Closing the gap on path-kipas Ramsey numbers

We dedicate this paper to the memory of Ralph Faudree,
one of the exponents of Ramsey theory who died on January 13, 2015

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Submitted: Jan 30, 2015; Accepted: Jul 26, 2015; Published: Aug 14, 2015
Mathematics Subject Classifications: 05C55, 05D10

Abstract

*The first author is partly supported by the Doctorate Foundation of Northwestern Polytechnical University (No. cx201202) and by the project NEXLIZ-CZ.1.07/2.3.00/30.0038, which is co-financed by the European Social Fund and the state budget of the Czech Republic.

†Research partly supported by project P202/12/G061 of the Czech Science Foundation.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G_1 is a subgraph of G, or G_2 is a subgraph of the complement of G. Let P_n denote a path of order n and $\widehat{K}_m$ a kipas of order $m + 1$, i.e., the graph obtained from a P_m by adding one new vertex v and edges from v to all vertices of the P_m. We close the gap in existing knowledge on exact values of the Ramsey numbers $R(P_n, \widehat{K}_m)$ by determining the exact values for the remaining open cases.

Keywords: Ramsey number; path; kipas

1 Introduction

We only consider finite simple graphs. A cycle, a path and a complete graph of order n are denoted by C_n, P_n and K_n, respectively. A complete k-partite graph with classes of cardinalities $n_1, n_2, \ldots, n_k$ is denoted by $K_{n_1,n_2,\ldots,n_k}$. For a nonempty proper subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. For a vertex $v \in V(G)$, we let $N_S(v)$ denote the set of neighbors of v that are contained in S. For two vertex-disjoint graphs H_1, H_2, we define $H_1 + H_2$ to be the graph with vertex set $V(H_1) \cup V(H_2)$ and edge set $E(H_1) \cup E(H_2) \cup \{uv \mid u \in V(H_1)$ and $v \in V(H_2)\}$. For two disjoint vertex sets X, Y, $e(X, Y)$ denotes the number of edges with one end in X and one end in Y. We use mG to denote m vertex-disjoint copies of G. A star $K_{1,n} = K_1 + nK_1$, a kipas $\widehat{K}_n = K_1 + P_n$ and a wheel $W_n = K_1 + C_n$. The term kipas and its notation were adopted from [8]. Kipas is the Malay word for fan; the motivation for the term kipas is that the graph $K_1 + P_n$ looks like a hand fan (especially if the path P_n is drawn as part of a circle) but the term fan was already in use for the graphs $K_1 + nK_2$.

We use $\delta(G)$ and $\Delta(G)$ to denote the minimum and maximum degree of G, respectively.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 or $\overline{G}$ contains G_2, where $\overline{G}$ is the complement of G. It is easy to check that $R(G_1, G_2) = R(G_2, G_1)$, and, if G_1 is a subgraph of G_3, then $R(G_1, G_2) \leqslant R(G_3, G_2)$. Thus, $R(P_n, K_{1,m}) \leqslant R(P_n, \widehat{K}_m) \leqslant R(P_n, W_m)$. In [7], an explicit formula for $R(P_n, K_{1,m})$ is given, while in [5], the Ramsey numbers $R(P_n, W_m)$ for all m, n have been obtained. It follows from these results that $R(P_n, K_{1,m}) = R(P_n, W_m)$ for $m \geqslant 2n$. Therefore, $R(P_n, \widehat{K}_m) = R(P_n, K_{1,m}) = R(P_n, W_m)$ for $m \geqslant 2n$, and the exact values of these Ramsey numbers can be found in both [5] and [7].

It is trivial that $R(P_1, \widehat{K}_m) = 1$ and $R(P_n, \widehat{K}_1) = n$. Many nontrivial exact values for $R(P_n, \widehat{K}_m)$ have been obtained by Salman and Broersma in [8]. Here we completely solve the case by determining all the remaining path-kipas Ramsey numbers. $R(P_n, \widehat{K}_m)$ can easily be determined for $m \geqslant 2n$ (and follows directly from earlier results, as indicated above). In this note we close the gap by proving the following theorem.

