Discrete Mathematics 338 (2015) 994–999

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Ramsey numbers of trees versus fans

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ARTICLE INFO

Article history:
Received 17 July 2014
Received in revised form 23 January 2015
Accepted 24 January 2015

Keywords:
Ramsey number
Tree
Star
Fan

ABSTRACT

For two given graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or the complement of G contains G_2 as a subgraph. Let T_n be a tree of order n, S_n a star of order n, and F_m a fan of order $2m + 1$, i.e., m triangles sharing exactly one vertex. In this paper, we prove that $R(T_n, F_m) = 2n - 1$ for $n \geq 3m^2 - 2m - 1$, and if $T_n = S_n$, then the range can be replaced by $n \geq \max\{m(m-1) + 1, 6(m-1)\}$, which is tight in some sense.

1. Introduction

In this paper we deal with finite simple graphs only. For a nonempty proper subset $S \subseteq V(G)$, let $G[S]$ and $G - S$ denote the subgraph induced by S and $V(G) - S$, respectively. Let $N_S(v)$ be the set of all the neighbors of a vertex v that are contained in S, $N_S[v] = N_S(v) \cup \{v\}$ and $d_S(v) = |N_S(v)|$. If $S = V(G)$, we write $N(v) = N_G(v)$, $N[v] = N(v) \cup \{v\}$ and $d(v) = d_G(v)$. For two vertex-disjoint graphs G_1 and G_2, $G_1 \cup G_2$ denotes their disjoint union and $G_1 + G_2$ is the graph obtained from $G_1 \cup G_2$ by joining every vertex of G_1 to every vertex of G_2. We use mG to denote the union of m vertex-disjoint copies of G. A path, a star, a tree, a cycle and a complete graph of order n are denoted by P_n, $S_n = K_1 + (n-1)K_1$, T_n, C_n and K_n, respectively. A book $B_n = K_2 + nK_1$, i.e., it consists of n triangles sharing exactly one common edge, and a fan $F_n = K_1 + nK_2$, i.e., it consists of n triangles sharing exactly one common vertex. We use $\Delta(G)$ and $\delta(G)$ to denote the maximum and minimum degree of a graph G.

Given two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ is the smallest integer N such that, for any graph G of order N, either G contains G_1 as a subgraph or $\overline{G}$ contains G_2 as a subgraph, where $\overline{G}$ is the complement of G. If both G_1 and G_2 are complete graphs, then $R(G_1, G_2)$ is called a classical Ramsey number, otherwise it is called a generalized Ramsey number. Because of the extreme difficulty encountered in the determination of classical Ramsey numbers, Chvátal and Harary [10–12] in a series of papers suggested studying generalized Ramsey numbers, both for their own sake, and for the light they might shed on classical Ramsey numbers. The following is a celebrated early result on generalized Ramsey numbers due to Chvátal.

Theorem 1 (*Chvátal [9]*). *$R(T_n, K_m) = (n-1)(m-1) + 1$ for all positive integers m and n.*

Let H be a connected graph of order p, $\chi(G)$ the chromatic number of G and $s(G)$ the chromatic surplus of G, i.e., the minimum number of vertices in some color class under all proper vertex colorings with $\chi(G)$ colors. Based on Chvátal's result, Burr [4]

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http://dx.doi.org/10.1016/j.disc.2015.01.030
0012-365X/

established the following general lower bound for $R(H, G)$ when $p \geq s(G)$: $R(H, G) \geq (p-1)(\chi(G)-1)+s(G)$. He also defined H to be G-good in case equality holds in this inequality. By Theorem 1, it is easy to see that T_n is K_m-good. This raises the natural questions whether and when T_n is G-good if G consists of ℓ complete graphs K_m sharing exactly one vertex. A special case of the question is whether T_n is F_ℓ-good. Another natural question is for what graphs G, T_n is G-good.

In 1982, Burr et al. determined the Ramsey numbers of sufficiently large trees versus odd cycles, by showing that T_n is C_m-good for odd $m \geq 3$ and $n \geq 756m^{10}$.

Theorem 2 (*Burr et al. [5]*). *$R(T_n, C_m) = 2n-1$ for odd $m \geq 3$ and $n \geq 756m^{10}$.*

In 1988, Erdős et al. confirmed the Ramsey numbers of relatively large trees versus books, by showing that T_n is B_m-good for $n \geq 3m-3$, a result that we will use in our proof of Lemma 2 in the next section.

