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Note on non-uniform bin packing games

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ABSTRACT

A non-uniform bin packing game is an N-person cooperative game, where the set N is defined by k bins of capacities $b_1, \dots, b_k$ and n items of sizes $a_1, \dots, a_n$. The objective function v of a coalition is the maximum total value of the items of that coalition which can be packed to the bins of that coalition. We investigate the taxation model of Faigle and Kern (1993) [2] and show that the 1/2-core is always nonempty for such bin packing games. If all items have size strictly larger than 1/3, we show that the 5/12-core is always non-empty. Finally, we investigate the limiting case $k \to \infty$, thereby extending the main result in Faigle and Kern (1998) [3] to the non-uniform case.

1. Introduction

A cooperative game is defined by a tuple $\langle N, v\rangle$, where N is a (finite) set of *players* and $v : 2^N \to \mathbb{R}$ is a characteristic (value) function satisfying $v(\emptyset) = 0$. A subset $S \subseteq N$ is called a *coalition* and N itself is the *grand coalition*. Usually, $v(S)$ stands for the total earning (or total cost) of a coalition S.

In a cooperative game, the players of the grand coalition N are agreed to cooperate if there is a "fair" allocation of the value $v(N)$ among the individual players. One of the most attractive solution concepts is the *core* of a game, defined as the set of vectors $x \in \mathbb{R}^N$ satisfying

(i) $x(N) = v(N)$,
(ii) $x(S) \geq v(S), \forall S \subseteq N$.

As usual, we denote by $x(S) = \sum_{i\in S} x_i$.

We say a game is *balanced* if it possesses a nonempty core. Unfortunately, many games are not balanced. This means players in a non-balanced game may not cooperate because there is no "fairness". For this case, one has to seek for a completely different solution concept (e.g. Shapley Value) or one has to modify the notion of "core". Several models for the latter have been established (see Shapley and Shubik [6], Tijs and Driessen [7]). In our paper, we analyze the (multiplicative) ϵ-core (*cf.* [2]), defined by the condition (i) above together with

(ii′) $x(S) \geq (1-\epsilon)v(S), \forall S \subseteq N$.

We can interpret the condition as a taxation rate ϵ in the sense that the players in S can keep only a $(1-\epsilon)$ fraction of their earnings on their own if they cooperate. This is the usual idea behind a sales tax and, therefore, appears to be quite realistic/acceptable for the players.

A game with non-empty ϵ-core is called ϵ-balanced. In this sense, ϵ-taxation provides an ϵ-approximation to balancedness. It can be easily seen that the 1-core is always non-empty for all games with $v \geq 0$. In general, we seek to find a "proper" (as small as possible) taxation rate ϵ such that the ϵ-core is non-empty for a given class of games.

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In [3], Faigle and Kern studied (uniform) bin packing games and provided a necessary and sufficient condition for the non-emptiness of the ϵ-core, based on the linear programming description of the core (*cf.* below). We extend this result to the more general class of superadditive games. Recall that a game is called *superadditive* if $v(S) + v(T) \leq v(S \cup T)$ for $S \cap T = \emptyset$ and $S, T \subseteq N$. We consider games with nonnegative characteristic function and with superadditivity. The corresponding "core allocation problem" is

$$\begin{aligned} &\min\ x(N) \\ &\text{s.t.}\ x(S) \geq v(S), \quad \forall S \subseteq N. \end{aligned} \tag{1.1}$$

Note that $x \geq 0$ is implied as v is nonnegative. Its dual problem can therefore be written as

$$\begin{aligned} &\max \sum_{S \subseteq N} v(S) y(S) \\ &\text{s.t.} \sum_{S \ni i} y(S) \leq 1, \quad \forall i \in N \\ &y(S) \geq 0, \quad \forall S \subseteq N. \end{aligned} \tag{1.2}$$

Note that the corresponding integral problem

$$\begin{aligned} &\max \sum_{S \subseteq N} v(S) y(S) \\ &\text{s.t.} \sum_{S \ni i} y(S) \leq 1, \quad \forall i \in N, \\ &y(S) \in \{0, 1\}, \quad \forall S \subseteq N, \end{aligned} \tag{1.3}$$

has optimal objective function value $v(N)$. Indeed, suppose $S_1, S_2, \ldots, S_t \subseteq N$ are the coalitions "selected" by an optimal solution of (1.3), i.e., $y(S_i) = 1$, for $i = 1, \ldots, t$ and $y(S) = 0$ for $S \neq S_1, \ldots, S_t$. Then $S_i \cap S_j = \emptyset$, for $i \neq j$. The optimal objective function value is $\sum_{i=1}^{t} v(S_i)$. But this must equal $v(N)$, since, by superadditivity,

$$\sum_{i=1}^{t} v(S_i) \leq v(N).$$

Let us denote by $v'(N)$ the optimal objective function value of (1.2). As explained above, $v(N)$ is the optimal objective function value of its 0-1 integer linear program (1.3). The necessary and sufficient condition for the non-emptiness of the ϵ-core is given below (*cf.* [3] for the uniform bin packing game). The proof is identical to the one given in [3]. We include it for convenience of the reader.

Lemma 1.1. *Assume a game $\langle N, v \rangle$ is superadditive and $v \geq 0$. Given $\epsilon \in [0, 1]$, the ϵ-core of N is nonempty if and only if $\epsilon \geq 1 - v(N)/v'(N)$.*

Proof. ($\Rightarrow$) Recall that $x \in \mathbb{R}^N$ is in the ϵ-core of N if and only if $x(S) \geq (1-\epsilon)v(S)$, $\forall S \subseteq N$ and $x(N) = v(N)$. Therefore, if x is in the ϵ-core, then $x/(1-\epsilon)$ must be a feasible solution to (1.1), implying

$$\frac{v(N)}{1-\epsilon} = \frac{x(N)}{1-\epsilon} \geq v'(N), \tag{1.4}$$

and it gives $\epsilon \geq 1 - v(N)/v'(N)$.

