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TWISTOR SPACE, MINKOWSKI SPACE AND THE CONFORMAL GROUP

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It is shown that the conformal group of compactified Minkowski space is isomorphic to a group of rays of semilinear transformations of twistor space. The action of the conformal group on twistor space is given by an explicit realisation of this isomorphism. In this way we determine the transformation of twistor space under space inversion and time inversion.

1. Introduction

The theory of twistors, developed by Penrose[1,2,3]) gives an alternative geometry of Minkowski space. It is therefore remarkable that the symmetry group of compactified Minkowski space, the conformal group C, is different from the symmetry group of twistor space, which is SU(2, 2). The relation between these two groups is that SU(2, 2) is 4:1 homomorphic to C_0, the identity-connected component of C.

The aim of this paper is to present an alternative approach to the symmetry transformations of twistor space which answers the following two questions:

i) Why is the homomorphism between the two symmetry groups a 4:1 homomorphism?

ii) What about the components of C which are not connected with the identity?

We will take the view that the symmetry group of the twistor formalism is the group G of transformations of projective twistor space which preserve the orthogonality of projective twistors. It will be shown that G is isomorphic to C. Also it will be shown that G is isomorphic to a group $\underset{\sim}{G}$ of rays of semilinear transformations of twistor space; this will reveal the relationship between C and SU(2, 2).

The transformation of twistor space under the conformal group is given by an explicit realization of the isomorphism between C and $\underset{\sim}{G}$. This explicit realisation will be established; in particular the transformation of twistor space under space inversion and time inversion is determined.

2. The symmetry group of twistor space

Twistor space, which is denoted by $\mathscr{T}$, is equal to $\mathbb{C}^4$ with the Hermitian form

$$\langle L, L' \rangle = \bar{L}_0 L'_2 + \bar{L}_1 L'_3 + \bar{L}_2 L'_0 + \bar{L}_3 L'_1 . \tag{2.1}$$

If

$$\eta = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \tag{2.2}$$

then we can write

$$\langle L, L' \rangle = (L, \eta L') , \tag{2.3}$$

where (,) denotes the usual scalar product on $\mathbb{C}^4$.

Elements of $\mathscr{T}$ are called twistors. The one-dimensional linear subspaces of $\mathscr{T}$ are called projective twistors. The set of all projective twistors is called projective twistor space and is denoted by $\mathscr{P}\mathscr{T}$. If $L \in \mathscr{T}$ and $L \neq 0$ then $\underline{L}$ denotes the projective twistor which contains L.

The projective twistors $\underline{L}_1$ and $\underline{L}_2$ are said to be orthogonal (denoted by $\langle \underline{L}_1 \cdot \underline{L}_2 \rangle = 0$) if and only if $\langle L_1, L_2 \rangle = 0$ for each $L_1 \in \underline{L}_1$ and each $L_2 \in \underline{L}_2$. Let C be the conformal group of compactified Minkowski space. C has the coset decomposition

$$C = C_0 + pC_0 + tC_0 + ptC_0 , \tag{2.4}$$

where C_0 is the normal subgroup of C consisting of the conformal transformations which are continuously connected with the identity transformation, p denotes space inversion and t denotes time inversion. Let G be the group of bijective mappings T of $\mathscr{P}\mathscr{T}$ onto itself which have the property that

$$\langle T\underline{L} \cdot T\underline{L}' \rangle = 0 \Leftrightarrow \langle \underline{L} \cdot \underline{L}' \rangle = 0 \quad \forall \underline{L}, \underline{L}' \in \mathscr{P}\mathscr{T} . \tag{2.5}$$

Since it are the projective twistors, not the twistors themselves, which have a physical meaning, and since the only structure in $\mathscr{P}\mathscr{T}$ is the orthogonality of projective twistors, it is natural to consider G as the symmetry group of the twistor formalism. We will show that C and G are isomorphic.

