Necessary and Sufficient Conditions for Global External Stochastic Stabilization of Linear Systems With Input Saturation

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***Abstract*—We consider the problem of global external stochastic stabilization for linear plants with saturating actuators, driven by a stochastic external disturbance, and having random Gaussian-distributed initial conditions. The objective is to control such plants by a possibly nonlinear static state feedback law that achieves global asymptotic stability in the absence of disturbances, while guaranteeing a bounded variance of the state vector for all time in the presence of disturbances and Gaussian distributed initial conditions. Results for continuous-time open-loop critically stable plants as well as for a chain of integrators have been obtained before. The goal of this technical note is to extend this result to critically unstable plants. We view our contribution in this technical note as a critical step in solving the LQG control problem for linear systems subject to saturated inputs which is a research problem with a long history in our field.**

***Index Terms*—Actuator constraints, control design, stochastic disturbances.**

I. Introduction

Internal and external stabilization of linear plants with actuators subject to saturation has been the subject of intense renewed interest among the control research community for the past two decades. The number of recent books and special issues of control journals devoted to this subject matter evidence this intense research focus, see for instance [1], [3], [10], [11], [17] and the references therein. It is now considered a classical fact that linear plants with saturating actuators can be globally internally stabilized if and only if all of the open-loop poles are located in the closed left half plane. These conditions on open-loop plants can be equivalently stated as a requirement that the plant be asymptotically null controllable with bounded control (ANCBC). It is also a classical fact that, in general, global internal stabilization of ANCBC plants requires nonlinear feedback laws. See, for instance, [2], [13], [18]. By weakening the notion of global internal stabilization to a semi-global framework, [4]–[6] showed that ANCBC plants with saturating actuators can be internally stabilized in the semi-global sense using only linear feedback laws. Such a relaxation from a global to semi-global framework is, from an engineering standpoint, both sensible and attractive.

With regard to external stabilization, the picture is complicated. Unlike linear plants, internal stability does not necessarily guarantee external stability when saturation is present. Hence, stabilization must be done simultaneously in both the internal and external sense. The literature points out that, for sustained disturbances, the induced L_∞ norm does not yield a suitable problem formulation. For instance, when considering the notion of, input-to-state stability (ISS) as a framework for simultaneous external and internal stability, a double-integrator system cannot achieve ISS stability with a linear feedback law, see [12]. It contains mathematical functions that are not reasonable models for disturbances, and it is impossible to get a good "stable" response if we use these functions as models for disturbance. In fact one would need to consider a class of sensible sustained disturbances from an engineering point of view. In the book [11], it was established that if the input is an L_∞ signal which is on average zero in an appropriately defined way then we can guarantee boundedness of the solution of the system. All these considerations lead us to believe that a suitable notion of external stability for linear plants with saturating actuators, and indeed for general nonlinear systems in the presence of disturbances and nonzero initial conditions, is yet to be developed.

In this technical note, we look at the simultaneous external and internal stabilization of ANCBC plants with saturating actuators when the external input is a stochastic disturbance. Specifically, we consider a linear time-invariant system subject to input saturation, stochastic external disturbances and random Gaussian distributed initial conditions. The aim will be to control this system by a possibly nonlinear static state feedback law that achieves global asymptotic stability in the absence of disturbances, while guaranteeing a bounded variance of the state vector for all time. This problem was first studied in [16]. A crucial result of [8] was first used in the context of this problem in [9]. In the book [11], a general result was presented for neutrally stable systems, i.e., the eigenvalues of the system matrix of the continuous-time system are in the closed left half plane and the eigenvalues on the imaginary axis have equal algebraic and geometric multiplicity. A preliminary version of this work for a chain of integrators was published in [14]. This technical note extends this to multi-input systems with arbitrary asymptotically null-controllable dynamics and solves this problem completely. We view our contribution in this technical note as a critical step in solving the LQG control problem for linear systems subject to saturated inputs which is a research problem with a long history in our field. In Section IV we illustrate our design by means of an example.