($\Leftarrow$) Assume $\epsilon \geq 1 - v(N)/v'(N)$ is true. Let $\bar{\epsilon} = 1 - v(N)/v'(N)$, hence $\epsilon \geq \bar{\epsilon}$ and let y be an optimal solution of (1.1). We claim $x = (1-\bar{\epsilon})y$ is in the ϵ-core of N, by verifying the two conditions as below (where we denote $y(S) = \sum_{S \subseteq N} y_S$ as before):

$$x(S) = (1-\bar{\epsilon})y(S) \geq (1-\bar{\epsilon})v(S) \geq (1-\epsilon)v(S), \ \forall S \subseteq N$$

and

$$x(N) = (1-\bar{\epsilon})y(N) = (1-\bar{\epsilon})v'(N) = v(N). \quad \square$$

This provides us with a powerful tool for analyzing the minimal taxation rate of bin packing games. In Section 2, we introduce non-uniform bin packing games and prove that the 1/2-core is always nonempty. In Section 3, we derive a somewhat stronger result for the special case where all item sizes are strictly larger than 1/3. There we will also try to point out why non-uniform bin packing games are much more complicated than uniform ones. Finally, in Section 4, we extend the main result of [3] about the limiting case (total number of bins $k \to \infty$).

2. Non-uniform bin packing games

Nowadays, as online shopping has become so popular, delivering goods by means of transport firms is a steadily growing business. The question therefore arises how transport costs should be compensated in a "fair way". Currently, usually weight and/or volume are used as indicators for transport costs. Motivated by this observation, it seems natural, to study bin packing games as defined below as a first step towards analyzing allocation problems of this kind. It is quite possible that more elaborate concepts like, e.g., knapsack or two-dimensional bin packing lead to even more insight also in real world scenarios.

Suppose there are two disjoint sets of players, say, A and B. Each player $i \in A$ possesses an item of value/size a_i, for $i = 1, \dots, n$, and each player $j \in B$ possesses a truck/bin of capacity b_j. The items produce a profit proportional to their size a_i if they are brought to the market place. The value $v(N)$ of the grand coalition thus represents the maximum profit achievable. How should $v(N)$ be allocated to the owners of the items and the owners of the trucks?

Faigle and Kern [2] first studied this problem and observed that the 1/2-core is always nonempty, provided that any item fits into each bin. It is also shown that for any $\epsilon < 1/7$, one can always find an instance such that the ϵ-core is empty. Hence, the minimal ϵ (ensuring a nonempty ϵ-core for all instances) is $\geq 1/7$.

Afterwards, researchers focused on bin packing games with uniform capacities ($b_j = 1$ for all j). Woeginger [8] showed that the 1/3-core is always nonempty—a result that was slightly improved later by Kern and Qiu [4], i.e., $(1/3 - 1/108)$-core is always nonempty. Kuipers [5] considered the special case of item sizes strictly larger than 1/3 and proved that the 1/7-core is nonempty and that this bound is tight. Faigle and Kern [3] showed that for any fixed ϵ, the ϵ-core is nonempty if the number of trucks is sufficiently large.

Results for the general (non-uniform) bin packing games are quite poor. Apparently, the problem becomes more difficult when capacities of trucks are distinct. In particular, the "matching approach" used in [3] and [5] cannot be applied any more and new ideas are needed even in the special case of large item sizes (*cf.* Section 4).

We start with some terminologies. The players of A are referred to as "items" and the players of B are "bins". A *feasible packing* of an item set $A' \subseteq A$ into a set of bins $B' \subseteq B$ is an assignment of some (or all) elements in A' to the bins in B' such that the total size of items assigned to any bin does not exceed its capacity. Items that are assigned to a bin are called *packed* and items that are not assigned are called *not packed*. The *value* of a feasible packing is the total size of packed items.

The player set N consists of all items and all bins. The value $v(S)$ of a coalition $S \subseteq N$, where $S = A_S \cup B_S$ with $A_S \subseteq A$ and $B_S \subseteq B$, is the maximum value of all feasible packings of A_S into B_S. A corresponding feasible packing is called an *optimum packing*.

We assume that the bins are ordered weakly decreasingly, i.e.,

$$1 = b_1 \geq b_2 \geq \cdots \geq b_k.$$

A set $F \subseteq A$ is called *feasible* for bin j, if the total size of items of F does not exceed the bin capacity b_j. Denote by $\mathcal{F}$ the collection of all feasible sets and $\mathcal{F}_j$ the collection of feasible sets for bin j, $j = 1, \dots, k$, thus,

$$\mathcal{F} = \mathcal{F}_1 \supseteq \mathcal{F}_2 \supseteq \cdots \supseteq \mathcal{F}_k.$$

Moreover, given a set of items, say F, denote by a_F the total size of F, i.e., $a_F = \sum_{i \in F} a_i$. Let $\mathcal{F}_{k+1} = \emptyset$. Hence, the value $v(N)$ of the grand coalition equals the optimal objective function value of the following integer linear program.

$$\begin{aligned}
&\max \sum_{F \in \mathcal{F}} a_F y_F, \\
&\text{s.t.} \sum_{F \ni i, F \in \mathcal{F}} y_F \leq 1 \quad (i = 1, \dots, n), \\
&\sum_{F \in \mathcal{F} \setminus \mathcal{F}_{j+1}} y_F \leq j \quad (j = 1, \dots, k), \\
&y_F \in \{0, 1\}, \quad \text{for all } F \in \mathcal{F}.
\end{aligned} \tag{2.1}$$

Its relaxation is

$$\begin{aligned}
&\max \sum_{F \in \mathcal{F}} a_F y_F, \\
&\text{s.t.} \sum_{F \ni i, F \in \mathcal{F}} y_F \leq 1 \quad (i = 1, \dots, n), \\
&\sum_{F \in \mathcal{F} \setminus \mathcal{F}_{j+1}} y_F \leq j \quad (j = 1, \dots, k), \\
&y_F \in [0, 1], \quad \text{for all } F \in \mathcal{F}.
\end{aligned} \tag{2.2}$$

A feasible solution to (2.2) is called a *fractional packing*. It is not difficult to see that the above problems (2.1) and (2.2) correspond to problems (1.3) and (1.2). Let v' be the optimal objective function value of (2.2). By Lemma 1.1, the ϵ-core is nonempty if and only if $\epsilon \geq 1 - v/v'$. Therefore, the minimal taxation rate is indeed $\epsilon_N = 1 - v(N)/v'(N)$.