According to Penrose[1]) there corresponds to each conformal transformation $h \in C$ a transformation $T(h) \in G$; this mapping $T: C \to G$ is a monomorphism. Let U be a semilinear transformation of $\mathscr{T}$ onto itself. The set $\{\lambda U \mid \lambda \in \mathbb{C}, \lambda \neq 0\}$ is called a ray of semilinear transformations and is denoted by $\underline{U}$. Let $\underline{G}$ be the group of rays of semilinear transformations of $\mathscr{T}$ which satisfy

$$\langle UL, UL' \rangle = C\zeta(\langle L, L' \rangle) \tag{2.6}$$

where $C \in \mathbb{R}$ and the mapping $\zeta : \mathbb{C} \to \mathbb{C}$ satisfies $\zeta(\lambda) = \lambda$ for each $\lambda \in \mathbb{C}$ if U is linear and $\zeta(\lambda) = \bar{\lambda}$ for each $\lambda \in \mathbb{C}$ if U is antilinear.

It is shown in [4]) that for each $T \in G$ there exists a unique $\underset{\sim}{U} \in \underset{\sim}{G}$ such that if $U \in \underset{\sim}{U}$ then

$$L \in \underset{\sim}{L} \Rightarrow UL \in T\underset{\sim}{L} \quad \forall L \in \mathscr{T} . \tag{2.7}$$

Moreover, G and $\underset{\sim}{G}$ are isomorphic; the correspondence above is an isomorphism between G and $\underset{\sim}{G}$, which we will denote by Q.

So the mapping $\tau = QT$ from C into $\underset{\sim}{G}$ is a monomorphism. Let $\underset{\sim}{G}_0$ denote the subgroup of $\underset{\sim}{G}$ which consists of the rays of linear transformations of $\mathscr{T}$ which satisfy eq. (2.6) with $C > 0$.

Then we have the coset decomposition

$$\underset{\sim}{G} = \underset{\sim}{G}_0 + \underset{\sim}{A}\underset{\sim}{G}_0 + \underset{\sim}{B}\underset{\sim}{G}_0 + \underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{G}_0 , \tag{2.8}$$

where the coset representatives $\underset{\sim}{A}$ and $\underset{\sim}{B}$ are chosen in such a way that they contain semilinear transformations A and B respectively which satisfy

$$\langle AL, AL' \rangle = -\langle L, L' \rangle , \quad \forall L, L' \in \mathscr{T} , \tag{2.9}$$

$$\langle BL, BL' \rangle = \langle L', L \rangle , \quad \forall L, L' \in \mathscr{T} . \tag{2.10}$$

The restriction τ' of τ to C_0 is, by continuity, a monomorphism from C_0 into $\underset{\sim}{G}$. We will show that τ is an isomorphism by showing that τ' is an isomorphism.

The group of linear transformations U and $\mathscr{T}$ which satisfy eq. (2.6) and have determinant equal to one is SU(2, 2). Each element of $\underset{\sim}{G}_0$ contain 4 elements of SU(2, 2). Let C_4 be the group consisting of $\{I, \mathrm{i}I, -I, -\mathrm{i}I\}$ where I is the identity operator on $\mathscr{T}$ then $\underset{\sim}{G}_0$ and SU(2, 2)/C_4 are isomorphic.

Let $\sigma : \underset{\sim}{G}_0 \to \mathrm{SU}(2,2)/C_4$ denote this isomorphism then $\sigma\tau'$ is a monomorphism from C_0 into SU(2, 2)/C_4. It is however well known that $\sigma\tau'$ is an isomorphism[5]). As a consequence, τ' is an isomorphism from C_0 into $\underset{\sim}{G}_0$ and thus τ is an isomorphism from C onto $\underset{\sim}{G}$. It follows that C, G and $\underset{\sim}{G}$ are isomorphic. As we have seen, SU(2, 2) is obtained as symmetry group of $\mathscr{T}$ by taking $\underset{\sim}{G}_0$ instead of G, i.e. by ignoring the inversions of space and time, and putting the (unnecessary) restriction that from the elements of $\underset{\sim}{G}_0$ only those transformations are taken which have determinant equal to one. This leads to the 4:1 homomorphism between SU(2, 2) and C_0 because $\mathscr{T}$ is a 4-dimensional space.