Theorem 1. $R(P_n, \widehat{K}_m) = \max\{2n-1, \lceil 3m/2 \rceil - 1, 2\lfloor m/2 \rfloor + n - 2\}$ *for* $m \leqslant 2n-1$ *and* $m, n \geqslant 2$.

2 Proof of Theorem 1

We first list the following eight useful results that we will use in our proof of Theorem 1, as separate lemmas.

Lemma 2. *(Gerencsér and Gyárfás [4]). For* $m \geqslant n \geqslant 2$, $R(P_m, P_n) = m + \lfloor n/2 \rfloor - 1$.

Lemma 3. *(Faudree et al. [3]). For* $n \geqslant 2$ *and even* $m \geqslant 4$, $R(C_m, P_n) = \max\{m + \lfloor n/2 \rfloor - 1, n + m/2 - 1\}$.

Lemma 4. *(Parsons [6]). For* $n \geqslant m \geqslant 2$, $R(K_{1,m}, P_n) = \max\{2m - 1, n\}$.

Lemma 5. *(Salman and Broersma [8]).* $R(P_4, \widehat{K}_6) = 8$.

Lemma 6. *(Dirac [2]). If G is a connected graph, then G contains a path of order at least* $\min\{2\delta(G) + 1, |V(G)|\}$.

Lemma 7. *(Bondy [1]). If* $\delta(G) \geqslant |V(G)|/2$, *then G contains cycles of every length between* 3 *and* $|V(G)|$, *or* $r = |V(G)|/2$ *and* $G = K_{r,r}$.

Lemma 8. *(Zhang et al. [9]). Let C be a longest cycle of a graph G and* $v_1, v_2 \in V(G) - V(C)$. *Then* $|N_{V(C)}(v_1) \cup N_{V(C)}(v_2)| \leqslant \lfloor |V(C)|/2 \rfloor + 1$.

Lemma 9. *Let G be a graph with* $|V(G)| \geqslant 6$ *and* $\delta(G) \geqslant 2$. *Then G contains two vertex-disjoint paths, one with order three and one with order two.*

Proof. If G is connected, by Lemma 6, G contains a path of order at least 5. Let $x_1x_2x_3x_4x_5$ be a path in G. Then G contains two vertex-disjoint paths $x_1x_2x_3$ and x_4x_5. If G is disconnected, then each component of G contains a path of order three. This completes the proof of Lemma 9. □

We proceed to prove Theorem 1. Let $N = \max\{2n-1, \lceil 3m/2 \rceil - 1, 2\lfloor m/2 \rfloor + n - 2\}$, and let $m \leqslant 2n - 1$ and $m, n \geqslant 2$. It suffices to show that $R(P_n, \widehat{K}_m) = N$.

If $n = 2$, then $m \leqslant 2n - 1$ and $m, n \geqslant 2$ imply $m = 2$ or $m = 3$. It is obvious that $R(P_2, \widehat{K}_m) = m + 1$, and one easily checks that $m + 1 = N$ for these values of m and n. Next we assume that $n \geqslant 3$. We first show that $R(P_n, \widehat{K}_m) \geqslant N$. For this purpose, we note that it is straightforward to check that any of the graphs $G \in \{K_{n-1,n-1}, K_{\lfloor m/2 \rfloor, \lceil m/2 \rceil - 1, \lceil m/2 \rceil - 1}, K_{n-1, \lfloor m/2 \rfloor - 1, \lfloor m/2 \rfloor - 1}\}$ contains no $\widehat{K}_m$, whereas $\overline{G}$ contains no P_n. Thus, $R(P_n, \widehat{K}_m) \geqslant \max\{2n - 1, \lceil 3m/2 \rceil - 1, 2\lfloor m/2 \rfloor + n - 2\} = N$.

It remains to prove $R(P_n, \widehat{K}_m) \leqslant N$. To the contrary, we assume there exists a graph G of order N such that neither G contains a $\widehat{K}_m$, nor $\overline{G}$ contains a P_n.

