Botchev, M.A., Proceedings of ICIAM-95, ICIAM 76 (suppl. 1) (1996) pp. 361-362

On Ad-hoc Upwind Implicit Difference Schemes for Parabolic P.D.E. of the Convection-Diffuson Type

Solving numerically initial boundary value problem $\frac{\partial u}{\partial t} + \mathcal{A}u = f$ in the assumption that finite difference analog A of the spatial differential operator $\mathcal{A}$ is representable as an M-matrix we present the stability analysis of so-called **replicatively splitted schemes***, i.e., two-layer schemes where difference operators on both time layers are amounting to the initial operator A and inherit its property (representability as M-matrix).*

For such schemes we give in the general form the stability sufficient condition in the norm $\|x\| = \max_i |x_i|$. In the important special case the condition coincides with Neumann's necessary feature.

The replicative splitting approach may naturally be extended on the ADI schemes, for example, here we also present the convergence sufficient condition to the Peaceman – Rachford's scheme for iterative solution of linear systems with non-symmetric M-matrix.

1. Introduction. FD replicative splitting schemes

Consider the FD (finite difference) scheme

$$\frac{u^{m+1} - u^m}{\tau} + Mu^{m+1} + Nu^m = f^m, \; m \geq 0, \quad \tau > 0, \tag{1}$$

$$M + N = A\,(\simeq \mathcal{A})\,, \tag{2}$$

which is FD analog of the equation $\frac{\partial u}{\partial t} + \mathcal{A}u = f$ with posed suitable initial and boundary conditions ($\mathcal{A}$ is an possibly nonself-adjoint differential operator). Let us agree that in (1) upper index m points out the moment of time $t = m\tau$ (τ is the time step).

We assume everywhere that matrix $A = (a_{i,j})$ satisfy the condition

$$a_{i,i} > 0, \qquad a_{i,j} \leq 0\;(i \neq j)\,, \qquad a_{i,i} \geq -\sum_{j \neq i} a_{i,j}\,, \tag{3}$$

Apparently the most frequently encountered case when assumption (3) holds is employing of the upwind differences (or equivalent finite element technique [1]) to discretize first derivatives in $\mathcal{A}$. Although the upwind difference approach has well-known drawbacks [1], there are a lot of specific cases in which the upwinding is preferable. Monotone FD schemes as usual resulting from upwinding were in detail investigated by many authors (see e.g. [2] and sourses cited in [1]). Note that as one can easily verify assumption (3) is necessary and sufficient condition for A to be (possibly singular) M-matrix (see e.g. [3]).

Definition 1. *In general, the splitting* (2) *is said to be* **replicative**[1] *if M and N* **inherit** *the certain property of A. In this specific case we will call the splitting* (2) **replicative** *if for matrices $M = (m_{i,j})$ and $N = (n_{i,j})$ the assumption* (3) *holds as soon as M (or N) is nonzero. Then also scheme* (1) *is said to be* **replicatively splitted scheme**.

Let $\|u\|$ everywhere represent the maximum vector norm $\max_i |u_i|$, whereas $\|A\| = \max_{\|x\|=1} \|Ax\| = \max_i \sum_j |a_{i,j}|$.

Theorem 1. *Let scheme* (1) *be replicatively splitted scheme with $N \neq 0$ and assume that time-step τ satisfy the constraint*

$$\tau \leq \left[\max_i n_{i,i}\right]^{-1}. \tag{4}$$

Then scheme (1) *is stable, so that*

$$\|u^m\| \leq \|u^0\| + \widehat{C} \max_{0 \leq k \leq m-1} \|f^k\|\,, \qquad \widehat{C} = m\tau\,, \qquad m \geq 1\,, \tag{5}$$

[1] From the genetics term "*replication*" which means creation of self-similiar structure (e.g. in the DNA molecule).

and monotone, i.e.

$$u^0 \geq 0 \quad \wedge \quad f^k \geq 0\,(\,k \leq m\,) \qquad \Longrightarrow \qquad u^m \geq 0 \qquad (\,m \geq 1\,)\,. \tag{6}$$

For the transfer matrix $G = (I + \tau M)^{-1}\,(I - \tau N)$ *of the scheme* (1)

$$\rho(G) \leq \|G\| \leq 1\,, \tag{7}$$

and in addition $\rho(G) < 1$ *if* A *is non-singular. In case when* $N = 0$ *scheme (1) is stable unconditionally with validity of* (5–7).

As one can verify in the case when (1) is transfer equation, i.e. $\mathcal{A} = v_1 u_x + v_2 v_y$ constraint (4) may be reduced applying von Newmann's spectral necessant condition (see, e.g. [3]).

2. Generalizations: ADI schemes based on replicative splittings. Convergence of the Peaceman–Rachford iterative method for solving linear system with an M-matrix

Consider the following ADI-type scheme

$$\begin{cases} \dfrac{u^{m+1/p} - u^m}{\tau_1} + Mu^{m+1/p} + Nu^m = f^m\,, \\ \dfrac{u^{m+1} - u^{m+1/p}}{\tau_2} + Mu^{m+1/p} + Nu^{m+1} = f^{m+1/p}\,, \qquad \tau_1 + \tau_2 = \tau, \quad p = \dfrac{\tau}{\tau_1}\,, \end{cases} \tag{8}$$

where M and N define replicative splitting (2). It is immediate from Theorem 1 that conditions like (5–7) are fulfilled for (8) as soon as $\tau_1 \leq \max_i n_{i,i}$ and $\tau_2 \leq \max_i m_{i,i}$.

To solve iteratively linear system $Au = b$ with non-symmetric M-matrix A consider Peaceman–Rachford iterative method:

$$\begin{cases} (\rho_1 I + M)\; u^{m+1/p} = (\rho_1 I - N)\; u^m + b\,, \\ (\rho_2 I + N)\; u^{m+1} = (\rho_2 I - M)\; u^{m+1/p} + b\,, \end{cases} \tag{9}$$

where M and N define replicative splitting (2).

The following simple result may occur not new. Nevertheless it is not known to us as published, so that it seems appropriate to place it here:

T h e o r e m 2. *The condition*

$$\rho_1 \geq \max_i n_{i,i}\,, \qquad \rho_2 \geq \max_i m_{i,i}\,, \tag{10}$$

is sufficient for method (9) *to converge, i.e. it follows from* (10) *that* $\rho(G) < 1$, $G = (\rho_2 I + N)^{-1}(\rho_2 I - M)(\rho_1 I + M)^{-1}(\rho_1 I - N)$.

Acknowledgements

Author should be grateful to his scientific advisers Prof. I.A. Nicolayev *and* Dr. L.A. Krukier *and also would like to thank* Prof. P.N. Vabishchevich *for helpful comments.*

3. References

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