \begin{pmatrix} K_\varepsilon & K_\varepsilon \\ 0 & 0 \end{pmatrix}, \\ C_i &= \begin{pmatrix} 0 & C_{ip} \end{pmatrix}, \quad D_i = \begin{pmatrix} 0 & 0 \end{pmatrix}. \end{aligned} \tag{38}$$

4. EXAMPLE

In this section, we illustrate our results on a network of  $N = 6$  agents under switching topologies, in which the agent models are as follows:

Agent 1, 2:

$$\begin{aligned} \dot{x}_{i_1a} &= -2x_{i_1a} + y_{i_1} + \sin(9t), \\ \dot{x}_{i_1d} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_{i_1d} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\tilde{u}_{i_1} + x_{i_1a} + (1 \ 2)x_{i_1d} \\ &\quad + \phi_{i_1d}(t, x_{i_1a}, x_{i_1d}) + \sin(9t)), \\ y_{i_1} &= (1 \ 0) x_{i_1d}, \end{aligned} \quad (i_1 = 1, 2)$$

Agent 3, 4:

$$\begin{aligned} \dot{x}_{i_2a} &= -3x_{i_2a} + 2y_{i_2} + 1.5, \\ \dot{x}_{i_2d} &= \tilde{u}_{i_2} + 2x_{i_2a} + 2x_{i_2d} + \phi_{i_2d}(t, x_{i_2a}, x_{i_2d}) + 0.5, \\ y_{i_2} &= x_{i_2d}, \quad (i_2 = 3, 4) \end{aligned}$$

Agent 5, 6:

$$\begin{aligned} \dot{x}_{i_3a} &= -x_{i_3a} + y_{i_3} + 0.5 \cos(3t), \\ \dot{x}_{i_3d} &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x_{i_3d} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} (\tilde{u}_{i_3} + x_{i_3a} + (1 \ 1)x_{i_3d} \\ &\quad + \phi_{i_3d}(t, x_{i_3a}, x_{i_3d}) + \cos(3t)), \\ y_{i_3} &= (1 \ 0) x_{i_3d}, \end{aligned} \quad (i_3 = 5, 6)$$

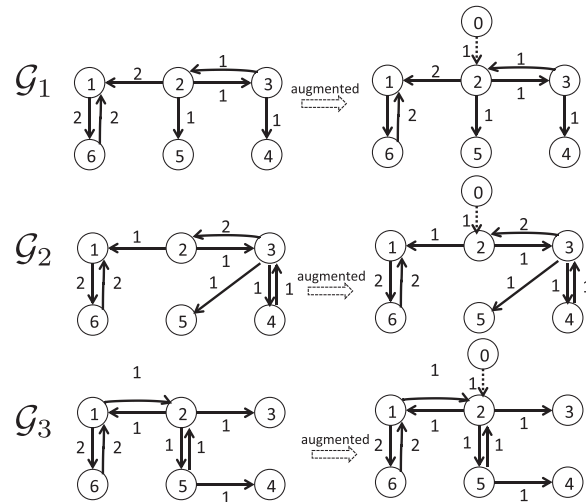


Figure 1. Topologies of the network and the augmented network.

where the nonlinearities are given by

$$\begin{aligned}\phi_{i_1 d}(t, x_{i_1 a}, x_{i_1 d}) &= 0.3 \sin(x_{i_1 a}) + \sin(0.1 x_{i_1 d1}) + 2 \ln(1 + x_{i_1 d2}^2), \\ \phi_{i_2 d}(t, x_{i_2 a}, x_{i_2 d}) &= \sin(t) x_{i_2 a} + \cos(x_{i_2 d}), \\ \phi_{i_3 d}(t, x_{i_3 a}, x_{i_3 d}) &= \cos(x_{i_3 d2}),\end{aligned}$$

which means each system has a sinusoidal disturbance (with different frequencies) as well. It is clear to see that Agents 1, 2 and 5, 6 have relative degree 2, and Agent 3, 4 have relative degree 1. We choose reference system as  $y_r = \sin(t)$ , of the form (7) with

$$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, R = (1 \ 0), x_r(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Assume the topology switches among three graphs in a circular manner, and the reference system is added as Agent 0 to the root Agent 2, shown in Figure 1. With methods given in Appendix A, a pre-compensator is generated for each agent such that all agents contain the mode of reference system and the relative degree of the interconnection of agents and pre-compensators is 3. The pre-compensators for each agent are as follows:

*Pre-compensator 1, 2:*

$$\begin{aligned}\dot{z}_{i_1} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z_{i_1} + \begin{pmatrix} 0.2001 \\ -0.2668 \end{pmatrix} u_{i_1}, \\ \tilde{u}_{i_1} &= \begin{pmatrix} -\frac{12}{5} & -\frac{9}{5} \end{pmatrix} z_{i_1},\end{aligned} \quad (i_1 = 1, 2)$$

*Pre-compensator 3, 4:*

$$\begin{aligned}\dot{z}_{i_2} &= \begin{pmatrix} 0 & 1 & 7 \\ -1 & 0 & 16 \\ 0 & 0 & 0 \end{pmatrix} z_{i_2} + \begin{pmatrix} 0 \\ 0 \\ -0.0164 \end{pmatrix} u_{i_2}, \\ \tilde{u}_{i_2} &= \begin{pmatrix} -\frac{16}{5} & \frac{7}{5} & 0 \end{pmatrix} z_{i_2},\end{aligned} \quad (i_2 = 3, 4)$$

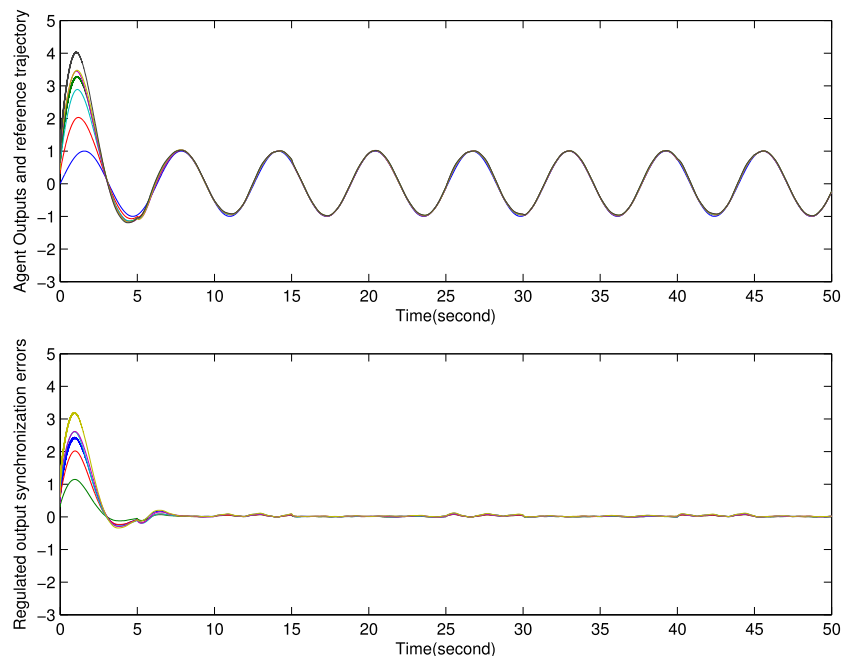


Figure 2. Agents' outputs and regulated output synchronization errors.

Pre-compensator 5, 6:

$$\begin{aligned} \dot{z}_{i_3} &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z_{i_3} + \begin{pmatrix} 0.0769 \\ -0.3845 \end{pmatrix} u_{i_3}, \\ \tilde{u}_1 &= \begin{pmatrix} 5 \\ -2 \end{pmatrix} z_{i_3}. \end{aligned} \quad (i_3 = 5, 6)$$

We choose  $\beta = 0.1$ , which is less than the real part of all eigenvalues of all augmented Laplacian matrix  $\bar{L}$  associated with topologies in Figure 1. By choosing  $K = (3 \ 7 \ 3 \ 3)'$ ,  $\delta = 10^{-9}$  and  $\varepsilon = 0.001$ , we obtain  $F_{\delta\varepsilon} = (-100.0000 \ -146.9475 \ -107.9673 \ -46.4688)$  for the protocol. Figure 2 shows that all agents' outputs are almost regulated to the reference sinusoid trajectory and regulated output synchronization error is squeezed harshly.