To analyze the relation between v and v', we first study a simple packing algorithm for constructing an integral packing: Consider a bin b_j and a set $\{a'_1, \dots, a'_s\}$ of items that fit into b_j (i.e., $a'_i \leq b_j$). The simple packing algorithm either packs all items into b_j (if $\sum_i a'_i \leq b_j$) or computes a subset A' of items that has total size $a_{A'} \geq \frac{1}{2} b_j$:

Algorithm Simple Packing

```
Input: bin b_j, items a'_1, ..., a'_s ≤ b_j
IF ∑_i a'_i ≤ b_j THEN return {a'_1, ..., a'_s}
ELSE
   let a'_1 + ··· + a'_r ≤ b_j, a'_1 + ··· + a'_{r+1} > b_j
   return the larger of {a'_1, ..., a'_r} and {a'_{r+1}}.
```

The simple packing algorithm is readily extended to a packing heuristic, constructing an integer packing for N: Let $A_j \subseteq A$ denote the set of items that fit into b_j. We first apply simple packing to b_k and A_k. Assume that the simple packing algorithm packs $F_k \subseteq A_k$ into b_k. We then apply simple packing to b_{k-1} and $A_{k-1} \setminus F_k$ and assume that $F_{k-1} \subseteq A_{k-1} \setminus F_k$ gets packed into b_{k-1} etc. Continuing this way, we find

Lemma 2.1. *The simple packing heuristic computes an integral packing $F_1, \ldots, F_k$ such that either*

(i) $a_{F_j} \geq \frac{1}{2} b_j$ *for* $j = 1, \ldots, k$ *or*
(ii) $a_{F_j} \geq \frac{1}{2} b_j$ *for* $j = 1, \ldots, r$ *and* $F_{r+1} \cup \cdots \cup F_k = A_{r+1}$ *for a suitable* $0 \leq r < k$ *(possibly* $r = 0$*).*

Proof. Apply the simple packing heuristic as described above, starting with b_k, b_{k-1} etc. If it never happens that all "remaining" items $A_j \setminus (F_{j+1} \cup \cdots \cup F_k)$ fit into b_j, then each bin gets filled to at least half its capacity (by simple packing). Otherwise, (ii) follows by letting r denote the smallest j such that indeed all "remaining" items were packed into b_{j+1}, and hence all of A_{j+1} was packed into bins $b_{j+1}, \ldots, b_k$. □

As a simple consequence, we obtain the following.

Theorem 2.2. *$\frac{1}{2}$-core(N) $\neq \emptyset$ for all N.*

Proof. Let v, v' denote the optimal integral resp. fractional packing value. Clearly, $v \geq a_{F_1} + \cdots + a_{F_k}$ for the simple packing $F_1, \ldots, F_k$. Thus, in case (i) of Lemma 2.1, we readily find

$$v \geq a_{F_1} + \cdots + a_{F_k} \geq \frac{1}{2}(b_1 + \cdots + b_k) \geq \frac{1}{2} v',$$

and the claim follows. If case (ii) occurs, then all of A_{r+1} gets packed into $b_{r+1}, \ldots, b_k$ by the simple packing heuristic. As a consequence, we find that the game N naturally splits into $N^{red} := (\{b_1, \ldots, b_r\}, A \setminus A_{r+1})$ and $N^{triv} := (\{b_{r+1}, \ldots, b_k\}, A_{r+1})$. Indeed, as no item in $A \setminus A_{r+1}$ fits into any bin $b_j, j \geq r+1$, an optimum fractional packing y' for N can assign items in $A \setminus A_{r+1}$ only to bins $b_1, \ldots, b_r$. As all of A_{r+1} can be packed (even integrally) into $b_1, \ldots, b_{r+1}$, an optimum fractional packing y' can be assumed to fractionally pack part of $A \setminus A_{r+1}$ into $b_1, \ldots, b_r$ and all of A_{r+1} into $b_{r+1}, \ldots, b_k$. Thus,

$$v' = v'_{red} + a_{A_{r+1}},$$

where v'_{red} is the fractional packing value for N^{red}. By Lemma 2.1, the simple packing heuristic yields a value

$$v \geq \frac{1}{2} v'_{red} + a_{A_{r+1}} \geq \frac{1}{2} v'$$

and the result follows.

We refer to N^{red} as defined in the proof of Theorem 2.2 the *reduced game*. More generally, let us call N *reducible* if, for suitable $r \leq k$, all items in A_{r+1} can be (integrally) packed into $b_{r+1}, \ldots, b_k$. Thus, as we have seen in the proof of Theorem 2.2, reducible games inherit ϵ-balancedness from their corresponding reductions $N^{red} = (\{b_1, \ldots, b_r\}, A \setminus A_{r+1})$.

3. Large item sizes $a_i > 1/3$

In the uniform case, instances with large items $a_i > 1/3$ have attracted much attention. In theoretical terms, the case $a_i > 1/3$ is critical for proving non-emptiness of the 1/3-core. In practice, such instances may occur in large express firms which only deal with large goods, i.e., small items are not delivered by them (as delivering small items gains less and causes almost the same administration cost). A standard proof technique for showing non-emptiness of the 1/3-core in the uniform case works as follows: First reduce the problem to the case where all items have size strictly larger than 1/3. In these reduced problem instances, at most two items fit into a bin. Hence a fractional packing is close to a fractional matching of items and can thus be treated with well-known techniques from matching theory. In the non-uniform case, this approach does not work, as we shall explain below. Indeed, it is even unclear whether (2.2) always has an optimal solution that is $\frac{1}{2}$-integral. (In the uniform case, this follows quite easily by standard arguments from (fractional) matching theory, *cf.*, *e.g.*, [3].)