We first claim that $\Delta(G) \geqslant N - \lfloor n/2 \rfloor$. To prove this claim, assume to the contrary that $\Delta(G) \leqslant N - \lfloor n/2 \rfloor - 1$. Then $\delta(\overline{G}) \geqslant \lfloor n/2 \rfloor$. Let H be a largest component of $\overline{G}$. If $|V(H)| \geqslant n$, then, since $\delta(H) \geqslant \lfloor n/2 \rfloor$, H contains a P_n by Lemma 6, a contradiction. Thus, $|V(H)| \leqslant n - 1$ and $|V(G)| - |V(H)| \geqslant N - n + 1$. Since $m \leqslant 2n - 1$, we have $n \geqslant \lfloor m/2 \rfloor$. From the definition of N we get that $N - n + 1 \geqslant n$ and $N - n + 1 \geqslant 2\lfloor m/2 \rfloor - 1$, so $N-n+1 \geqslant \max\{2\lfloor m/2\rfloor-1, n\}$. Since $\overline{G}-V(H)$ contains no P_n, by Lemma 4, $G-V(H)$ contains a $K_{1,\lfloor m/2\rfloor}$. If $|V(H)| \geqslant \lceil m/2\rceil$, since every vertex of $V(H)$ is adjacent to every vertex of $V(G)-V(H)$ in G, then G contains a $\widehat{K}_m$, a contradiction. This implies that $|V(H)| \leqslant \lceil m/2\rceil-1$. Recall that H is a largest component of $\overline{G}$. Thus $\overline{G}$ contains at least four components; otherwise $|V(\overline{G})| \leqslant 3(\lceil m/2\rceil-1) < \lceil 3m/2\rceil-1 \leqslant N$, a contradiction. Let H' be a smallest component of $\overline{G}$. Then $|V(H')| \leqslant N/4$ and $|V(G)|-|V(H')| \geqslant 3N/4 \geqslant 3/4(\lceil 3m/2\rceil-1) \geqslant 9m/8-3/4 > m-3/4$. That is, $|V(G)|-|V(H')| \geqslant m$. Since every component in $\overline{G}-V(H')$ is of order at most $\lceil m/2\rceil-1$, then every vertex in $\overline{G}-V(H')$ is of degree at most $\lceil m/2\rceil-2$. Thus, we have $\delta(G-V(H')) > (|V(G)|-|V(H')|)/2$. By Lemma 7, $G-V(H')$ contains a P_m, which together with any vertex of $V(H')$ forms a $\widehat{K}_m$ in G, a contradiction. This proves our claim that $\Delta(G) \geqslant N-\lfloor n/2\rfloor$.

Let u be a vertex of G with $d(u)=d=\Delta(G)$, let $F=G[N(u)]$ and $Z=V(G)-V(F)-\{u\}$. Then $|V(F)|=d \geqslant N-\lfloor n/2\rfloor = \max\{n+\lceil n/2\rceil-1, \lceil 3m/2\rceil-\lfloor n/2\rfloor-1, 2\lfloor m/2\rfloor+\lceil n/2\rceil-2\}$. We claim that $R(P_m,P_n) > d$; otherwise $R(P_m,P_n) \leqslant d$, and either F contains a P_m, which together with u forms a $\widehat{K}_m$, a contradiction; or $\overline{F}$ contains a P_n, also a contradiction. If $m \leqslant n$, or if $m=n+1$ and m is even, then by Lemma 2, $R(P_m,P_n)=\max\{n+\lfloor m/2\rfloor-1, m+\lfloor n/2\rfloor-1\} \leqslant n+\lceil n/2\rceil-1 \leqslant d$, a contradiction. Therefore, it remains to deal with the cases that $m \geqslant n+2$, and that $m=n+1$ and m is odd. We first deal with the latter case.