II. Problem Formulation and Main Results

We consider the stochastic differential equation

$$\mathrm{d}x(t) = Ax(t)\mathrm{d}t + B\sigma(u(t))\,\mathrm{d}t + E\mathrm{d}w(t) \tag{1}$$

where the state x, the control u and the disturbance w are vector-valued signals of dimension n, m and ℓ, respectively. Here w is a Wiener process (a Brownian motion) with mean 0 and rate Q, that is, $\mathrm{Var}[w(t)] = tQ$ and the initial condition x_0 of (1) is a Gaussian random vector which is independent of w. Its solution x is rigorously defined through Wiener integrals and is a Gauss-Markov process. See, for instance, [7]. σ denotes the standard saturation function and admissible

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feedbacks are possibly nonlinear static state feedbacks $u = f(x)$ where f is a Lipschitz-continuous mapping with $f(0) = 0$.

Problem II.1: Given the system (1), the *global external stochastic stabilization problem* is to find an admissible feedback $u = f(x)$ such that, for all possible values for the rate Q of the stochastic process w, the following properties hold:

(i) in the absence of the external input w, the equilibrium point $x = 0$ of the system (1) with $u = f(x)$ is globally asymptotically stable.

(ii) the variance $\text{Var}(x(t))$ of the state of the controlled system (1) with $u = f(x)$ is bounded over $t \geqslant 0$.

Remark: It has been shown that we cannot, in general, find a linear controller which achieves global external stochastic stabilization problem. For instance, for the double integrator it has been shown (see [11]) that the best we can hope for with linear controllers is semiglobal external stochastic stabilization, i.e. the case where the controller explicitly depends on an upper bound for the rate Q even though this upper bound can be arbitrarily large.

We are going to look at a class of scheduled low-gain feedbacks which is known to achieve objective (*i*) described above. Let $P_\varepsilon \geqslant 0$ for any $\varepsilon > 0$ be the unique solution of the following algebraic Riccati equation:

$$A'P_\varepsilon + P_\varepsilon A - P_\varepsilon BB'P_\varepsilon + \varepsilon P_\varepsilon = 0 \tag{2}$$

such that $A - BB'P_\varepsilon$ is asymptotically stable.

This specific parameterized Riccati equation was first introduced in [20]. The following properties are well-known, see for instance [20]:

Lemma II.2: Consider (2). In that case:

(i) Equation (2) has a unique solution $P_\varepsilon \geqslant 0$ such that $A - BB'P_\varepsilon$ is asymptotically stable provided (A, B) is stabilizable.

(ii) If (A, B) is stabilizable and A has no asymptotically stable eigenvalues then $P_\varepsilon > 0$ and we have that $R_\varepsilon = P_\varepsilon^{-1}$ satisfies the following Lyapunov equation:

$$R_\varepsilon A' + AR_\varepsilon - BB' + \varepsilon R_\varepsilon = 0.$$

(iii) P_ε is a polynomial function in ε provided (A, B) is controllable and we have a scalar input. In general, P_ε is a rational function in ε.

(iv) We have $P_\varepsilon \to 0$ as $\varepsilon \to 0$.

We are going to schedule this parameter ε according to

$$\varepsilon(x) = \max\left\{\varepsilon \in [0,1] | \varepsilon x'P_\varepsilon x \leqslant c\right\}.$$

The first step is to show that this feedback respects the saturation bound:

Lemma II.3: We have that

$$\left\|F_{\varepsilon(x)}x\right\| < 1 \quad \text{for all } x \in \mathbb{R}^n$$

provided c is sufficiently small where $F_\varepsilon = -B'P_\varepsilon$.

Proof: Since P_ε is a rational function in ε with a zero for $\varepsilon = 0$ we find that there exists d such that $\|P_\varepsilon\| < d\varepsilon$ for all $\varepsilon \in [0, 1]$. But then

$$\begin{aligned}\left\|-B'P_{\varepsilon(x)}x\right\|^2 &\leqslant \|B'\|^2 \left\|P_{\varepsilon(x)}\right\| x'P_{\varepsilon(x)}x \\ &< \|B'\|^2 d\varepsilon(x)x'P_{\varepsilon(x)}x \leqslant cd\|B'\|^2\end{aligned}$$

which is less than 1 for

$$c \leqslant \left(d\|B'\|^2\right)^{-1}$$

■

The following is the main result of this technical note:

Theorem II.4: Consider the system (1) and suppose that (A, B) is stabilizable, while the eigenvalues of A are in the closed left half plane. In that case, the feedback

$$u = F_{\varepsilon(x)}x$$

solves the global external stochastic stabilization problem as defined in Problem II.1 where c is chosen according to Lemma II.3.