Theorem 3 (*Erdős et al. [13]*). *$R(T_n, B_m) = 2n-1$ for $n \geq 3m-3$.*

Other results on Ramsey numbers concerning trees can be found in [1–3,6–8,14], see [15] for a survey. In this paper, we first show that S_n is F_m-good for all integers $n \geq \max\{m(m-1)+1, 6(m-1)\}$, by proving the following result.

Theorem 4. *$R(S_n, F_m) = 2n-1$ for $n \geq m(m-1)+1$ and $m \neq 3, 4, 5$, and the lower bound $n \geq m(m-1)+1$ is best possible. $R(S_n, F_m) = 2n-1$ for $n \geq 6(m-1)$ and $m = 3, 4, 5$.*

We postpone the proof of Theorem 4 to the last section. Next we show that T_n is F_m-good for positive integers $n \geq 3m^2-2m-1$, which is the main theorem of our paper.

Theorem 5. *$R(T_n, F_m) = 2n-1$ for all integers $n \geq 3m^2-2m-1$.*

We also postpone the proof of Theorem 5 to the last section. We next show that the following more general result can be obtained from Theorem 5 by induction.

Corollary 1. *$R(T_n, K_{\ell-1}+mK_2) = \ell(n-1)+1$ for $\ell \geq 2$ and $n \geq 3m^2-2m-1$.*

Proof. By Theorem 5, the statement is valid for $\ell = 2$. Assume that $k \geq 3$ and that the statement holds for all integers ℓ with $2 \leq \ell < k$. We prove that it also holds for $\ell = k$.

Since kK_{n-1} contains no T_n and its complement contains no K_{k+1}, hence no $K_{k-1}+mK_2$, we have $R(T_n, K_{k-1}+mK_2) \geq k(n-1)+1$. Let G be a graph of order $k(n-1)+1$. If $\delta(G) \geq n-1$, then by the following folklore lemma that is straightforward to prove using a Greedy approach, G contains T_n and the proof is complete. We present the lemma in a more specific form since we will use it in this form in the sequel.

Lemma 1. *Let G be a graph with $\delta(G) \geq k$, and let $u \in V(G)$. Let T be a tree of order $k+1$ with $v \in V(T)$. Then T can be embedded into G in such a way that v is mapped to u.*

Let us now assume that $\delta(G) \leq n-2$. Then $\Delta(\overline{G}) \geq (k-1)(n-1)+1$. Let v be a vertex with $d_{\overline{G}}(v) = \Delta(\overline{G})$. Then, by the induction hypothesis either $G[N_{\overline{G}}(v)]$ contains a T_n, or $\overline{G}[N_{\overline{G}}(v)]$ contains a $K_{k-2}+mK_2$, which together with v forms a $K_{k-1}+mK_2$ in $\overline{G}$. This completes the proof of Corollary 1. ■

We finish this section by posing a conjecture on the best possible lower bound for n for which T_n is F_m-good.

Conjecture 1. *$R(T_n, F_m) = 2n-1$ for $n \geq m^2-m+1$.*

Let G be any given graph. It is believed that $R(T_n, G) \leq R(S_n, G)$ in general, and all known results point in this direction. Based on this and Theorem 4, we believe that the above conjecture holds, at least for $m \geq 6$.

2. Two preliminary lemmas

In the next section we use the following lemma in our proof of Theorem 4. It is the special case of the statement of Theorem 4 when $m = 2$.

Lemma 2. *$R(S_n, F_2) = 2n-1$ for $n \geq 3$.*

Proof. The lower bound $R(S_n, F_2) \geq 2n-1$ is implied by the fact that $2K_{n-1}$ contains no S_n and its complement contains no triangle, hence no F_2. It remains to prove that $R(S_n, F_2) \leq 2n-1$ for $n \geq 3$.

Let G be a graph of order $2n-1$. Suppose that G contains no F_2 and $\overline{G}$ has no S_n. Then $\Delta(\overline{G}) \leq n-2$ and so $\delta(G) \geq n$. By Theorem 3, G contains B_2. Let $x_1x_2x_3x_4$ be a C_4 with diagonal x_2x_4 in G. Set $X = \{x_1, x_2, x_3, x_4\}$ and $Y = V(G)-X$. If $n = 3$, then $|Y| = 1$ and the vertex in Y has at least three neighbors in X, and so G has F_2, a contradiction. Hence, $n \geq 4$. If $x_1x_3 \in E(G)$, then $N_Y(x_i) \cap N_Y(x_j) = \emptyset$ for $1 \leq i < j \leq 4$, otherwise G contains F_2. Thus, we have $4(n-2) \leq \sum_{k=1}^{4} d_Y(x_k) + 4 \leq 2n-1$, which implies that $n \leq 3$, a contradiction. If $x_1x_3 \notin E(G)$, then since G has no F_2, we get that $N_Y(x_1) \cap N_Y(x_i) = \emptyset$ for $i = 2, 4$ and $N_Y(x_1)$ is an independent set of cardinality at least $n-2$. In this case, we have $d(y) \leq n-1$ for any $y \in N_Y(x_1)$, which contradicts that $\delta(G) \geq n$. ■

We use the following lemma in our proof of Theorem 5. It deals with Ramsey numbers of trees versus mK_2 instead of F_m and might be of some interest by itself.