Still, the reduction to large item sizes can be extended to the non-uniform case, which might be of independent interest: As it turns out, in the non-uniform case we have to distinguish between small and large items, where "small" and "large" are defined relative to the *average bin size* $\bar{b} = \sum_{j=1}^{k} b_j / k$.

Lemma 3.1. *Let N be a bin packing game and assume N is ϵ-balanced for some $\epsilon < 1/2$. Then adding "small" items of size $a_i \leq \epsilon\bar{b}$ does not affect ϵ-balancedness.*

Proof. First note that it suffices to prove the claim in the case where a single small item a_i is added. Let $N^+ := N \cup \{a_i\}$ denote the extended game. We fix an optimum integral packing y^* for N and distinguish two cases:

Case 1: The new item a_i can be packed "on top of" the optimum integral packing for N (i.e. some bin j is filled only up to at most $b_j - a_i$). In this case, we conclude that $v_{N^+} = v_N + a_i$, whereas clearly, $v'_{N^+} \leq v'_N + a_i$ (Take an optimal fractional packing for N^+ and remove item a_i from each feasible set to obtain a corresponding feasible fractional packing for N.) Hence, $\epsilon_{N^+} \leq \epsilon_N \leq \epsilon$ follows.

Case 2: In the optimum integral packing for N, each bin with capacity $b_j \geq \epsilon\bar{b}$ is filled to more than $b_j - a_i$. In this case, the total content of each bin is at least $b_j - \epsilon\bar{b}$, hence

$$\frac{v_{N^+}}{v'_{N^+}} \geq \frac{v_N}{\sum_{j=1}^{k} b_j} \geq \frac{\sum_{j=1}^{k}(b_j - \epsilon\bar{b})}{\sum_{j=1}^{k} b_j} = \frac{k\bar{b} - k\epsilon\bar{b}}{k\bar{b}} = 1 - \epsilon,$$

proving ϵ-balancedness of N^+. $\square$

Unfortunately, Lemma 3.1 is of not much help in simplifying matters: Indeed, by adding a number of small dummy bins (plus corresponding items if we like), the average bin size can be made arbitrarily small – and hence the item sizes become relatively large – without significant change in the instance.

If we instead restrict ourselves to item sizes that are large in an absolute sense, the bound $\epsilon \leq 1/2$ can be somewhat improved (although, as compared to the uniform case, with considerably more effort and weaker result):

Proposition 3.2. *If all items have size $a_i > 1/3$, the 5/12-core is nonempty.*

Proof. Let $y' = (y'_F)$ be an optimum fractional solution with value v'. We seek to "round" y' to an integral packing y of value $v \geq \frac{7}{12}v'$. The method we use is a modification of the rounding technique proposed in [4]. Let $\mathcal{F}' = \{F_1, \ldots, F_m\}$ denote the support of y' and assume that

$$a_{F_1} \geq \cdots \geq a_{F_m}.$$

We think of $F_1, \ldots, F_m$ as being assigned to bins $b_1 \geq \cdots \geq b_k$ in this order, so that every bin except possibly the last ones are assigned feasible sets of total y-value equal to 1. Thus a feasible set F_s may get assigned to two consecutive bins j and $j+1$ if $y_{F_1} + \cdots + y_{F_{s-1}} < j$ and $y_{F_1} + \cdots + y_{F_s} > j$. We seek to achieve the following simplifications:

(i) $y'_F < 1$ for all $F \in \mathcal{F}'$.
(ii) All item sizes are less than 2/3.
(iii) At least one two-element set F_j is assigned to b_k. (Hence, in particular, $b_k > 2/3$.)

Proof of (i): We proceed by induction on the number of players. If $y'_{F_j} = 1$ for some j, remove all items contained in F_j and the bin to which F_j is assigned. (If F_j is assigned to two bins, choose the smaller one.) Let $\tilde{N}$ denote the resulting instance. Obviously, y' induces a feasible fractional packing $\tilde{y}'$ for $\tilde{N}$ of value $\tilde{v}' = v' - a_{F_j}$. By induction, there exists a corresponding integral packing $\tilde{y}$ of value $\tilde{v} \geq \frac{7}{12}\tilde{v}'$. Extend this integral packing to an integral packing for N by packing F_j into the removed bin. The resulting integral packing has value $v = \tilde{v} + a_{F_j} \geq \frac{7}{12}\tilde{v}' + a_{F_j} \geq \frac{7}{12}v'$. Thus, in what follows, we may (and will) assume that $y'_F < 1$ for all $F \in \mathcal{F}'$.

Proof of (ii): Assume to the contrary that some item has size $a = a_{max} \geq 2/3$. Then a cannot be combined with any other item into a feasible set. Hence there must be a single-item set $F_s = \{a\}$. (We tacitly assume that item a is used at all—otherwise the Theorem follows by induction on the number of items.) According to (i), we may assume $y'_{F_j} < 1$. Remove the (smallest) bin, say, b_j, to which $F_s = \{a\}$ is assigned, together with subsequent feasible sets $F_{s+1}, F_{s+2}, \ldots$ assigned to b_j so that the removed feasible sets have a y'-value of exactly 1. The resulting fractional packing $\tilde{v}'$ for the instance $\tilde{N} = N \setminus \{a, b_j\}$ has value $\tilde{v}' \geq v' - 1$ and, by induction, there exists an integral packing of size $\tilde{v} \geq \frac{7}{12}(v' - 1)$. Adding item a filled into bin b_j, we obtain an integral packing for N of value

$$v \geq \frac{7}{12}(v' - 1) + a \geq \frac{7}{12}(v' - 1) + \frac{2}{3} \geq \frac{7}{12}v'.$$

Thus, in what follows, we may (and will) assume that (ii) holds *w.l.o.g.*

Proof of (iii): According to (ii), the one-element sets have smaller size than the two-element sets, and, hence, appear last in the ordering $F_1, \ldots, F_m$. Now assume that all sets F_m, F_{m-1} *etc.* assigned to b_k are one-element sets. (If no F_j is assigned to b_k, the claim of the Theorem follows by induction on the number of bins.) Let $F_m = \{a\}$.