Remark: Note that the above theorem provided necessary and sufficient conditions for solvability of the global external stochastic stabilization problem. After all, if (A, B) is not stabilizable or if the eigenvalues of A are not in the closed left half plane then we can no longer achieve global internal stabilization. Moreover, if (A, E) is controllable, then it is easily verified that an eigenvalue in the open right half plane immediately implies that the variance of the state grows unbounded. This is a consequence of the fact that there exists a nonzero probability that we reach a state which can no longer be steered back to the origin with a bounded input and which subsequently results in an exponential growth of the state.

In our previous work [14] we have shown, using some fundamental insight from [8], that the above theorem is valid provided the following conjecture holds:

Conjecture II.1: There exists α_2 such that:

$$\varepsilon \frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} \leqslant \alpha_2 P_\varepsilon$$

for all $\varepsilon \in [0, 1]$.

In this technical note, we will prove that the above conjecture always holds which completes the proof of Theorem II.4.

III. Proof of Main Result

We first note that without loss of generality we can ignore the asymptotically stable dynamics since a subsystem

$$\mathrm{d}x_1 = A_s x_1 \mathrm{d}t + B_s \sigma(u)\mathrm{d}t + E_s \mathrm{d}w$$

will always yield a finite variance for x_1, provided that A_s is asymptotically stable, independent of the controller, (since $\sigma(u)$ is bounded). Therefore, a controller that achieves a finite variance for the unstable dynamics will automatically achieve a finite variance for the full system. Note that our scheduled low gain feedback is a feedback that only takes into account the unstable part since the kernel of P_ε is precisely the stable dynamics.

Therefore, we assume from now on that all eigenvalues of A are on the imaginary axis.

As stated before, in order to prove Theorem II.4, we only need to prove the following key lemma. The rest of the argument then follows from our previous work [14].

Lemma III.1: There exists α_2 such that

$$\varepsilon \frac{\mathrm{d}P_\varepsilon}{\mathrm{d}\varepsilon} \leqslant \alpha_2 P_\varepsilon$$

for all $\varepsilon \in [0, 1]$.

Proof: As indicated before we can, without loss of generality, assume that A has only eigenvalues on the imaginary axis. The lemma however holds in general as a direct consequence of the fact that the stable dynamics are contained in the kernel of P_ε and hence also contained in the kernel of its derivative with respect to ε.

We assume in a suitable basis that A and B allow for the following decomposition:

$$A = \begin{pmatrix} J_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & J_{k-1} & 0 \\ 0 & \cdots & 0 & J_k \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ \vdots \\ \vdots \\ B_k \end{pmatrix}$$

where $J_1,\ldots,J_k$ are Jordan blocks associated with eigenvalues $\lambda_1,\ldots,\lambda_k$. Clearly our matrices have become complex and hence we use complex conjugate A^* instead of simply conjugate A' from now on. We define diagonal matrices

$$S_i^{\varepsilon} = \begin{pmatrix} \varepsilon^{j_i-1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \varepsilon & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

whose dimension j_i is equal to the dimension of J_i for $i = 1,\ldots,k$. Define

$$S^{\varepsilon} = \begin{pmatrix} S_1^{\varepsilon} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & S_{k-1}^{\varepsilon} & 0 \\ 0 & \cdots & 0 & S_k^{\varepsilon} \end{pmatrix}.$$

Since (A,B) is controllable it is easily verified that (A,S^0B) is controllable. Moreover, note that

$$\varepsilon \frac{\mathrm{d}S^{\varepsilon}}{\mathrm{d}\varepsilon} = \begin{pmatrix} D_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & D_{k-1} & 0 \\ 0 & \cdots & 0 & D_k \end{pmatrix} S^{\varepsilon}$$

where

$$D_i = \begin{pmatrix} j_i-1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & 0 \end{pmatrix}$$

which clearly does not depend on ε. Combining all Jordan blocks associated to the same eigenvalue, we get a different decomposition of A

$$A = \begin{pmatrix} \widetilde{\lambda}_1 I + N_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \widetilde{\lambda}_v I + N_v \end{pmatrix}, \; B = \begin{pmatrix} \widetilde{B}_1 \\ \vdots \\ \vdots \\ \widetilde{B}_v \end{pmatrix}$$