Lemma 3. *$R(T_n, mK_2) = n + m - 1$ for $n \geq 4(m-1)$.*

Proof. The result is trivial for $m = 1$, thus we assume that $m \geq 2$. Since $K_{n-1} \cup (m-1)K_1$ contains no T_n and its complement contains no mK_2, we conclude that $R(T_n, mK_2) \geq n+m-1$. It remains to prove that $R(T_n, mK_2) \leq n+m-1$ for $n \geq 4(m-1)$.

Let G be a graph of order $n + m - 1$, and suppose to the contrary that neither G contains a T_n nor $\overline{G}$ contains mK_2. Let $M = \{x_1y_1, \ldots, x_ty_t\} \subseteq E(\overline{G})$ be a maximum matching in $\overline{G}$ and $X = V(G) - V(M)$. Then, obviously $t \leq m - 1$ since $\overline{G}$ contains no mK_2, and $G[X]$ is a complete graph by the maximality of M. Assume without loss of generality that $d_X(x_i) \leq d_X(y_i)$ for $1 \leq i \leq t$ in $\overline{G}$. By the maximality of M, $d_X(x_i) \leq 1$ for $1 \leq i \leq t$ in $\overline{G}$. Let Y be the subset of X containing all adjacent vertices of $\{x_1, \ldots, x_t\}$ in $\overline{G}$. Then, by the previous arguments $|Y| \leq t \leq m-1$. Since T_n is a bipartite graph, we may assume without loss of generality that $V(T_n) = (X', Y')$ with $|X'| \leq |Y'|$. Since $n \geq 4(m-1)$, we get that $|Y'| \geq n/2 \geq 2(m-1) \geq |Y| + t$. Now we can embed T_n into G using the following procedure. First map $|Y| + t$ vertices of Y' to $Y \cup \{x_1, \ldots, x_t\}$ arbitrarily, and then map the remaining vertices of T_n to $X - Y$ arbitrarily. This is possible because $|X| + t = n+m-1-2t+t = n+m-(t+1) \geq n$ and every vertex of $X - Y$ is adjacent to every vertex of $X \cup \{x_1, \ldots, x_t\}$ except itself. Thus, G contains T_n, a contradiction. This completes the proof of Lemma 3. ∎

3. Proofs of the main results

We use the lemmas of the previous sections to prove our main results in separate subsections.

3.1. Proof of Theorem 4

The result is easy to prove for $m = 1$ and in this case follows also from Theorem 1, and it holds for $m = 2$ by Lemma 2, thus we may assume that $m \geq 3$.

We are first going to show that if $n \leq m(m-1)$, then $R(S_n, F_m) \geq 2n$, showing that the lower bound $n \geq m(m-1)+1$ is in some sense best possible. Since K_{2m-1} contains no F_m and its complement contains no S_n, we have $R(S_n, F_m) \geq 2m$, so we only need to consider the case that $n \geq m+1$. There exist positive integers p, q such that $n = pm + q$ and $1 \leq q \leq m$. Let $H = pS_m \cup S_q$ if $q \neq 1$, and $H = (p-1)S_m \cup S_{m-1} \cup S_2$ if $q = 1$. Since $n \leq m(m-1)$, then $p \leq m-2$. It is easy to check that H is a graph of order n with $\delta(H) \geq 1$, and that H contains neither S_{m+1} nor mK_2. Let $H' = K_{n-1} \cup \overline{H}$. Then H' contains no S_n and $\overline{H'}$ contains no F_m. Thus, if $n \leq m(m-1)$, then $R(S_n, F_m) \geq 2n$.