#### APPENDIX A: DESIGN OF A PRE-COMPENSATOR

In this section, we will design pre-compensators such that agent model (1) plus pre-compensators can be represented in (13). To fulfill this target, we need a pre-compensator for each agent. This pre-compensator is designed in two steps.

*Step A: Design a pre-compensator such that the interconnection of agent model and pre-compensator contains the dynamics of the reference system.*

The design of this pre-compensator is quite straightforward if, a priori, the agent has no dynamics in common with the exosystem. In that case,  $\Pi_i$  and  $\Gamma_i$  are uniquely determined by the so-called regulator equations:

$$\begin{aligned} \hat{A}_i \Pi_i + \hat{B}_i \Gamma_i &= \Pi_i S, \\ \hat{C}_i \Pi_i &= R. \end{aligned} \quad (39)$$

Then the pre-compensator is given by

$$\begin{aligned} \dot{p}_{i,1} &= S p_{i,1} + B_{i,1} u_i^1, \\ \hat{u}_i &= \Gamma_i p_{i,1}, \end{aligned} \quad (40)$$

where  $B_{i,1}$  is chosen according to the technique presented in [18] such that (40) has no finite zeros. Because agent (1) is minimum phase, the interconnection of agent (2) and pre-compensator (40), indicated by system  $\check{\Sigma}_i(\check{A}_i, \check{B}_i, \check{C}_i, \check{G}_i)$ , is minimum phase. If  $\check{A}_i$  and  $S$  have common eigenvalues, the design needs to be refined by deleting the unobservable dynamics from (40). For details, we refer to [18].

*Step B: Design another pre-compensator such that the cascade of system  $\check{\Sigma}_i$  and pre-compensator has relative degree  $\rho$ .*

Let  $\rho_i$  denote the relative degree of system  $\check{\Sigma}_i$  and  $\rho = \max\{\rho_i\}$ ,  $i = 1, \dots, N$ . We can add  $\rho - \rho_i$  integrators before agent  $i$  in system  $\check{\Sigma}_i$ . Then the second pre-compensator is equal to

$$\begin{aligned} \dot{p}_{i,2} &= \begin{pmatrix} 0 & I_{\rho-\rho_i-1} \\ 0 & 0 \end{pmatrix} p_{i,2} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u_i^2, \\ u_i^1 &= (1 \ 0) p_{i,2}, \end{aligned} \quad (41)$$

which guarantees that each agent has relative degree  $\rho$ . The interconnection system of system  $\check{\Sigma}_i$  and pre-compensator (41) is denoted by  $\check{\check{\Sigma}}_i(\check{\check{A}}_i, \check{\check{B}}_i, \check{\check{C}}_i, \check{\check{G}}_i)$ . The dimension of system  $\check{\check{\Sigma}}_i$  is  $n_i$ , and the relative degree is  $\rho$ . Again, we can ignore the nonlinear dynamics in this construction.

If the original system is of the form (2) and satisfies Assumption 2, then the interconnection of the original system interconnected with the aforementioned two pre-compensators can be transformed through nonsingular state and input transformations into the form presented in (14) where  $\phi_{id}$  still satisfies the lower-triangular structure of Assumption 2.