where $\widetilde{\lambda}_1,\ldots,\widetilde{\lambda}_v$ are mutually distinct eigenvalues of A while $N_1,\ldots,N_v$ are nilpotent matrices. Because of the selected structure, we have

$$\begin{aligned} S^{\varepsilon}A &= \varepsilon \begin{pmatrix} \varepsilon^{-1}\widetilde{\lambda}_1 I + N_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \varepsilon^{-1}\widetilde{\lambda}_v I + N_v \end{pmatrix} S^{\varepsilon} \\ &= \varepsilon \widetilde{A}_{\varepsilon} S^{\varepsilon} \\ S^{\varepsilon}B &= \begin{pmatrix} \widetilde{B}_{\varepsilon,1} \\ \vdots \\ \vdots \\ \widetilde{B}_{\varepsilon,v} \end{pmatrix} = \widetilde{B}_{\varepsilon} \end{aligned}$$

The above structure enables us to establish that

$$P_{\varepsilon} = \varepsilon S^{\varepsilon} \widetilde{P}_{\varepsilon} S^{\varepsilon}$$

where $\widetilde{P}_{\varepsilon}$ satisfies

$$\widetilde{A}_{\varepsilon}^* \widetilde{P}_{\varepsilon} + \widetilde{P}_{\varepsilon}\widetilde{A}_{\varepsilon} - \widetilde{P}_{\varepsilon}\widetilde{B}_{\varepsilon}\widetilde{B}_{\varepsilon}^*\widetilde{P}_{\varepsilon} + \widetilde{P}_{\varepsilon} = 0$$

and hence $\widetilde{R}_{\varepsilon} = \widetilde{P}_{\varepsilon}^{-1}$ satisfies

$$\widetilde{R}_{\varepsilon}\widetilde{A}_{\varepsilon}^* + \widetilde{A}_{\varepsilon}\widetilde{R}_{\varepsilon} - \widetilde{B}_{\varepsilon}\widetilde{B}_{\varepsilon}^* + \widetilde{R}_{\varepsilon} = 0$$

or

$$\widetilde{R}_{\varepsilon}\left(\widetilde{A}_{\varepsilon} + \frac{1}{2}I\right)^* + \left(\widetilde{A}_{\varepsilon} + \frac{1}{2}I\right)\widetilde{R}_{\varepsilon} = \widetilde{B}_{\varepsilon}\widetilde{B}_{\varepsilon}^*$$

This is a simple Lyapunov equation and since the real part of all eigenvalues of $\widetilde{A}_{\varepsilon} + (1/2)I$ equals 1/2 and $\widetilde{B}_{\varepsilon}$ is bounded, we obtain from [19] that the solution $\widetilde{R}_{\varepsilon} > 0$ is bounded. However, we also want to prove that $\widetilde{R}_{\varepsilon}$ is bounded away from zero. In this regard, we decompose $\widetilde{R}_{\varepsilon}$ compatible with the decomposition of $\widetilde{A}_{\varepsilon}$. We have

$$\widetilde{R}_{\varepsilon} = \begin{pmatrix} \widetilde{R}_{\varepsilon}^{11} & \cdots & \widetilde{R}_{\varepsilon}^{1v} \\ \vdots & & \vdots \\ \widetilde{R}_{\varepsilon}^{v1} & \cdots & \widetilde{R}_{\varepsilon}^{vv} \end{pmatrix}$$

We note

$$\varepsilon^{-1}(\widetilde{\lambda}_j - \widetilde{\lambda}_i)\widetilde{R}_{\varepsilon}^{ij} = -\widetilde{R}_{\varepsilon}^{ij}\left(\frac{1}{2}I + N_j\right) - \left(\frac{1}{2}I + N_i\right)^* \widetilde{R}_{\varepsilon}^{ij} + \widetilde{B}_{\varepsilon,i}\widetilde{B}_{\varepsilon,j}^*.$$

Since we know that the right-hand side is bounded, we find for $i \neq j$ (and hence $\widetilde{\lambda}_j \neq \widetilde{\lambda}_i$), there exists M_{ij} such that

$$\left\|\widetilde{R}_{\varepsilon}^{ij}\right\| \leqslant \varepsilon M_{ij}.$$