It remains to show that $R(S_n, F_m) = 2n - 1$ for $n \geq \max\{m(m-1)+1, 6(m-1)\}$ and $m \geq 3$. First we note that since $2K_{n-1}$ contains no S_n and its complement contains no F_m, we conclude that $R(S_n, F_m) \geq 2n-1$. To prove $R(S_n, F_m) \leq 2n-1$, let G be a graph of order $2n-1$ and suppose to the contrary that G contains no F_m and $\overline{G}$ contains no S_n. Then $\Delta(\overline{G}) \leq n-2$ and $\delta(G) \geq n$. For any vertex u of $V(G)$, let $M_u \subseteq E(G)$ be a maximum matching in $G[N(u)]$ and $X_u = N(u) - V(M_u)$. Then, obviously $G[X_u]$ contains no edges, and $|M_u| \leq m-1$; otherwise $G[N[u]]$ contains an F_m, a contradiction. Moreover, by the maximality of M_u, for $xy \in M_u$, if $d_{X_u}(x) \geq 2$, then $d_{X_u}(y) = 0$; and if $d_{X_u}(x) = d_{X_u}(y) = 1$, then x and y are adjacent to the same vertex in X_u. Let $Y_u \subseteq V(M_u)$ be the set of vertices that have at least two neighbors in X_u, and let $Z_u = N(u) - Y_u$. It is obvious that $|Y_u| \leq m-1$ and $|Z_u| \geq n-m+1$.

Since $X_u \subseteq Z_u$ and $|X_u| \geq n - 2(m-1) \geq m$, there exists a vertex $v \in X_u$ with $d_{Z_u}(v) = 0$. We define M_v, X_v, Y_v, Z_v in a completely analogous way. Since $d_{Z_u}(v) = 0$ and $Z_v \subseteq N(v)$, we get that $Z_u \cap Z_v = \emptyset$. Hence, $X_u \cap X_v = \emptyset$. We first prove the following two claims.

Claim 1. *Let $V_1 = \{w \mid |X_w \cap X_u| \geq |X_u| - 2m+3$ and $X_w \cap X_v = \emptyset\}$, $V_2 = \{w \mid |X_w \cap X_v| \geq |X_v| - 2m+3$ and $X_w \cap X_u = \emptyset\}$. Then for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$. Moreover, $Z_v \subseteq V_1$, $Z_u \subseteq V_2$.*

Proof. For any vertex w of $V(G)$, if $X_w \cap X_u = \emptyset$ and $X_w \cap X_v = \emptyset$, then $2n-1 \geq |X_u| + |X_v| + |X_w| \geq 3(n-2(m-1))$, and hence $n \leq 6(m-1)-1$, a contradiction. Thus, either $X_w \cap X_u \neq \emptyset$ or $X_w \cap X_v \neq \emptyset$. If $X_w \cap X_u \neq \emptyset$, since both $G[X_w]$ and $G[X_u]$ are edgeless graphs, then for any vertex z in $X_w \cap X_u$, we have $d(z) \geq |X_w| + |X_u| - |X_w \cap X_u| - 1$ in $\overline{G}$. Since $d(z) \leq \Delta(\overline{G}) \leq n-2$, we obtain $|X_w \cap X_u| \geq |X_w| + |X_u| - 1 - (n-2)$. Hence, $|X_w \cap X_u| \geq |X_u| - 2m + 3$ and $|X_w \cap X_u| \geq |X_w| - 2m+3$. For the same reason, if $X_w \cap X_v \neq \emptyset$, then $|X_w \cap X_v| \geq |X_v| - 2m+3$ and $|X_w \cap X_v| \geq |X_w| - 2m+3$. If both $X_w \cap X_u \neq \emptyset$ and $X_w \cap X_v \neq \emptyset$, then $|X_w| \geq |X_w \cap X_u| + |X_w \cap X_v| \geq 2(|X_w| - 2m+3)$, and hence $|X_w| \leq 4m-6$, which contradicts $|X_w| \geq n - 2(m-1) \geq 4m-4$. Therefore, for any vertex w of $V(G)$, either $w \in V_1$ or $w \in V_2$.

Any vertex w of Z_v has at most one adjacent vertex in X_v, hence $w \in V_1$. Thus, $Z_v \subseteq V_1$. By symmetry, $Z_u \subseteq V_2$. ∎

Claim 2. *For any two vertices $w_1, w_2 \in V_1$, $|X_{w_1} \cap X_{w_2}| \geq 2m-1$. For any two vertices $w_3, w_4 \in V_2$, $|X_{w_3} \cap X_{w_4}| \geq 2m-1$.*

Proof. It is sufficient to prove the first statement. For any two vertices $w_1, w_2 \in V_1$, since $|X_{w_i} \cap X_u| \geq |X_u| - 2m+3$ for $i = 1, 2$, we get that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1} \cap X_u| + |X_{w_2} \cap X_u| - |X_u| \geq 2$. Since both $G[X_{w_1}]$ and $G[X_{w_2}]$ are edgeless graphs, for any vertex z in $X_{w_1} \cap X_{w_2}$, we have $d(z) \geq |X_{w_1}| + |X_{w_2}| - |X_{w_1} \cap X_{w_2}| - 1$ in $\overline{G}$. Since $d(z) \leq \Delta(\overline{G}) \leq n-2$, we obtain that $|X_{w_1} \cap X_{w_2}| \geq |X_{w_1}| + |X_{w_2}| - 1 - (n-2)$. Hence, $|X_{w_1} \cap X_{w_2}| \geq 2m-1$. ∎