3. Conformal transformations in twistor space

In this section we will first give the relation between compactified Minkowski space M and twistor space $\mathscr{T}$ and then describe how for each ray of semilinear transformations of $\mathscr{T}$ which satisfy eq. (2.6) the corresponding conformal transformation of M is obtained. This gives us an explicit realisation of the isomorphism of C and $\underset{\sim}{G}$. We will explicitly consider the case of time inversion and space inversion since these results are new; the results for the identity-connected conformal transformations and the inversion for both space and time have been previously obtained in [1]) and [5]). However these results are also given here, both for completeness and for the fact that, due to reasons of notation and conventions, these results take different forms with different authors.

Let the Hermitian 2 × 2-matrices σ^μ ($\mu = 0, 1, 2, 3$) be equal to $2^{-1/2}$ times the unit matrix and the Pauli spin matrices:

$$\sigma^0 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \sigma^1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$\sigma^2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}; \quad \sigma^3 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3.1}$$

The equation

$$x_{ab} = \sigma^{\mu}_{ab} x^{\mu} \tag{3.2}$$

and its inverse

$$x^{\mu} = \overline{\sigma^{\mu}_{ab}} x_{ab} \tag{3.3}$$

give a one-to-one correspondence between the vectors of Minkowski space and the Hermitian 2 × 2-matrices. Here and in the sequel we adopt the convention that there is a summation over repeated indices.

Let L be a twistor, define the 2-vectors ω and π by

$$(L_0, L_1, L_2, L_3) = (\omega_1, \omega_2, \pi_1, \pi_2) \tag{3.4}$$

and consider the equation

$$\omega_a = \mathrm{i} x_{ab} \pi_b \,. \tag{3.5}$$

If $\langle L, L \rangle = 0$ the real solutions x^μ of eqs. (3.2) and (3.5) form a null line in m which is determined uniquely by $\underset{\sim}{L}$, and conversely to every null line in M corresponds via eqs. (3.2) and (3.5) uniquely a projective twistor $\underset{\sim}{L}$ with $\langle \underset{\sim}{L} \cdot \underset{\sim}{L} \rangle = 0$. Twistors with $\pi = 0$ correspond to null lines "at infinity".

If $\langle L, L \rangle \neq 0$ then eqs. (3.2) and (3.5) have no real solutions for x^μ.

The isomorphism τ of C and $\underset{\sim}{G}$ which was introduced in the previous section is obtained in the following way. A semilinear transformation U of $\mathscr{T}$ belonging to a ray $\underset{\sim}{U} \in \underset{\sim}{G}$ transforms null twistors (twistors L which satisfy $\langle L, L \rangle = 0$) into null twistors. So there corresponds via eqs. (3.2) and (3.5) a transformation of null lines in M. This transformation of null lines in M determines the conformal transformation $d \in C$ which corresponds to $\underset{\sim}{U} \in \underset{\sim}{G}$, i.e. $\tau(d) = \underset{\sim}{U}$.

The transformed twistor UL will be denoted by $\tilde{L}$; the transforms of ω and π by $\tilde{\omega}$ and $\tilde{\pi}$, respectively.

Let A be the semilinear transformation of $\mathscr{T}$ given by

$$\tilde{\omega} = -\omega\,, \quad \tilde{\pi} = \pi\,. \tag{3.6}$$

Then $\underset{\sim}{A} \in \underset{\sim}{G}$ and A satisfies eq. (2.9). From eqs. (3.2) and (3.5) it is clear that the corresponding conformal transformation is pt, the transformation of space and time. This result was also obtained in [1]). It follows that the coset ptC_0 corresponds to the coset $\underset{\sim}{A}\underset{\sim}{G}_0$ of $\underset{\sim}{G}$.