On the other hand, for $i = j$, we have

$$\left(\frac{1}{2}I + N_i\right)^* \widetilde{R}_{\varepsilon}^{ii} + \widetilde{R}_{\varepsilon}^{ii}\left(\frac{1}{2}I + N_i\right) = \widetilde{B}_{\varepsilon,i}\widetilde{B}_{\varepsilon,i}^*.$$

Since $((1/2)I + N_i, \widetilde{B}_{\varepsilon,i})$ is controllable for all ε in the closed interval [0, 1], the matrix $\widetilde{R}_{\varepsilon}^{ii}$, which is polynomial in ε, is invertible for all $\varepsilon \in [0,\ 1]$. Therefore, there exists m_i such that

$$\left\|\left(\widetilde{R}_{\varepsilon}^{ii}\right)^{-1}\right\| \leqslant m_i$$

Clearly this means that the matrix $\widetilde{R}_{\varepsilon}$ is block diagonal dominant for small ε since the off-diagonal elements are of order ε while the diagonal elements are bounded away from zero. This implies that there exists ε^* such that

$$\widetilde{P}_{\varepsilon} = (\widetilde{R}_{\varepsilon})^{-1}$$

is bounded for $\varepsilon \in (0, \varepsilon^*]$. However, it is clear that $\widetilde{P}_{\varepsilon}$ is a rational function for $[\varepsilon^*, 1]$ without singularities and hence is a bounded function for $[\varepsilon^*, 1]$. This implies that $\widetilde{P}_{\varepsilon}$ is bounded for all $\varepsilon \in (0, 1]$. Next, we note that

$$\varepsilon \frac{\mathrm{d}P_{\varepsilon}}{\mathrm{d}\varepsilon} = \varepsilon \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\varepsilon S^{\varepsilon}\widetilde{P}_{\varepsilon}S^{\varepsilon}\right) \tag{3}$$

$$= \varepsilon S^{\varepsilon}\left(D\widetilde{P}_{\varepsilon} + \widetilde{P}_{\varepsilon}D + \widetilde{P}_{\varepsilon} + \varepsilon \frac{\mathrm{d}\widetilde{P}_{\varepsilon}}{\mathrm{d}\varepsilon}\right) S^{\varepsilon}. \tag{4}$$

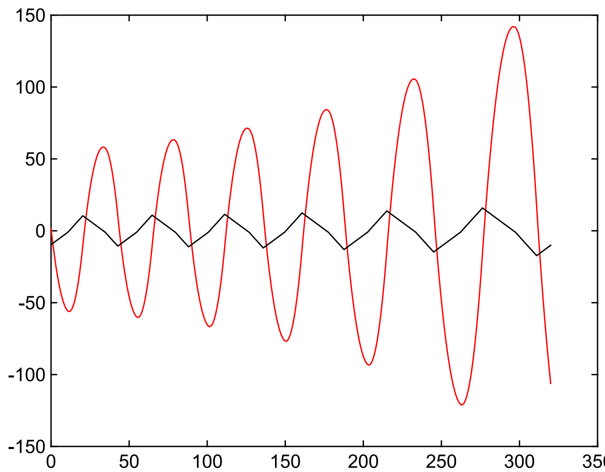

Fig. 1. Linear feedback with bounded disturbances.

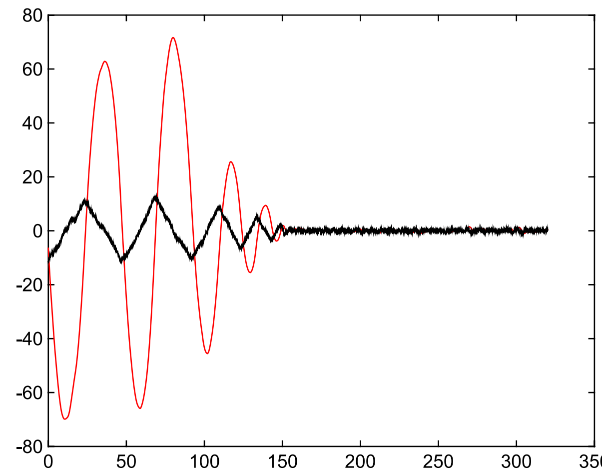

Fig. 2. Linear feedback with stochastic noise.