We first aim at showing that we may assume $\sum_{F\in\mathcal{F}'} y'_F = k$ *w.l.o.g.* Indeed, if $\sum_{F\in\mathcal{F}'} y'_F < k$, we first try to increase this sum by increasing y'_{F_m} as much as possible until either $\sum_{F\in\mathcal{F}'} y'_F = k$ holds (and we are done) or item a gets fully packed in the sense that $\sum_{F\in\mathcal{F}'}\sum_{F\ni a} y'_F = 1$. We then seek to increase y'_{F_m} further by *splitting* a suitable feasible $F_j = \{a, a_i\}$, *i.e.*, we increase both $y'_{\{a_i\}}$ and $y'_{\{a\}}$ and decrease y'_{F_j} by the same amount. Note that this modification keeps y' feasible, as basically F_j is replaced by the smaller feasible set $\{a_i\}$. Proceeding this way we eventually end up with a modified feasible (!) fractional packing (which we again denote by y') of equal value v' that satisfies $\sum_{F\in\mathcal{F}'} y'_F = k$ (unless, in between, either $y'_{\{a\}}$ or $y'_{\{a_i\}}$ is increased to 1 and induction applies anyway). Thus, we may indeed assume $\sum_{F\in\mathcal{F}'} y'_F = k$ in the following.

As $\sum_{F\in\mathcal{F}'} y'_F = k$ holds, the total y'-value of sets assigned to b_k equals 1. Thus there are at least two one-element sets $F_m = \{a\}$ and $F_{m-1} = \{a'\}$, say, assigned to b_k (as we assume $y'_{F_m} < 1$). Since $a_{F_{m-1}} \geq a_{F_m}$, we have $a' \geq a$. We seek to reduce y'_{F_m} to 0. To this end, we first increase $y'_{F_{m-1}}$ and decrease y'_{F_m} as much as possible until either $y'_{F_m} = 0$ (and claim (iii) follows by induction on the number of single-element sets in the support of y'—under the additional assumption that $\sum_{F\in\mathcal{F}'} y'_F = k$ (!)) or a' gets fully packed, *i.e.*, $\sum_{F\in\mathcal{F}'}\sum_{F\ni a'} y'_F = 1$. In the latter case we seek to reduce y'_{F_m} further by replacing a' with a as much as possible in any set $F_j = \{a', a_i\}$ with $a_i \neq a$. More precisely, as long as there is some $F_j = \{a', a_i\}$ with $y'_{F_j} > 0$ and $a_i \neq a$, we decrease y'_{F_j} and y'_{F_m} and increase $y'_{\{a,a_i\}}$ and $y'_{\{a'\}}$ by the same amount. Note that this modification keeps y' feasible, since $a' \leq a$, so F_j is (partially) replaced by a smaller feasible set in the fractional packing. This modification stops when the only feasible two-element set containing a' is $F_j = \{a', a\}$. Note that, at that point of our modification, we have $y'_{\{a,a'\}} + y'_{\{a'\}} = 1$.

Assume for a moment that there is a third single element set $F_{m-2} = \{a''\}$ assigned to b_k with $a'' \geq a'$. We could then repeat the above modification *w.r.t.* a'' and a' (instead of a' and a), thereby either succeeding in reducing $y'_{F_{m-1}}$ to 0 (in which case induction on the number of single-element sets in the support of y' applies) or getting stuck in a situation where a'' is fully packed but the only two-element set containing a'' is $\{a'', a'\}$. But this would contradict our assumption that a' is only combined with a in a feasible set $F_j = \{a', a\}$.

Summarizing, we may assume that F_{m-2} is assigned to b_{k-1} and, consequently, $y'_{\{a'\}} + y'_{\{a\}} \geq 1$. Hence $y'_{\{a',a\}} + y'_{\{a'\}} = 1$ and $y'_{\{a',a\}} + y'_{\{a\}} \leq 1$ imply $y'_{\{a'\}} \geq y'_{\{a',a\}}$ and, therefore, $y'_{\{a'\}} \geq \frac{1}{2}$ and $y'_{\{a',a\}} \leq \frac{1}{2}$. Removing b_k with all its content and item a' from F_j (the only two-element set containing a') results in a feasible fractional packing $\tilde{y}'$ for $\tilde{N} := N \setminus \{a', b_k\}$ of value

$$\tilde{v}' = v' - a' - y'_{F_m} a \geq v' - a' - \frac{1}{2}a \geq v' - \frac{3}{2}a'.$$

By induction, there is a corresponding integral solution of value $\tilde{v} \geq \frac{7}{12}\tilde{v}'$. Adding item a' (assigned to b_k), we obtain a packing for N of value

$$v \geq \frac{7}{12}\tilde{v}' + a' \geq \frac{7}{12}\left(v' - \frac{3}{2}a'\right) + a' \geq \frac{7}{12}v'.$$

This completes the proof of (iii).