By Claim 1, every vertex belongs to either V_1 or V_2, but not both. Since $|V(G)| = 2n-1$, we have $|V_1| \ge n$ or $|V_2| \ge n$. Without loss of generality, we may assume that $|V_1| \ge n$. For any vertex z of V_1, if $d_{V_1}(z) \ge m$, we choose m adjacent vertices of z from V_1, denoted by $z_1, \dots, z_m$. By Claim 2, for $1 \le i \le m$, z_i has at least m adjacent vertices in $X_z - \{z_1, \dots, z_m\}$. Thus, we may find a matching of m edges in $G[N(z)]$, which together with z forms an F_m, a contradiction. Therefore, for any vertex z of V_1, we have $d_{V_1}(z) \le m-1$. If $|Z_v| \ge n$, since $X_v \subseteq Z_v$ and $|X_v| \ge n-2(m-1) \ge m$, there exists a vertex of degree 0 in $G[Z_v]$, that is, $\overline{G}[Z_v]$ contains a vertex of degree at least $n-1$, a contradiction. This implies that $|Z_v| \le n-1$. Since $Z_v \subseteq V_1$, we choose a subset of V_1 containing Z_v and any $n - |Z_v|$ vertices of $V_1 - Z_v$. For simplicity, this subset of V_1 is also denoted by V_1 in the sequel. Thus, $|V_1| = n$.

In the remainder, we prove that there exists a vertex z_0 of V_1 such that $d_{V_1}(z_0) = 0$ in G, and then $d_{V_1}(z_0) = n-1$ in $\overline{G}$, which is a contradiction. Since $|Z_v| \ge n-m+1$, we distinguish three cases: $|Z_v| \ge n-m+3$, $|Z_v| = n-m+1$ and $|Z_v| = n-m+2$, separately.

If $|Z_v| \ge n-m+3$, X_v contains at most $m-1$ vertices which are adjacent to $Z_v - X_v$, and every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of X_v. Since $|X_v| \ge n-2(m-1)$, $|V_1 - Z_v| \le m-3$ and $n-2(m-1)-(m-1)-(m-3)(m-1) \ge 1$, so we may find the required z_0 in X_v, that is, with $d_{V_1}(z_0) = 0$ in G.

If $|Z_v| = n-m+1$, then $|Y_v| \ge n - |Z_v| = m-1$, and by the maximality of M_v, $G[Z_v]$ is an edgeless graph. Since every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-1$ and $n-m+1-(m-1)^2 \ge 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G.

If $|Z_v| = n-m+2$, then $|Y_v| \ge n - |Z_v| = m-2$, and by the maximality of M_v, $G[Z_v]$ contains at most one edge of M_v. Let x_1y_1 be the possible edge both in $G[Z_v]$ and M_v, and suppose that $d_{V_1}(x_1) \ge d_{V_1}(y_1)$. If there is at most one vertex in $Z_v - \{x_1, y_1\}$ which is adjacent to x_1 or y_1, then, since every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-2$ and $n-m+2-3-(m-2)(m-1) \ge 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G. If there are at least two vertices in $Z_v - \{x_1, y_1\}$ which are adjacent to x_1 or y_1, then, by the maximality of M_v, they are all adjacent to x_1. Since $d_{Z_v}(x_1) \le m-1$, every vertex of $V_1 - Z_v$ is adjacent to at most $m-1$ vertices of Z_v, $|V_1 - Z_v| = m-2$ and $n-m+2-m-(m-2)(m-1) \ge 1$, so we may find the required z_0 in Z_v, that is, with $d_{V_1}(z_0) = 0$ in G. This completes the proof.

We recall that we have shown that $R(S_n, F_m) \ge 2n$ for $n \le m(m-1)$, so the lower bound $n \ge m(m-1)+1$ is best possible for $m \ge 6$. ■

3.2. Proof of Theorem 5

Recall that we want to prove that $R(T_n, F_m) = 2n-1$ for all integers $n \ge 3m^2 - 2m - 1$. The lower bound $R(T_n, F_m) \ge 2n-1$ is implied by the fact that $2K_{n-1}$ contains no T_n while its complement contains no F_m. Now we prove the upper bound.

