Relative Length of Long Paths and Cycles in Graphs with Large Degree Sums

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ABSTRACT

For a graph G, $p(G)$ denotes the order of a longest path in G and $c(G)$ the order of a longest cycle. We show that if G is a connected graph on $n \geq 3$ vertices such that $d(u) + d(v) + d(w) \geq n$ for all triples u, v, w of independent vertices, then G satisfies $c(G) \geq p(G) - 1$, or G is in one of six families of exceptional graphs. This generalizes results of Bondy and of Bauer, Morgana, Schmeichel, and Veldman.

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1. INTRODUCTION

We use Bondy and Murty [4] for terminology and notation not defined here and consider only finite undirected graphs with no loops or multiple edges.

For a graph G and an integer k with $1 \leq k \leq \alpha(G)$, define $\sigma_k(G)$ by

$$\sigma_k(G) = \min\{\sum_{v \in S} d(v) \mid S \subseteq V(G) \text{ an independent set, } |S| = k\}.$$

For $k > \alpha(G)$ we set $\sigma_k(G) = k(|V(G)| - \alpha(G))$.

G is called *1-tough* if $|S| \geq \omega(G - S)$ for every subset $S \subseteq V(G)$ with $\omega(G - S) > 1$, where $\omega(H)$ denotes the number of components of a graph H. We use "+" to denote the disjoint union of graphs and $G[S]$ to denote the subgraph of G induced by a nonempty set $S \subseteq V(G)$.

A cycle C of G is called a *dominating cycle* if every edge of G has at least one of its end vertices on C, or, equivalently, if $G - V(C)$ contains no edges. The order of a longest path and a longest cycle in G is denoted by $p(G)$ and $c(G)$, respectively.

There are now several results in graph theory that relate degree sums to the structure of long cycles. Two such results are the following.

Theorem 1 (Bondy [3]). Let G be a 2-connected graph on n vertices such that $\sigma_3(G) \geq n + 2$. Then every longest cycle in G is a dominating cycle.

Theorem 2 (Bauer, Morgana, Schmeichel, and Veldman [1]). Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. Then every longest cycle in G is a dominating cycle.

Our main results were inspired by the following easy observation.

Lemma 3. Let G be a connected graph that satisfies $c(G) \geq p(G) - 1$. Then every longest cycle in G is a dominating cycle.

Proof. Let G be a connected graph such that $c(G) \geq p(G) - 1$ and let C be a longest cycle in G. Suppose $G - C$ contains a component H with $|V(H)| \geq 2$. Now it is easy to construct a path in G that contains all vertices on the cycle and at least 2 vertices of H, hence contradicting $p(G) \leq c(G) + 1$. ∎

Our first result is the next theorem, the proof of which will be given in Section 2.

Theorem 4. Let G be a connected graph on n vertices such that $\sigma_3(G) \geq n$. Then G contains a Hamilton path, or $c(G) \geq p(G) - 1$.

Theorem 4 improves the result in Enomoto, Kaneko, and Tuza [5] that if G is a connected graph on n vertices with $\sigma_3(G) \geq n$, then G contains a Hamilton path, or every longest cycle in G is a dominating cycle.

Theorem 4 is best possible in the sense that the condition $\sigma_3(G) \geq n$ cannot be relaxed, even under a strong connectivity constraint. To see this we construct two classes of graphs. For integers a, b, c with $a \geq b \geq c \geq 2$, define the graph $G_{a,b,c}$ as the join of K_1 and $K_a \cup K_b \cup K_c$. Then $G_{a,b,c}$ is connected and satisfies $\sigma_3(G_{a,b,c}) = a + b + c = |V(G_{a,b,c})| - 1$. Furthermore, $c(G_{a,b,c}) = a + 1$ and $p(G_{a,b,c}) = a + b + 1$, hence $c(G_{a,b,c}) = p(G_{a,b,c}) - b \leq p(G_{a,b,c}) - 2$, but $G_{a,b,c}$ also contains no Hamilton path.

Next define, for an integer $t \geq 1$, the graph H_t as the join of K_t and $(t + 2)K_2$. Then H_t is t-connected and satisfies $\sigma_3(H_t) = 3t + 3 = |V(H_t)| - 1$. Furthermore, a longest path in H_t has order $3t + 2$ and a longest cycle in H_t has order $3t$. So H_t contains no Hamilton path and also $c(H_t) = p(H_t) - 2$.