Let $m=n+1$ and m is odd. Then n is even, hence $n \geqslant 4$. We claim that $|Z| \geqslant 1$; otherwise $d=N-1=2n-2$, and then $R(P_m,P_n)=m+n/2-1 \leqslant 2n-2=d$ by Lemma 2, a contradiction. By Lemma 3, $R(C_{m-1},P_n)=m-1+n/2-1=n+n/2-1 \leqslant d$. Since $\overline{F}$ contains no P_n, then F contains a C_{m-1}. Let $C_{m-1}=x_1x_2\dots x_{m-1}x_1$, $Y=V(F)-V(C_{m-1})=\{y_1,y_2,\dots,y_k\}$. Then $k \geqslant n/2-1$. If $e(V(C_{m-1}),Y) \geqslant 1$, say $x_1y_1 \in E(G)$, then $y_1x_1x_2\dots x_{m-1}$ is a path in G, which together with u forms a $\widehat{K}_m$, a contradiction. Thus, $e(V(C_{m-1}),Y)=0$. If there is an edge in $\overline{G}[V(C_{m-1})]$, say $x_ix_j \in E(\overline{G})$ $(1 \leqslant i < j \leqslant m-1)$, then $x_ix_jy_1x'_1y_2x'_2\dots y_{n/2-1}x'_{n/2-1}$ with $\{x'_k : 1 \leqslant k \leqslant n/2-1\} \subseteq V(C_{m-1})-\{x_i,x_j\}$ is a path of order n in $\overline{G}$, a contradiction. Thus, $G[V(C_{m-1})]$ is a complete graph. Set $z \in Z$. If $e(\{z\},V(C_{m-1})) \geqslant 1$ in $\overline{G}$, say $zx_1 \in E(\overline{G})$, then $uzx_1y_1\dots x_{n/2-1}y_{n/2-1}$ is a path of order n in $\overline{G}$, a contradiction. Thus, $e(\{z\},V(C_{m-1}))=0$ in $\overline{G}$, and G contains a path $ux_1zx_2x_3\dots x_{m-2}$, which together with x_{m-1} forms a $\widehat{K}_m$, another contradiction. This completes the case that $m=n+1$ and m is odd. We proceed with the case that $n+2 \leqslant m \leqslant 2n-1$, and first consider the small values of n.

For $n=3$ and $m=5$, or $n=4$ and $m=7$, or $n=5$ and $7 \leqslant m \leqslant 9$, we get that $R(P_m,P_n)=m+\lfloor n/2\rfloor-1 \leqslant \lceil 3m/2\rceil-\lfloor n/2\rfloor-1 \leqslant d$, a contradiction. By Lemma 5, $R(P_4,\widehat{K}_6)=8=N$. Hence it remains to consider the case that $m \geqslant n+2 \geqslant 8$.

We first claim that $|Z| \geqslant 2$. If not, $|Z| \leqslant 1$ and $d=N-1-|Z| \geqslant N-2$. By Lemma 2, $R(P_m,P_n)=m+\lfloor n/2\rfloor-1$. If $m \geqslant n+3$, then $m+\lfloor n/2\rfloor-1 \leqslant \lceil 3m/2\rceil-3 \leqslant N-2 \leqslant d$, a contradiction; if $n \geqslant 7$ or $(n,m)=(6,8)$, then $m+\lfloor n/2\rfloor-1 \leqslant 2\lfloor m/2\rfloor+n-4 \leqslant N-2 \leqslant d$, also a contradiction. Thus, for $m \geqslant n+2 \geqslant 8$, we have $|Z| \geqslant 2$.

Since $m \geqslant n+2 \geqslant 8$, by Lemma 3, $R(C_{2\lfloor m/2\rfloor-2},P_n)=\max\{2\lfloor m/2\rfloor+\lfloor n/2\rfloor-$

$3, n + \lfloor m/2 \rfloor - 2\} < 2\lfloor m/2 \rfloor + \lceil n/2 \rceil - 2 \leqslant d$. Since $\overline{F}$ contains no P_n, F contains a $C_{2\lfloor m/2 \rfloor - 2}$. Let C be a longest cycle in F. Then $|V(C)| \geqslant m-3$. If $|V(C)| \geqslant m$, then F contains a P_m, which together with u forms a $\widehat{K}_m$ in G, a contradiction. Thus, $m-3 \leqslant |V(C)| \leqslant m-1$. We complete the proof by distinguishing the three cases that $|V(C)| = m-1$, $|V(C)| = m-2$ or $|V(C)| = m-3$. In each case, let $C = x_1x_2 \ldots x_{|V(C)|}x_1$ and $Y = V(F) - V(C) = \{y_1, y_2, \ldots, y_k\}$.