We may assume that $m \ge 2$ since the result is easy to prove for $m = 1$ and in this case follows also from Theorem 1. Let G be a graph of order $2n-1$ with $n \ge 3m^2 - 2m - 1$ and $m \ge 2$. Suppose to the contrary that G contains no F_m and its complement contains no T_n. We first claim that $\Delta(G) \le n+m-2$. If not, let v be a vertex with $d(v) = \Delta(G) \ge n+m-1$. Since $n \ge 3m^2 - 2m - 1$ and $m \ge 2$, this implies that $n \ge 4(m-1)$. By Lemma 3, either $G[N(v)]$ contains mK_2, which together with v forms an F_m, or $\overline{G}[N(v)]$ contains a T_n. Therefore, we have $\Delta(G) \le n+m-2$.

Next we prove that Theorem 5 holds when $\Delta(T_n) \ge 13n/24$. Let u be a vertex of largest degree in T_n, let A denote the set of vertices of T_n that are adjacent to u and have degree one in T_n, and let B denote the set of vertices of T_n that are adjacent to u and have degree at least two in T_n. Then, obviously since T_n is a tree, $|V(T_n)| \ge 1 + |A| + 2|B|$ and $\Delta(T_n) = |A| + |B|$. Since $|V(T_n)| = n$ and we assume that $\Delta(T_n) \ge 13n/24$, we obtain that $|A| + n = 1 + 2|A| + 2|B| = 1 + 2\Delta(T_n) \ge 1 + 13n/12$, hence $|A| \ge n/12 + 1$. Then $T_n - A$ is a tree of order at most $11n/12 - 1$. We want to apply Lemma 1 to embed $T_n - A$ in $\overline{G}$ such that u is mapped to the vertex of degree $n-1$ of an S_n. Since $|V(T_n - A)| \le 11n/12 - 1$, it is sufficient to show that $\delta(\overline{G}) \ge 11n/12 - 2$ and that $\overline{G}$ contains an S_n.

Since $\Delta(G) \le n+m-2$, we get that $\delta(\overline{G}) \ge (2n-1) - 1 - (n+m-2) = n-m$. Using $m \ge 2$, it is easy to check that $3m^2 - 2m - 1 \ge 12m - 24$. By the condition of the theorem, $n \ge 3m^2 - 2m - 1 \ge 12m - 24$, so $n/12 \ge m-2$, and hence $n - m \ge 11n/12 - 2$. Furthermore, again using $m \ge 2$, $3m^2 - 2m - 1 \ge \max\{m(m-1)+1, 6(m-1)\}$. By Theorem 4, $\overline{G}$ contains an S_n. By Lemma 1, $T_n - A$ can be embedded in $\overline{G}$ such that u is mapped to the vertex with degree $n-1$ of the S_n. Because u now has at least $n-1$ adjacent vertices in $\overline{G}$, the embedding of $T_n - A$ can easily be extended to T_n in $\overline{G}$. This contradicts the assumption that $\overline{G}$ contains no T_n, completing this case. So, in the remainder of the proof we assume that $\Delta(T_n) < 13n/24$.

By Lemma 1, $\delta(\overline{G}) \le |V(T_n)| - 2 = n-2$; otherwise we can embed T_n in $\overline{G}$. So we obtain that $\Delta(G) \ge n$. Let x be a vertex with $d(x) = \Delta(G) \ge n$, let $M = \{x_1y_1, \dots, x_ty_t\} \subseteq E(G[N(x)])$ be a maximum matching in $G[N(x)]$, and let $U = V(G[N(x)]) - V(M)$. Then $G[U]$ is an edgeless graph, and $t \le m-1$; otherwise $G[N(x)]$ contains mK_2, which together with x forms an F_m, a contradiction. Without loss of generality, suppose that $d_U(x_i) \le d_U(y_i)$ for $1 \le i \le t$, and suppose that k and the order of vertices are chosen such that $d_U(y_i) \le 1$ for $1 \le i \le k$, and $d_U(y_i) \ge 2$ for $k+1 \le i \le t$. (We assume that the degenerate cases that all $d_U(y_i) \le 1$ or all $d_U(y_i) \ge 2$ do not occur, but these can be dealt with similarly.) By the maximality of M, $d_U(x_i) = 0$ for $k+1 \le i \le t$, $d_U(x_i) \le 1$ for $1 \le i \le k$, and if $d_U(x_i) = d_U(y_i) = 1$, then x_i and y_i are adjacent to the same vertex of U. Let Y consist of the set $V(M) - \{y_{k+1}, \dots, y_t\}$ and its adjacent vertex set in U, and let

$X = U - Y$. It is easy to check that $|X| \geq n - 2t - k$, $|Y| \leq t + 2k$ and $|X| + |Y| \geq n - t + k$. Next we prove the following claim.