Now we can state our main result, in which we characterize the connected graphs G on n vertices with $\sigma_3(G) \geq n$ that do not satisfy $c(G) \geq p(G) - 1$.

Theorem 5. Let G be a connected graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. Then G satisfies $c(G) \geq p(G) - 1$, or $G \in \mathcal{F}(n)$.

Here $\mathcal{F}(n)$ is the class of graphs defined below. Theorem 5 is an immediate consequence of the following result, the proof of which will be given in Section 2. Theorem 6 gives some more information on the relation between paths and cycles in the graphs satisfying the hypothesis of Theorem 5.

Theorem 6. Let G be a connected graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$ and suppose $G \notin \mathcal{F}(n)$. Then for every path P in G, there exists a cycle C in G such that $|V(P) - V(C)| \leq 1$.

$\mathcal{F}(n)$ is a class of graphs on n vertices consisting of six subclasses:

$$\mathcal{F}(n) = \mathcal{F}_{1,1}(n) \cup \mathcal{F}_{1,2}(n) \cup \mathcal{F}_{2,1}(n) \cup \mathcal{F}_{2,2}(n) \cup \mathcal{F}_{2,3}(n) \cup \mathcal{F}_{2,4}(n).$$

The subclasses $\mathcal{F}_{1,2}(n), \ldots, \mathcal{F}_{2,4}(n)$ are defined as follows.

- $\mathcal{F}_{1,1}(n)$: $G \in \mathcal{F}_{1,1}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, $V(G) = A_1 \cup A_2$ with $A_1 \cap A_2 = \emptyset$, $G[A_1]$ and $G[A_2]$ are hamiltonian or isomorphic to K_2, and there exists exactly one edge between A_1 and A_2.
- $\mathcal{F}_{1,2}(n)$: $G \in \mathcal{F}_{1,2}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, and $V(G) = A_1 \cup A_2$ with $A_1 \cap A_2 = \{a\}$, $G[A_1]$ and $G[A_2]$ are both hamiltonian or both isomorphic to K_2, and there exists no edge between $A_1 - \{a\}$ and $A_2 - \{a\}$.

$\mathcal{F}_{2,1}(n)$: $G \in \mathcal{F}_{2,1}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, and G is a 2-connected spanning subgraph of the join of K_2 and $K_a + K_b + K_c$, with $a, b, c \geq 2$ $(n = a + b + c + 2)$.

$\mathcal{F}_{2,2}(n)$: $G \in \mathcal{F}_{2,2}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, and G is a 2-connected spanning subgraph of the join of K_3 and $aK_2 + bK_3$, with $a, b \geq 0$ and $a + b = 4$ $(n = 2a + 3b + 3,\ 11 \leq n \leq 15)$.

$\mathcal{F}_{2,3}(n)$: $G \in \mathcal{F}_{2,3}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, and G is a 2-connected spanning subgraph of the join of K_s and $sK_2 + K_3$, with $s \geq 4$ $(n = 3s + 3)$.

$\mathcal{F}_{2,4}(n)$: $G \in \mathcal{F}_{2,4}(n)$ if $|V(G)| = n$, $\sigma_3(G) \geq n$, and G is a 2-connected spanning subgraph of the join of K_s and $(s + 1)K_2$, with $s \geq 4$ $(n = 3s + 2)$.

The graphs in $\mathcal{F}(n)$ are not 1-tough, the graphs in $\mathcal{F}_{1,1}(n) \cup \mathcal{F}_{1,2}(n)$ are not 2-connected, and the graphs in $\mathcal{F}_{2,1}(n) \cup \mathcal{F}_{2,2}(n) \cup \mathcal{F}_{2,3}(n) \cap \mathcal{F}_{2,4}(n)$ are 2-connected but satisfy $\sigma_3(G) \leq n + 1$. These observations show that Theorem 5 implies the following results, which are, by Lemma 3, generalizations of Theorems 1 and 2, respectively.

Corollary 7.

(a) Let G be a 2-connected graph on n vertices such that $\sigma_3(G) \geq n + 2$. Then G satisfies $c(G) \geq p(G) - 1$.

(b) Let G be a 1-tough graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. Then G satisfies $c(G) \geq p(G) - 1$.

Theorem 5 also implies the following improvement of Bauer, Morgana, Schmeichel, and Veldman [1, Lemma 8]. The proof of Corollary 8 will be given in Section 2 too.