Having achieved the above three simplifications, we are now ready to proceed to the main part of the proof, which consists in "rounding" y' to an integer packing y, with value $v \geq \frac{7}{12}v'$. The basic idea is a greedy selection rule similar to the one in [4]. The main difference is that, here, we construct pairwise disjoint feasible sets $F_{j_1}, \ldots, F_{j_r}$ in a reverse order, *i.e*, starting with the *smallest* feasible two-element set F_j rather than with the largest (as we did in [4]). Thus we let $F_{j_1} \in \mathcal{F}$ (assigned to b_k !) denote the smallest two-element set in the support of y', and choose it as a feasible set of our integral packing (i.e., $y_{F_{j_1}} = 1$). Then we look for the next feasible set among $F_{j_1-1}, \ldots, F_1$ that is disjoint from F_{j_1} and call it F_{j_2} *etc.* Thus in each step we determine the smallest feasible set that is disjoint from all previously selected ones. As each of the selected feasible set F_{j_ρ} contains exactly two items, say, $F_{j_\rho} = \{a_i, a_l\}$, the total y'-value of feasible sets intersecting F_{j_ρ} is bounded by $2 - y_{F_{j_\rho}}$, for $\rho = 1, \ldots, r$. (This is straightforward from $\sum_{F\cap F_{j_\rho}\neq\emptyset} y'_F \leq \sum_{F\ni a_i} y'_F + \sum_{F\ni a_l} y'_F - y'_{F_{j_\rho}}$.) For that reason, $F_{j_1}, \ldots, F_{j_r}$ can be assigned to bins $b_k, b_{k-2}, \ldots, b_{k-2(r-1)}$ (in that order). Due to (iii), summation yields

$$\sum_{j=1}^{r}(2 - y_{F_{i_j}}) > k - 1,$$

implying $2r \geq k$.

For the remaining $k - r$ bins, w.l.o.g., we assume 1/2 capacity of each bin can be filled by greedily packing items to those bins (as $a_i < 2/3 < b_k$ for all i, *cf.* also [2] or apply the simple packing heuristic). Let R be the index set of the remaining $k - r$ bins and $\bar{b}(R)$ be the corresponding average bin size. In case k is even, we have $\bar{b}(R) \geq \bar{b}$. Hence,

$$v \geq \frac{2}{3}r + \sum_{j\in R}\frac{b_j}{2} = \frac{2}{3}r + (k-r)\frac{\bar{b}(R)}{2} \geq \frac{2}{3}\cdot\frac{k}{2} + \frac{k}{2}\cdot\frac{\bar{b}}{2} \geq \frac{v'}{3} + \frac{v'}{4} = \frac{7}{12}v'.$$

For k odd, the approximation is even better as we have in addition b_1 filled to at least $\frac{2}{3}$ of its capacity. This completes the proof. □

4. The limiting case: $k \to \infty$

In this section, we seek to extend the result of [3], saying that the ϵ-core is non-empty provided the game is "large enough", to the nonuniform case. As in [3], our arguments are based on the bin packing approach initially introduced by de la Vega and Lueker [1]. Consider the class of bin packing games where the number of distinct item sizes and the number of distinct bin sizes are bounded by m. Assume that the item sizes are $a_1, \ldots, a_m$ and occur with multiplicities $\alpha_1, \ldots, \alpha_m$, and assume that the bin sizes are $b_1, \ldots, b_m$ and occur with multiplicities $\beta_1, \ldots, \beta_m$. Each feasible set $F \in \mathcal{F}$ can be described by its *type vector* $T = (t_1, \ldots, t_m)$ indicating the number t_i of items of size a_i that occur in F. Let

$$a_T = \sum_{i=1}^{m} t_i a_i$$

and let $\mathcal{T}$ be the set of type vectors. Moreover, for each bin size b_j, denote by $\mathcal{T}_j$ the set of type vectors, with $a_T \leq b_j$ for all $T \in \mathcal{T}_j$. Hence,

$$\mathcal{T} = \mathcal{T}_1 \supseteq \mathcal{T}_2 \supseteq \cdots \supseteq \mathcal{T}_m.$$

Let $\mathcal{T}_{m+1} = \emptyset$. Now v and v' can be computed by the following (integer) linear programs.

$$\begin{aligned} &\max \sum_{T \in \mathcal{T}} a_T z_T, \\ &\text{s.t.} \sum_{T \in \mathcal{T} \setminus \mathcal{T}_{i+1}} z_T \leq \sum_{j=1}^{i} \beta_j \quad (i = 1, \ldots, m), \\ &\sum_{T \in \mathcal{T}} t_i z_T \leq \alpha_i \quad (i = 1, \ldots, m), \\ &z_T \in \mathbb{N}^+, \text{ for all } T \in \mathcal{T}, \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} &\max \sum_{T \in \mathcal{T}} a_T z_T, \\ &\text{s.t.} \sum_{T \in \mathcal{T} \setminus \mathcal{T}_{i+1}} z_T \leq \sum_{j=1}^{i} \beta_j \quad (i = 1, \ldots, m), \\ &\sum_{T \in \mathcal{T}} t_i z_T \leq \alpha_i \quad (i = 1, \ldots, m), \\ &z_T \in \mathbb{R}^+, \text{ for all } T \in \mathcal{T}. \end{aligned} \tag{4.2}$$

Given an instance N, let $gap(N) = v'(N) - v(N)$. Let a_A, a_B be the total size of $A(N)$ and the total capacity of $B(N)$, respectively.

Lemma 4.1. *If the item sizes and bin sizes take on at most m different values, then $gap \leq m$.*

Proof. Let $z^* = (z_T^*)_{T \in \mathcal{T}}$ be an optimal fractional packing which is a basic feasible solution of (4.2). As there are only $2m$ constraints in (4.2), we conclude that $|supp(z^*)| \leq 2m$, where $supp(z^*) = \{z_T > 0,\ T \in \mathcal{T}\}$. Furthermore, we may assume that $z^* \leq 1$ (componentwise). Indeed, assume $z_T^* > 1$ and let b_j denote the smallest bin size to which a set of type T is assigned by z^*. Reducing the multiplicities of all items in T by 1 and, similarly, replacing β_j by $\beta_j - 1$, we obtain a modified instance $\tilde{N}$ with fractional packing value $\tilde{v}' = v' - a_T$ and, by induction, a corresponding integral packing of value $\tilde{v} \geq \tilde{v}' - m$. Extending this to an integral packing for N in the obvious way (by assigning a set of type T to a bin of type j), the claim follows. Thus we may indeed assume that $z^* \leq 1$, and hence

$$v' = \sum_{T \in \mathcal{T}} a_T z_T^* \leq \sum_{T \in \mathcal{T}} z_T^* \leq \left|supp(z^*)\right| \leq 2m.$$