APPENDIX B: GENERATION OF THE FINAL COMPLETE NETWORK DYNAMICS

Define  $\xi_{ia} = \bar{x}_{ia}$  and  $\xi_{id} = S_\varepsilon \bar{x}_{id}$ . Then dynamics (19) can be written as

$$\begin{aligned} \dot{\xi}_{ia} &= A_{ia}\xi_{ia} + V_{iad}\xi_{id} + \tilde{G}_{ia}\tilde{w}_i, \\ \varepsilon \dot{\xi}_{id} &= A_d\xi_{id} + B_d F_\delta \hat{\xi}_{id} + V_{ida}^\varepsilon \xi_{ia} + V_{idd}^\varepsilon \xi_{id} + \varepsilon \tilde{G}_{id}^\varepsilon \tilde{w}_i, \\ e_i &= C_d \xi_{id}, \end{aligned}$$

where  $V_{iad} = L_{ia}C_d$ ,  $V_{ida}^\varepsilon = \varepsilon^\rho B_d E_{ida} + \varepsilon S_\varepsilon \Phi_{ia}(t)$ ,  $V_{idd}^\varepsilon = \varepsilon^\rho B_d E_{idd} S_\varepsilon^{-1} + \varepsilon S_\varepsilon \Phi_{id}(t) S_\varepsilon^{-1}$  and  $\tilde{G}_{id}^\varepsilon = S_\varepsilon \tilde{G}_{id}$ . Similarly, the controller (17) can be rewritten as

$$\varepsilon \dot{\hat{\xi}}_{id} = A_d \hat{\xi}_{id} + K \sum_{j=1}^N \bar{\ell}_{ij}(t) C_d \xi_{jd} - K C_d \hat{\xi}_{id},$$

where  $\hat{\xi}_{id} = S_\varepsilon \hat{\lambda}_{id}$  while  $\zeta_i + \psi_i = \sum_{j=1}^N \ell_{ij}(t)(y_j - y_r) + \iota_i(y_i - y_r) = \sum_{j=1}^N \bar{\ell}_{ij}(t)e_j$ . Let

$$\xi_a = \text{col}\{\xi_{ia}\}, \quad \xi_d = \text{col}\{\xi_{id}\}, \quad \hat{\xi}_d = \text{col}\{\hat{\xi}_{id}\}, \quad \tilde{w} = \text{col}\{\tilde{w}_i\}.$$

Then we have,

$$\begin{aligned} \dot{\xi}_a &= A_a \xi_a + V_{ad} \xi_d + G_a \tilde{w}, \\ \varepsilon \dot{\xi}_d &= (I_N \otimes A_d) \xi_d + (I_N \otimes B_d F_\delta) \hat{\xi}_d + V_{da}^\varepsilon \xi_a + V_{dd}^\varepsilon \xi_d + \varepsilon G_d^\varepsilon \tilde{w}, \\ \varepsilon \dot{\hat{\xi}}_d &= (I_N \otimes A_d - K C_d) \hat{\xi}_d + (\bar{L}(t) \otimes K C_d) \xi_d, \end{aligned}$$

where  $A_a = \text{blkdiag}\{A_{ia}\}$ , and  $V_{ad}, V_{da}^\varepsilon, V_{dd}^\varepsilon, G_a, G_d^\varepsilon$  are similarly defined.

Define  $U_t^{-1} \bar{L}(t) U_t = J_t$ , where  $J_t$  is the Jordan form of  $\bar{L}(t)$ , and let

$$v_a = \xi_a, \quad v_d = (J_t U_t^{-1} \otimes I_{\rho\rho}) \xi_d, \quad \tilde{v}_d = v_d - (U_t^{-1} \otimes I_{\rho\rho}) \hat{\xi}_d.$$