Case 1: $|V(C)| = m-1$.
We have $k = d - (m-1) \geqslant \lceil n/2 \rceil - 2$. If $e(V(C), Y) \geqslant 1$, say $x_1y_1 \in E(G)$, then $y_1x_1x_2 \ldots x_{m-1}$ is a path in G, which together with u forms a $\widehat{K}_m$, a contradiction. Thus, $e(V(C), Y) = 0$. Let $z_1, z_2 \in Z$. If $e(\{z_1\}, V(C)) \geqslant 1$ in $\overline{G}$, say $z_1x_1 \in E(\overline{G})$, then $z_2uz_1x_1y_1 \ldots x_{\lceil n/2 \rceil - 2}y_{\lceil n/2 \rceil - 2}x_{\lceil n/2 \rceil - 1}$ is a path of order at least n in $\overline{G}$, a contradiction. This implies that $e(\{z_1\}, V(C)) = 0$ in $\overline{G}$. For the same reason, $e(\{z_2\}, V(C)) = 0$ in $\overline{G}$.

We claim that $\delta(\overline{G}[V(C)]) \leqslant 1$. If not, $\delta(\overline{G}[V(C)]) \geqslant 2$. Since $m \geqslant 8$, by Lemma 9, there are two vertex-disjoint paths in $\overline{G}[V(C)]$, one with order three and one with order two. Without loss of generality, let $x'_1x'_2x'_3$ and $x'_4x'_5$ be the two paths in $\overline{G}[V(C)]$. Because $m-1 \geqslant \lceil n/2 \rceil + 2$, we may assume that $x'_6, \ldots, x'_{\lceil n/2 \rceil + 2} \in V(C) - \{x'_1, \ldots, x'_5\}$. Then $x'_1x'_2x'_3y_1x'_4x'_5y_2x'_6y_3 \ldots x'_{\lceil n/2 \rceil + 1}y_{\lceil n/2 \rceil - 2}x'_{\lceil n/2 \rceil + 2}$ is a path of order at least n in $\overline{G}$, a contradiction. This proves our claim that $\delta(\overline{G}[V(C)]) \leqslant 1$. That is, there exists a vertex of $V(C)$ which is adjacent to at least $|V(C)| - 2$ vertices of $V(C)$. Without loss of generality, let x_1 be a vertex with maximum degree in $G[V(C)]$, and let x_3 be the possible vertex that is nonadjacent to x_1. Then $ux_2z_1x_4z_2x_5x_6 \ldots x_{m-1}$ is a path of order m, which together with x_1 forms a $\widehat{K}_m$ in G, our final contradiction in Case 1.

Case 2: $|V(C)| = m-2$.
We have $k = d - (m-2)$. Note that $k \geqslant \lceil n/2 \rceil - 1$ for odd m, and $k \geqslant \lceil n/2 \rceil$ for even m. Let X be the set of all vertices of $V(C)$ that are nonadjacent to Y in G. For $1 \leqslant i \leqslant m-2$, either $x_i \in X$, or $x_{i+1} \in X$. Here, $x_{m-1} = x_1$. This is because, if x_i and x_{i+1} have a common neigbor in Y, say y_1, then by replacing x_ix_{i+1} by $x_iy_1x_{i+1}$ in C, we obtain a cycle longer than C, a contradiction; if x_i and x_{i+1} are adjacent to different vertices of Y, say $x_iy_1, x_{i+1}y_2 \in E(G)$, then $y_2x_{i+1}x_{i+2} \ldots x_{m-2}x_1 \ldots x_iy_1$ is a path of length m, which together with u forms a $\widehat{K}_m$ in G, also a contradiction. Thus, at least one end of each edge of C is nonadjacent to Y in G. Note that $|X| \geqslant \lceil n/2 \rceil$ and $|Y| \geqslant \lceil n/2 \rceil - 1$ for odd m and $|Y| \geqslant \lceil n/2 \rceil$ for even m. If m is even or n is odd, then we get a path P_n in $\overline{G}[X \cup Y]$. This implies it remains to consider the case that n is even and m is odd, with $m \geqslant n+3$.