Note that

$$D\widetilde{P}_\varepsilon + \widetilde{P}_\varepsilon D + \widetilde{P}_\varepsilon + \varepsilon \frac{d\widetilde{P}_\varepsilon}{d\varepsilon} \quad (5)$$

is bounded. After all, $\widetilde{P}_\varepsilon$ is a bounded rational function in ε on the interval (0, 1] and therefore can be extended to a rational function on the interval [0, 1]. Then clearly the derivative of $\widetilde{P}_\varepsilon$ is again a rational function and since the original function has no poles on [0, 1] also the derivative has no poles on [0, 1] and is hence continuous on [0, 1]. This implies that $\widetilde{P}_\varepsilon$ and its derivative are both bounded.

We know that $\widetilde{P}_\varepsilon$ is bounded away from zero (since its inverse $\widetilde{R}_\varepsilon$ is bounded) while (5) is bounded from above. This implies that there exists α_2 such that $\alpha_2 \widetilde{P}_\varepsilon$ is an upper bound for the expression in (5). In other words

$$D\widetilde{P}_\varepsilon + \widetilde{P}_\varepsilon D + \widetilde{P}_\varepsilon + \varepsilon \frac{d\widetilde{P}_\varepsilon}{d\varepsilon} \leqslant \alpha_2 \widetilde{P}_\varepsilon$$

but then (4) implies

$$\varepsilon \frac{dP_\varepsilon}{d\varepsilon} \leqslant \alpha_2 \varepsilon S^{\varepsilon} \widetilde{P}_\varepsilon S^{\varepsilon} = \alpha_2 P_\varepsilon$$

which completes the proof. ■

IV. Example

The double integrator has long been a benchmark example for stabilization problems of linear systems with saturation

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \sigma(u) + w.$$

It is well-known that any linear feedback $u = -ax_1 - bx_2$ with a, b both positive achieves global internal stabilization for this system. However, for a nonlinear system, internal stability does not guarantee external stability. It was shown in [15], that there exists arbitrarily small inputs in both L_2 and L_∞ which still cause instability of the system for certain initial conditions. This is illlustrated by the simulation depicted in Fig. 1 where the disturbance was bounded by 0.2 and hence much smaller than the saturation bounds. On the other hand, as mentioned in the introduction, it has been shown in [11], that, although worst case disturbances can destabilize the system, disturbances which are on "average" zero can not destabilize the system. For the above system with a linear feedback and *stochastic* disturbance we have not proven that the variance of the state remains bounded but extensive simulation has not contradicted the claim that the system is stable. One such simulation is given in Fig. 2. The design presented in this technical note is proven to yield a bounded variance of the state. Simulations indicate that it also yields better performance than linear feedbacks with often faster convergence and an asymptotic variance which is often a factor of 2 or more smaller. One such simulation using the design of this technical note is given in Fig. 3.

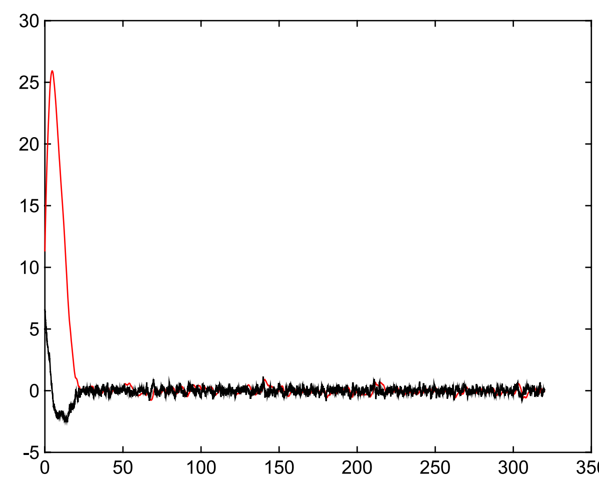

Fig. 3. Nonlinear feedback with stochastic noise.

V. Discussion and Conclusions

This technical note studies linear systems subject to stochastic disturbances with bounded control. The main objective of this line of research is to minimize the variance of the state using a saturated, possible nonlinear, control law. The first step is to establish that we can achieve a finite variance. This was established earlier for open-loop critically stable linear plants. The current technical note expands this to all systems whose eigenvalues are in the closed left-half plane. The next objective in our research is focused on minimizing the variance and obtaining a solution of th LQG control problem for this class of systems.

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