Claim 3. *Let T' be an arbitrary tree of order $|X| + |Y|$ with $w_1 \in V(T')$, and let w_2 be a vertex of X. Then T' can be embedded in $\overline{G}[X \cup Y]$ such that w_1 is mapped to w_2.*

Proof. Since T' is a bipartite graph, we may assume $V(T') = (X_1, Y_1)$ with $|X_1| \leq |Y_1|$. Since $n \geq 3m^2 - 2m - 1$, $m \geq 2$ and $k \leq t \leq m - 1$, it is not difficult to check that $|Y_1| - 1 \geq \lceil (|X| + |Y|)/2 \rceil - 1 \geq |Y|$. Now we can embed T' in $\overline{G}[X \cup Y]$ through the following procedure. First map w_1 to w_2; then map $|Y|$ vertices of Y_1 to Y arbitrarily. Finally, map the remaining vertices of T' to X arbitrarily. Because in $\overline{G}$ every vertex of X is adjacent to every vertex of $X \cup Y$ except itself, the embedding can succeed. ■

If $|X| + |Y| \geq n$, then by Claim 3, $\overline{G}$ contains T_n, a contradiction. So we may assume $|X| + |Y| \leq n - 1$. Let T' be a largest subtree of T_n that can be embedded in $\overline{G}$. Then T' is a proper subgraph of T_n. This implies there exists a vertex in T', say x', such that x' is adjacent to every vertex of $V(G) - V(T')$ in G. Hence, $d_{G-V(T')}(x') \geq n$.

In $G[N(x')] - (X \cup Y)$, we define M', U', t', k', X', Y' in a completely analogous way as we have defined M, U, t, k, X, Y in $G[N(x)]$. Now we distinguish two cases.

Case 1. In $\overline{G}$, $d_X(z) \geq m/2$ for some $z \in X'$, or $d_{X'}(z) \geq m/2$ for some $z \in X$.

By symmetry, we may assume that $d_X(z) \geq m/2$ for some $z \in X'$ in $\overline{G}$. For $v \in V(T_n)$, let $H_1, \ldots, H_\ell$ be all the components of $T_n - v$ with at most $m - 1$ vertices, and ordered in such a way that $m - 1 \geq |V(H_1)| \geq \cdots \geq |V(H_\ell)|$. We distinguish two subcases.

Subcase 1.1. There exists a vertex v of T_n such that $\sum_{i=1}^{p} |V(H_i)| \geq m - 1$, where $p = \min\{\lceil m/2 \rceil, \ell\}$.

We give an embedding of T_n in $\overline{G}$. First we map v to z. Let v_i be the vertex of H_i adjacent to v in T_n. Since $d_X(z) \geq m/2$ and $p \leq \lceil m/2 \rceil$, we map $v_1, \ldots, v_p$ sequentially to the adjacent vertices of z in $\overline{G}[X]$. Since $|X| \geq n-2t-k$, $n \geq 3m^2-2m-1$ and $k \leq t \leq m-1$, we have $|X| \geq \lceil m/2 \rceil (m-1) \geq \sum_{i=1}^{p} |V(H_i)|$. Since $\overline{G}[X]$ is a complete graph, $H_1, \ldots, H_p$ can be embedded in $\overline{G}[X]$ easily. Since $\sum_{i=1}^{p} |V(H_i)| \geq m-1$, $T_n - \bigcup_{i=1}^{p} V(H_i)$ is a tree of order at most $n-m+1$. Since $|X|+|Y| \geq n-m+1$, we have $|X'|+|Y'| \geq n-m+1$ by symmetry. By Claim 3 and the symmetry of $\overline{G}[X \cup Y]$ and $\overline{G}[X' \cup Y']$, $\overline{G}[X' \cup Y']$ contains $T_n - \bigcup_{i=1}^{p} V(H_i)$ such that v is mapped to z. Therefore, $\overline{G}$ contains T_n, a contradiction.

Subcase 1.2. For any vertex v of T_n, $\sum_{i=1}^{p} |V(H_i)| < m - 1$, where $p = \min\{\lceil m/2 \rceil, \ell\}$.