Theorem 2.2 then implies $v \geq m$ and the claim follows. $\square$

Lemma 4.2. *Let $\epsilon > 0$ be such that $\epsilon^{-1} \in \mathbb{N}$. Then $a_B \geq \epsilon n$ implies $gap \leq \epsilon^{-2} + 4\epsilon a_B$.*

Proof. Assume items are given by the following non-decreasingly ordered list,

$$A: \ a_1 \leq a_2 \leq \cdots \leq a_n.$$

Given $m > 0$, $m \in \mathbb{N}$ and $h = \lfloor n/m \rfloor$, divide A into $m + 1$ consecutive sublists

$$A = A_1, \ldots, A_m, R$$

satisfying $|A_i| = h, i = 1, \dots, m$ and $|R| < h$. Let a_{i_j} be the first element of A_j. We consider the modified item list

$$A^- = A_1^-, \dots, A_{m-1}^-, A_m^-, R,$$

where the sublist $A_j^- = a_{i_j}, \dots, a_{i_j}$ arises from A_j by replacing each element of A_j with a copy of the smallest item in the sublist.

On the one hand, any feasible (integral) packing relative to A^- yields a feasible (integral) packing of A if we replace elements of A_j^- by the corresponding elements of A_{j-1}, for $j = 2, \dots, m$ and remove all elements of A_1^-. The decrease in value is then bounded by

$$h(a_{i_2} - a_{i_1}) + h(a_{i_3} - a_{i_2}) + \cdots + h(a_{i_m} - a_{i_{m-1}}) + a_{A_1^-} \leq ha_{i_m} \leq h.$$

Denote by v_A, v'_A the integral resp. fractional optimum, with respect to an item list A. Hence,

$$v_A \geq v_{A^-} - h. \tag{4.3}$$

On the other hand, each feasible fractional packing relative to A also yields a feasible packing of A^- if we replace elements of A_j by the corresponding elements of A_j^-, for $j = 1, \dots, m$. Because $\sum_{F \in \mathcal{F}} a_F z_F = \sum_{i=1}^n \sum_{F \ni i} z_F a_i$, the resulting decrease in value is bounded by

$$h(a_{i_2} - a_{i_1}) + h(a_{i_3} - a_{i_2}) + \cdots + h(1 - a_{i_m}) \leq h.$$

Thus,

$$v'_A \leq v'_{A^-} + h. \tag{4.4}$$

Let $gap = gap_A = v'_A - v_A$ and $gap_{A^-} = v'_{A^-} - v_{A^-}$. Then inequalities (4.3) and (4.4) imply

$$gap_A \leq gap_{A^-} + 2h. \tag{4.5}$$

Now consider the bin packing game relative to A^-. Assume bin sizes are ordered non-increasingly, i.e.,

$$B: \ b_1 \geq b_2 \cdots \geq b_k,$$

We also divide B into $m + 1$ consecutive sublists

$$B = B_1, B_2, \dots, B_m, R'.$$

Let $h' = \lfloor k/m \rfloor$, hence $\left|B_j\right| = h'$ for $j = 1, \dots, m$ and $\left|R'\right| < h'$. Define the modified lists

$$B^- = B_1^-, B_2^-, \dots, B_m^-, R'$$

by letting $B_j^- = b_{i_j}, \dots, b_{i_j}$, where b_{i_j} is the smallest bin size in B_j.

Denote by v_B, v'_B the integral resp. fractional optimum corresponding to a bin list B (and item set A^-). It is straightforward to see that

$$v_B \geq v_{B^-}. \tag{4.6}$$

Indeed, any feasible (integral) packing of B^- is a feasible (integral) packing of B if we simply pack the feasible sets (which are packed to bins) of B_j^- to (the bins of) B_j, for $j = 1, \dots, m$.

On the other hand, each feasible fractional packing relative to B also yields a feasible fractional packing relative to B^- if we pack the feasible sets of B_j to B_{j-1}^-, for $j = 2, \dots, m$ and remove all feasible sets assigned to B_1. The resulting decrease in value is then bounded by $a_{B_1} \leq h'$.

This shows

$$v'_B \leq v'_{B^-} + h'. \tag{4.7}$$

Let $gap_B = v'_B - v_B$ and $gap_{B^-} = v'_{B^-} - v_{B^-}$. Inequalities (4.6) and (4.7) yield

$$gap_B \leq gap_{B^-} + h'. \tag{4.8}$$

As gap_B and gap_{B^-} are both defined relative to item set A^-, we may combine (4.5) and (4.8) to yield

$$gap \leq gap_{B^-} + 2h + h'. \tag{4.9}$$

Now observe that B^- has at most $m + h'$ different bin sizes and, similarly, A^- contains at most $m + h$ different item sizes. Furthermore, we may assume *w.l.o.g.* that $k \leq n$, hence $h' \leq h$. Lemma 4.1 implies

$$gap \leq m + h + 2h + h' \leq m + 4h.$$

Let $m = \epsilon^{-2}$. Then $h \leq \epsilon^2 n \leq \epsilon a_B$ and, correspondingly,

$$gap \leq \epsilon^{-2} + 4\epsilon a_B. \quad \square$$

Lemma 4.3. *Let* $0 < \epsilon < a_1 \leq \cdots \leq a_n$. *Then* $gap \leq 4\epsilon^{-4} + 2\epsilon^2 a_B$.

Proof. Recall the optimization problem (2.2) and let $y^* = (y^*_F)_{F\in\mathcal{F}}$ be an optimal solution of the problem. By induction on the number n of items, we may assume that each item i occurs in some feasible set F with $y^*_F \neq 0$. Because each feasible set contains at most $(\epsilon^{-1} - 1)$ items, we obtain the upper bound

$$n \leq \left|supp(y^*)\right| (\epsilon^{-1} - 1)$$

on the number of items.