Let B be the semilinear transformation of $\mathscr{T}$ given by

$$\tilde{\omega}_a = \epsilon_{ab}\bar{\omega}_b\,, \quad \tilde{\pi}_a = \epsilon_{ab}\bar{\pi}_b\,, \tag{3.7}$$

where the matrix ϵ is given by

$$\epsilon = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{3.8}$$

Then $\underset{\sim}{B} \in \underset{\sim}{G}$ and B satisfies eq. (2.10).

It is easily seen that if x^μ is a real solution of eqs. (3.2) and (3.5) then

$$\tilde{x}^\mu = (\tilde{x}^0, \tilde{x}^1, \tilde{x}^2, \tilde{x}^3) = (-x^0, x^1, x^2, x^3) \tag{3.9}$$

is a real solution of the equations

$$\tilde{x}_{ab} = \sigma^\mu_{ab}\tilde{x}^\mu \tag{3.10}$$

and

$$\tilde{\omega}_a = \mathrm{i}\tilde{x}_{ab}\tilde{\pi}_b\,. \tag{3.11}$$

So the conformal transformation corresponding to $\underset{\sim}{B}$ is the time inversion and the coset tC_0 of C corresponds to the coset $\underset{\sim}{B}\underset{\sim}{G}_0$ of $\underset{\sim}{G}$.

From this results it follows that the semilinear transformation AB of $\mathscr{T}$ given by

$$\tilde{\omega}_a = -\epsilon_{ab}\bar{\omega}_b\,, \quad \tilde{\pi}_a = \epsilon_{ab}\bar{\pi}_b \tag{3.12}$$

corresponds to the space inversion p; the coset pC_0 of C corresponds to the coset $\underset{\sim}{A}\underset{\sim}{B}\underset{\sim}{G}_0$ of $\underset{\sim}{G}$.

A conformal transformation belonging to this coset is the inversion

$$\tilde{x}^\mu = \frac{x^\mu}{x_\nu x^\nu}\,. \tag{3.13}$$

A straightforward calculation shows that it corresponds to the ray of semilinear transformations which contains the transformation

$$\tilde{\omega}_a = \epsilon_{ab}\bar{\pi}_b\,, \quad \tilde{\pi}_a = 2\epsilon_{ab}\bar{\omega}_b\,. \tag{3.14}$$

We conclude this paper by giving the correspondence between C_0 and $\underset{\sim}{G}_0$. The translation

$$\tilde{x}^\mu = x^\mu + a^\mu \quad (a^\mu \text{ real}) \tag{3.15}$$

correspond to

$$\tilde{\omega}_a = \omega_a + \mathrm{i}a_{ab}\pi_b\,, \quad \tilde{\pi}_a = \pi_a\,, \tag{3.16}$$

where

$$a_{ab} = \sigma^\mu_{ab}a^\mu\,. \tag{3.17}$$

The dilations

$$\tilde{x}^\mu = cx^\mu \quad (c > 0) \tag{3.18}$$

correspond to

$$\tilde{\omega}_a = c\omega_a\,, \quad \tilde{\pi}_a = \pi_a\,. \tag{3.19}$$

The accelerations

$$x^\mu = \frac{x^\mu - \frac{1}{2}a^\mu x_\nu x^\nu}{1 - a_\nu x^\nu + \frac{1}{4}a_\nu a^\nu x_\rho x^\rho} \tag{3.20}$$

correspond to

$$\tilde{\omega}_a = \omega_a\,, \quad \tilde{\pi}_a = \pi_a + \mathrm{i}a_{ba}\omega_b\,, \tag{3.21}$$

where a_{ba} is given by eq. (3.17).

Finally, the restricted Lorentz transformation

$$\tilde{x}^\mu = \Lambda^\mu{}_\nu x^\nu \tag{3.22}$$

corresponds to

$$\tilde{\omega}_a = Q_{ab}\omega_b\,, \quad \tilde{\pi}_a = -(\epsilon Q \epsilon)_{ab}\pi_b\,, \tag{3.23}$$

where $Q \in \mathrm{SL}(2, \mathbb{C})$ and is determined (up to a sign) by

$$\tilde{x}_{ab} = Q_{ac}\bar{Q}_{bd}x_{cd}\,. \tag{3.24}$$

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