Then,

$$\begin{aligned} \dot{v}_a &= A_a v_a + W_{ad,t} v_d + G_a \tilde{w}, \\ \varepsilon \dot{v}_d &= (I_N \otimes A_d) v_d + (J_t \otimes B_d F_\delta)(v_d - \tilde{v}_d) + W_{da,t}^\varepsilon v_a + W_{dd,t}^\varepsilon v_d + \varepsilon \bar{G}_{d,t}^\varepsilon \tilde{w}, \\ \varepsilon \dot{\tilde{v}}_d &= (I_N \otimes (A_d - K C_d)) \tilde{v}_d + (J_t \otimes B_d F_\delta)(v_d - \tilde{v}_d) + W_{da,t}^\varepsilon v_a + W_{dd,t}^\varepsilon v_d + \varepsilon \bar{G}_{d,t}^\varepsilon \tilde{w}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} W_{ad,t} &= V_{ad} (U_t J_t^{-1} \otimes I_\rho), \\ W_{da,t}^\varepsilon &= (J_t U_t^{-1} \otimes I_\rho) V_{da}^\varepsilon, \\ W_{dd,t}^\varepsilon &= (J_t U_t^{-1} \otimes I_\rho) V_{dd}^\varepsilon (U_t J_t^{-1} \otimes I_\rho) \end{aligned}$$

and  $\bar{G}_{d,t}^\varepsilon = (J_t U_t^{-1} \otimes I_\rho) G_d^\varepsilon$ . Note that  $v_d$  and  $\tilde{v}_d$  exhibit discontinuous jumps when the network changes. Finally, let  $\eta_a = v_a$ , and define  $N_d$  such that

$$\eta_d \triangleq N_d \begin{pmatrix} v_d \\ \tilde{v}_d \end{pmatrix} = \begin{pmatrix} v_{1d} \\ \tilde{v}_{1d} \\ \vdots \\ v_{Nd} \\ \tilde{v}_{Nd} \end{pmatrix} \text{ where } N_d = \begin{pmatrix} e_1 & 0 \\ 0 & e_1 \\ \vdots & \vdots \\ e_N & 0 \\ 0 & e_N \end{pmatrix} \otimes I_\rho,$$

where  $e_i \in \mathbb{R}^N$  is the  $i$ -th standard basis vector whose elements are all zero except for the  $i$ -th element, which is equal to 1. Then (42) can be written as

$$\begin{aligned}\dot{\eta}_a &= A_a \eta_a + \tilde{W}_{ad,t} \eta_d + G_a \tilde{w}, \\ \varepsilon \dot{\eta}_d &= \tilde{A}_{\delta,t} \eta_d + \tilde{W}_{da,t}^\varepsilon \eta_a + \tilde{W}_{dd,t}^\varepsilon \eta_d + \varepsilon \tilde{G}_{d,t}^\varepsilon \tilde{w},\end{aligned}\quad (43)$$

where

$$\tilde{A}_{\delta,t} = I_N \otimes \begin{pmatrix} A_d & 0 \\ 0 & A_d - KC_d \end{pmatrix} + J_t \otimes \begin{pmatrix} B_d F_\delta & -B_d F_\delta \\ B_d F_\delta & -B_d F_\delta \end{pmatrix},$$

and

$$\begin{aligned}\tilde{W}_{ad,t} &= (W_{ad,t} \ 0) N_d^{-1}, & \tilde{G}_{d,t}^\varepsilon &= N_d \begin{pmatrix} \tilde{G}_{d,t}^\varepsilon \\ \tilde{G}_{d,t}^\varepsilon \end{pmatrix}, \\ \tilde{W}_{da,t}^\varepsilon &= N_d \begin{pmatrix} W_{da,t}^\varepsilon \\ W_{da,t}^\varepsilon \end{pmatrix}, & \tilde{W}_{dd,t}^\varepsilon &= N_d \begin{pmatrix} W_{dd,t}^\varepsilon & 0 \\ W_{dd,t}^\varepsilon & 0 \end{pmatrix} N_d^{-1}.\end{aligned}$$