Note that each item i with $\sum_{F\in\mathcal{F}} \sum_{F\ni i} y^*_F = 1$ contributes more than ϵ to the objective function value. So there can be no more than a_B/ϵ such items i. Hence,

$$\left|supp(y^*)\right| \leq a_B\epsilon^{-1} + k.$$

This shows that $a_B \geq \eta n$ holds with $\eta = \epsilon^2/2$. Therefore, Lemma 4.2 yields the bound

$$gap \leq 4\epsilon^{-4} + 2\epsilon^2 a_B. \quad \square$$

Theorem 4.4. *Let* $0 < \epsilon < 1/4$. *Then* $k \geq 8(\epsilon\bar{b})^{-5}$ *implies* $gap \leq \epsilon a_B$.

Proof. By induction on $|N|$. If all items of N have size $a_i > \epsilon\bar{b}$, then Lemma 4.3 implies

$$\begin{aligned} & gap \leq 4(\epsilon\bar{b})^{-4} + 2(\epsilon\bar{b})^2 a_B \leq \epsilon a_B \\ \Leftrightarrow\ & [\epsilon - 2(\epsilon\bar{b})^2]a_B \geq 4(\epsilon\bar{b})^{-4} \\ \Leftrightarrow\ & [\bar{b}^{-1} - 2(\epsilon\bar{b})]a_B \geq 4(\epsilon\bar{b})^{-5} \\ \Leftrightarrow\ & k(1 - 2\epsilon\bar{b}^2) \geq 4(\epsilon\bar{b})^{-5}. \end{aligned}$$

As $\epsilon < 1/4$, the latter follows from the assumed lower bound on k.

If N contains some item $a_i \leq \epsilon\bar{b}$, consider $\tilde{N} = N \setminus \{a_i\}$. By induction, we have $gap(\tilde{N}) \leq \epsilon a_B$. Let $\tilde{v}$ be the value of an optimum integral packing for $\tilde{N}$. If a_i can be placed into any bin on "on top of" a corresponding packing of $\tilde{v}$, then $v \geq \tilde{v} + a_i$ and $v' \leq \tilde{v}' + a_i$ imply $gap(N) \leq gap(\tilde{N}) \leq \epsilon a_B$. Otherwise, if a_i does not fit anywhere, then each bin is filled to at least $b_j - a_i$ in the optimum integral solution for $\tilde{N}$, hence

$$v \geq \tilde{v} \geq \sum_{j=1}^{k}(b_j - a_i) = a_B - ka_i \geq a_B - \epsilon k\bar{b} = (1 - \epsilon)a_B$$

and, again, $gap \leq \epsilon a_B$ follows. $\square$

We seek to prove that ϵ-core(N)$\neq \emptyset$ provided the game defined by N is "large" enough. In [3], in the uniform case, a sufficient condition in terms of a lower bound $k = \Omega(\epsilon^{-5})$ was given. Note, however, that we cannot expect such a result to hold for the non-uniform case. Indeed, consider a fixed instance N_0 with minimal tax rate $\epsilon_0 = \epsilon_{N_0}$. Adding arbitrarily many small bins (smaller than a_{min}, the minimum item size), we find that $k \to \infty$ (as well as $a_B \to \infty$), while ϵ_N remain unaffected. The same argument shows that even the assumptions in Theorem 4.4 cannot guarantee ϵ-balancedness.

Thus, it seems that we should restrict our attention to irreducible games. Alternatively, given an arbitrary game N, we first apply the simple packing algorithm to split N into a reduced game N^{red} and a (possibly empty) trivial game N^{triv}. Then, if the reduced part is (still) large, a lower bound on the minimum taxation rate for N_{red} (and hence for N) follows:

Corollary 4.5. *Let* $0 < \epsilon < 1/2$ *with* $\epsilon^{-1} \in \mathbb{N}$. *If* N *is reduced (in particular, if* N *is irreducible), then* $k \geq 2^8(\epsilon\bar{b})^{-5}$ *implies* ϵ*-core(*N*)* $\neq \emptyset$.

Proof. Straightforward: As $k \geq 8(\frac{\epsilon}{2}\bar{b})^{-5}$, we get $gap \leq \frac{\epsilon}{2}a_B$ from Theorem 4.4 and since N is reduced, greedy packing yields $v \geq \frac{1}{2}a_B$. Hence

$$\epsilon_N = \frac{gap}{v'} \leq \frac{gap}{v} \leq \epsilon. \quad \square$$

Thus, roughly speaking, games with empty ϵ-core are either "small" or arise from small games by trivial extensions.

5. Remarks and open problems

Our results reveal a certain tradeoff between the taxation rate ϵ and the average bin size $\bar{b}$. This is most evident in Corollary 4.5, but also applies elsewhere. For example, the condition $a_i > 1/3$ in Proposition 3.2 could equally be replaced by $\bar{b} \geq 4/5$, since for $\epsilon = 5/12$, we have $\bar{b} \geq 4/5 \Leftrightarrow \epsilon\bar{b} \geq 1/3$ and hence the result can be obtained via Lemma 3.1. It is not clear to us whether this phenomenon is inherent to the non-uniform case. In particular, if we consider

$$\epsilon^* := \inf_N \{\epsilon \mid \epsilon\text{-core}(N) \neq \emptyset\}$$

where the infimum is taken over all uniform bin packing games, then it is clear (from [3]) that it suffices to consider only games up to a certain fixed size of $|N|$. Is this no longer true in the non-uniform case?

A challenging conjecture of G. J. Woeginger states that, for uniform games, the gap is bounded by a universal constant. Are there any counterexamples at least in the non-uniform case?

Finally, of course a natural question to ask is whether one can improve upon Theorem 2.2 (saying that $\epsilon^* \leq 1/2$ in the non-uniform case). In particular, it is also worthwhile to know whether one can improve the bound 5/12 in Proposition 3.2 for large instances, i.e. $a_i > 1/3$ for all i.

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