If $|V(C) - X| \geqslant 2$, say $x_i, x_j \notin X$, then $x_{i+1}, x_{j+1} \in X$. Moreover, $x_{i+1}x_{j+1} \notin E(G)$; otherwise we may obtain either a cycle longer than C in F, or a path of length m in F, which together with u forms a $\widehat{K}_m$ in G, both of which are contradictions. Now let $x'_1, x'_2, \ldots, x'_{|X|-2} \in X - \{x_{i+1}, x_{j+1}\}$. Since $|X| - 2 \geqslant \lceil |V(C)|/2 \rceil - 2 \geqslant n/2 - 1$, let $P = x_{i+1}x_{j+1}y_1x'_1y_2x'_2 \ldots y_{n/2-1}x'_{n/2-1}$. Note that P is a path of order n in $\overline{G}$, a contradiction. Thus, $m-3 \leqslant |X| \leqslant m-2$ and there exists a vertex in $V(C)$, say x_1, such that $e(V(C) - \{x_1\}, Y) = 0$.

Since $m \geqslant n+3 \geqslant 9$, we have $m-3 \geqslant \lceil n/2 \rceil + 2$. If there is an edge in $\overline{G}[V(C)-\{x_1\}]$, say $x_ix_j \in E(\overline{G})$, then $\overline{G}[X \cup Y]$ contains a path P_n, a contradiction. Thus, $G[V(C)-\{x_1\}]$ is a complete graph of order $m-3$.

Let $z_1, z_2 \in Z$. We claim that $e(\{z_1\}, V(C)-\{x_1\}) = 0$ in $\overline{G}$; otherwise, say for $z_1x_2 \in E(\overline{G})$, $z_2uz_1x_2y_2x_3y_3 \ldots x_{n/2-1}y_{n/2-1}x_{n/2}$ is a path of order n in $\overline{G}$, a contradiction. For the same reason, $e(\{z_2\}, V(C)-\{x_1\}) = 0$ in $\overline{G}$.

It is easy to check that $x_1ux_3z_1x_4z_2x_5 \ldots x_{m-2}$ is a path of order m, which together with x_2 forms a $\widehat{K}_m$ in G, our final contradiction in Case 2.

Case 3: $|V(C)| = m-3$.
If $m = n+2 \geqslant 8$, then m and n have the same parity. In that case, $R(C_{2\lfloor (m-1)/2 \rfloor}, P_n) = 2\lfloor (m-1)/2 \rfloor + \lfloor n/2 \rfloor - 1 \leqslant 2\lfloor m/2 \rfloor + \lceil n/2 \rceil - 2 \leqslant d$. Since $\overline{F}$ contains no P_n, F contains a $C_{2\lfloor (m-1)/2 \rfloor}$. This contradicts the fact that C with $|V(C)| = m-3$ is a longest cycle in F. It remains to consider the case that $m \geqslant n+3 \geqslant 9$.

We have $k = d-(m-3) \geqslant \lceil n/2 \rceil$. By Lemma 8, any two vertices of Y have at least $\lceil (m-3)/2 \rceil - 1 \geqslant \lceil n/2 \rceil - 1$ common nonadjacent vertices of $V(C)$ in G. Since C is a longest cycle in G, any vertex of Y has at least $\lceil (m-3)/2 \rceil \geqslant \lceil n/2 \rceil$ nonadjacent vertices of $V(C)$ in G. By these observations, y_1 and y_2 have a common nonadjacent vertex in $V(C)$, say x_1; for $2 \leqslant i \leqslant \lceil n/2 \rceil - 1$, y_i and y_{i+1} have a common nonadjacent vertex in $V(C) - \{x_1, x_2, \ldots, x_{i-1}\}$, say x_i; $y_{\lceil n/2 \rceil}$ have a nonadjacent vertex in $X - \{x_1, x_2, \ldots, x_{\lceil n/2 \rceil - 1}\}$, say $x_{\lceil n/2 \rceil}$. Then $y_1x_1y_2x_2 \ldots y_{\lceil n/2 \rceil}x_{\lceil n/2 \rceil}$ is a path of order at least n in $\overline{G}$. This final contradiction completes the proof of Case 3 and of Theorem 1. □

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