We first show that we may assume that for any vertex $v \in V(T_n)$, the largest component of $T_n - v$ is of order at least m. If not, each component of $T_n - v$ is of order at most $m - 1$. Since Subcase 1.1 does not occur and each nontrivial component is of order at least two, then the number of nontrivial components is at most $m/2 - 1$, and the total order of the nontrivial components is at most $m - 2$. Thus, the total order of the trivial components is at least $n - m + 1$. This implies that $d(v) \geq n - m + 1$. Using $n \geq 3m^2 - 2m - 1$ and $m \geq 2$, we easily obtain that $d(v) \geq 13n/24$, but we have already shown that Theorem 5 holds when $\Delta(T_n) \geq 13n/24$. Thus, henceforth we may assume that for any vertex $v \in V(T_n)$, the largest component of $T_n - v$ is of order at least m.

Choose a vertex v from T_n such that the order of the largest component of $T_n - v$ is as small as possible. Let H_0 be a largest component of $T_n - v$ with $v_0 \in V(H_0)$ being adjacent to v in T_n. Then we claim that $|V(H_0)| \leq n - m + 1$. Suppose to the contrary that $|V(H_0)| \geq n - m + 2$. By the choice of v, the largest component of $T_n - v_0$ has at least $|V(H_0)| \geq n - m + 2$ vertices, so this is the component of $T_n - v_0$ containing v. In that case, $|V(H_0)| \leq m - 2$, a contradiction to our assumption. Therefore, there exists a vertex v such that $m \leq |V(H_0)| \leq n - m + 1$, where H_0 is the largest component of $T_n - v$.

Let $zz' \in E(\overline{G})$ with $z \in X'$ and $z' \in X$. By symmetry and by Claim 3, we may embed H_0 in $\overline{G}[X \cup Y]$ such that v_0 is mapped to z', and $T_n - V(H_0)$ in $\overline{G}[X' \cup Y']$ such that v is mapped to z. Thus, $\overline{G}$ contains T_n, a contradiction. This completes Case 1.

Case 2. In $\overline{G}$, $d_X(z) < m/2$ for every $z \in X'$, and $d_{X'}(z) < m/2$ for every $z \in X$.

First consider an arbitrary vertex $v \in V(G) - (X \cup Y \cup X' \cup Y')$. Suppose $d_X(v) \geq \lceil 3m/2 \rceil - 1$ and $d_{X'}(v) \geq \lceil 3m/2 \rceil - 1$. Then, since every vertex of $N_X(v)$ has at most $\lceil m/2 \rceil - 1$ adjacent vertices of X' in $\overline{G}$, every vertex of $N_X(v)$ has at least m adjacent vertices of $N_{X'}(v)$ in G. Thus, in that case we may find a matching of m edges in $N_{X \cup X'}(v)$, which together with v forms an F_m, a contradiction. Therefore, for every vertex $v \in V(G) - (X \cup Y \cup X' \cup Y')$, if $d_X(v) \leq \lceil 3m/2 \rceil - 2$, then put $v \in Z$; if this is not the case, then put $v \in Z'$. Now (X, Y, Z, X', Y', Z') is a partition of G. Since $|V(G)| = 2n - 1$, either $|X| + |Y| + |Z| \geq n$, or $|X'| + |Y'| + |Z'| \geq n$. Without loss of generality, assume that $|X| + |Y| + |Z| \geq n$. Let Z'' be a subset of Z with exactly $n - |X| - |Y| \leq t - k$ vertices. Then every vertex of Z'' has at most $\lceil 3m/2 \rceil - 2$ adjacent vertices in X. Since $n \geq 3m^2-2m-1$, $|X|-(\lceil 3m/2 \rceil-2)|Z''| \geq (n-2t-k)-(\lceil 3m/2 \rceil-2)(t-k) \geq n-\lceil 3m/2 \rceil t \geq n-\lceil 3m/2 \rceil (m-1) \geq n/2$. Since T_n is a bipartite graph, we may assume $V(T_n) = (X_2, Y_2)$ and $|X_2| \leq |Y_2|$. Now we can embed T_n in $\overline{G}[X \cup Y \cup Z'']$ through the following procedure. First map $|Y| + |Z''| + |N_X(Z'')|$ vertices of Y_2 to $Y \cup Z'' \cup N_X(Z'')$ arbitrarily; then map the remaining vertices of T_n to $X - N_X(Z'')$ arbitrarily. Because in $\overline{G}$, every vertex of $X - N_X(Z'')$ is adjacent to every vertex of $X \cup Y \cup Z''$ except itself, and $|X - N_X(Z'')| \geq n/2$, the embedding can succeed. Thus, $\overline{G}$ contains T_n, our final contradiction. ■

Acknowledgments

Many thanks to the anonymous referees for their careful comments that improved the presentation of this paper. This research was supported by NSFC under grant numbers 11071115, 11371193 and 11101207, and in part by the Priority Academic Program Development of Jiangsu Higher Education Institutions.

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