### APPENDIX C: DERIVATIVE OF LYAPUNOV FUNCTION DURING TIME INTERVAL

The derivative of  $V_a$  is bounded by

$$\begin{aligned}\dot{V}_a &= -\nu V_a - \varepsilon^2 \|\eta_a\|^2 + 2\varepsilon^2 \operatorname{Re}(\eta_a' P_a \tilde{W}_{ad,t} \eta_d) + 2\varepsilon^2 \operatorname{Re}(\eta_a' P_a \tilde{G}_a \tilde{w}) \\ &\leq -\nu V_a + \varepsilon c_1 V_d + 2\varepsilon^2 r_2^2 \|\tilde{w}\|^2,\end{aligned}\quad (44)$$

where  $r_1, r_2$ , and  $c_1$  are such that

$$\begin{aligned}2\operatorname{Re}(\eta_a' P_a \tilde{W}_{ad,t} \eta_d) &\leq 2r_1 \|\eta_a\| \|\eta_d\| \leq \frac{1}{2} \|\eta_a\|^2 + 2r_1^2 \|\eta_d\|^2 \leq \frac{1}{2} \|\eta_a\|^2 + \varepsilon^{-1} c_1 V_d, \\ 2\operatorname{Re}(\eta_a' P_a \tilde{G}_a \tilde{w}) &\leq 2r_2 \|\eta_a\| \|\tilde{w}\| \leq \frac{1}{2} \|\eta_a\|^2 + 2r_2^2 \|\tilde{w}\|^2.\end{aligned}$$

Note that we can choose  $r_1, r_2$ , and  $c_3$  independent of the network graph but only depending on our bounds on the eigenvalues and condition number of our expand Laplacian  $\tilde{L}(t)$ .

Next, the derivative of  $V_d$  is bounded by

$$\begin{aligned}\dot{V}_d &= -\nu \varepsilon^{-1} V_d - 4 \|\eta_d\|^2 + 2\operatorname{Re}(\eta_d' P_\delta \tilde{W}_{da,t}^\varepsilon \eta_a) + 2\operatorname{Re}(\eta_d' P_\delta \tilde{W}_{dd,t}^\varepsilon \eta_d) + 2\varepsilon \operatorname{Re}(\eta_d' P_\delta \tilde{G}_{d,t}^\varepsilon \tilde{w}) \\ &\leq c_2 V_a - \left(\nu \varepsilon^{-1} + \nu - \varepsilon^2 \frac{c_1 c_2}{\nu}\right) V_d + \varepsilon^2 r_3^2 \|\tilde{w}\|^2,\end{aligned}\quad (45)$$

where  $2\operatorname{Re}(\eta_d' P_\delta \tilde{W}_{da,t}^\varepsilon \eta_a) \leq \|\eta_d\|^2$  for small  $\varepsilon$ , and

$$\begin{aligned}2\varepsilon \operatorname{Re}(\eta_d' P_\delta \tilde{G}_{d,t}^\varepsilon \tilde{w}) &\leq 2\varepsilon r_3 \|\eta_d\| \|\tilde{w}\| \leq \|\eta_d\|^2 + \varepsilon^2 r_3^2 \|\tilde{w}\|^2, \\ 2\operatorname{Re}(\eta_d' P_\delta \tilde{W}_{dd,t}^\varepsilon \eta_d) &\leq 2\varepsilon r_4 \|\eta_d\| \|\eta_d\| \leq \varepsilon^2 r_4^2 \|\eta_d\|^2 + \|\eta_d\|^2 \leq c_2 V_a + \|\eta_d\|^2,\end{aligned}$$

provided  $r_3, r_4$  is such that we have  $\varepsilon r_3 \geq \varepsilon \|P_\delta \tilde{G}_{d,t}^\varepsilon\|$ ,  $\varepsilon r_4 \geq \|P_\delta \tilde{W}_{da,t}^\varepsilon\|$ , and  $c_2$  sufficiently large. Then, we obtain

$$\begin{pmatrix} \dot{V}_a \\ \dot{V}_d \end{pmatrix} \leq A_e \begin{pmatrix} V_a \\ V_d \end{pmatrix} + \varepsilon^2 \begin{pmatrix} 2r_2^2 \\ r_3^2 \end{pmatrix} \|\tilde{w}\|^2.$$

where

$$A_e = \begin{pmatrix} -\nu & \varepsilon c_1 \\ c_2 & -\varepsilon^{-1} \nu - \nu + \varepsilon^2 \frac{c_1 c_2}{\nu} \end{pmatrix}.$$

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