Corollary 8. Let G be a connected graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$. Suppose G contains a longest cycle C which is a dominating cycle. Let $A = \bigcup_{v \in V(G) - V(C)} N(v)$. Fix an orientation $\vec{C}$ on C and let A^+ denote the set of vertices immediately following the vertices of A on $\vec{C}$. Then $(V(G) - V(C)) \cup A^+$ is an independent set of vertices.

In Van den Heuvel [6] it is shown that the conclusions from Corollary 8 can be extended in order to obtain a version of the Hopping Lemma from Woodall [7] that uses all vertices outside the cycle. Using these results, in [6] several new lower bounds for the lengths of longest cycles in graphs with large degree sums are proved.

2. PROOFS OF THE RESULTS

First we introduce some additional notation.

If P is a path in a graph G, then we denote by $\vec{P}$ the path P with a given orientation, and by $\overleftarrow{P}$ the same path with reverse orientation. If $u, v \in V(P)$ and u precedes v on $\vec{P}$, then $u\vec{P}v$ denotes the consecutive vertices of P from u to v. The same vertices in reverse order are given by $v\overleftarrow{P}u$. We will consider $u\vec{P}v$ and $v\overleftarrow{P}u$ both as paths and as vertex sets. If $u \in V(P)$, then u^+ denotes the successor of u on $\vec{P}$ and u^- its predecessor. For $U \subseteq V(P)$, $U^+ = \{u^+ \mid u \in U\}$ and $U^- = \{u^- \mid u \in U\}$. Similar notation is used for cycles.

An *extension* of P is a path P' with $V(P) \subseteq V(P')$ and $V(P) \neq V(P')$. P is called *nonextendable* if there exists no extension of P.

First we prove Theorem 4. It is a consequence of the following result, in the same way as Theorem 5 follows from Theorem 6.

Theorem 9. Let G be a connected graph on $n \geq 3$ vertices such that $\sigma_3(G) \geq n$ and let P be a nonextendable path in G. Then P is a Hamilton path, or there exists a cycle C in G such that $|V(P) - V(C)| \leq 1$.

Proof. Let G be a connected graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$. Let $P = x_1\vec{P}x_p$ be a nonextendable path in G. Suppose P is not a Hamilton path and there exists no cycle C in G such that $|V(P) - V(C)| \leq 1$. Since G is connected and $n \geq 3$, we may assume $|V(P)| \geq 3$. Let $y \in V(G) - V(P)$. Since P is nonextendable, we have $N(x_1) \subseteq V(P) - \{x_1\}$ and $N(x_p) \subseteq V(P) - \{x_p\}$. Set

$$A = N(x_1)^-, \quad B = N(x_p)^+, \quad \text{and } D = N(y).$$

If $x \in A \cap D$, then the path $yx\overleftarrow{P}x_1x^+\vec{P}x_p$ is an extension of P, contradicting the assumption. Therefore, we have $A \cap D = \varnothing$ and, similarly, $B \cap D = \varnothing$. If $x \in A \cap B$, then $C = x^-\overleftarrow{P}x_1x^+\vec{P}x_px^-$ is a cycle with $|V(P) - V(C)| = 1$, a contradiction. Thus we have $A \cap B = \varnothing$. Finally, if $x_1x_p \in E(G)$, then $C = x_1\vec{P}x_px_1$ is a cycle with $V(P) = V(C)$, again a contradiction. So $\{x_1, x_p, y\}$ is an independent set and we have, since $y \notin A \cup B \cup D$,

$$\begin{aligned} n \leq \sigma_3(G) &\leq d(x_1) + d(x_p) + d(y) = |A^+| + |B^-| + |D| \\ &= |A| + |B| + |D| = |A \cup B \cup D| \leq n - 1. \end{aligned}$$

This contradiction completes the proof. ∎

The remainder of this section is devoted to the proof of Theorem 6.

Proof of Theorem 6. Let G be a graph on $n \geq 3$ vertices with $\sigma_3(G) \geq n$ and suppose there exists a path $P = x_1\vec{P}x_p$ in G such that there is no cycle C with $|V(P) - V(C)| \leq 1$. If P' is an extension of P and C' is a cycle such that $|V(P') - V(C')| \leq 1$, then also $|V(P) - V(C')| \leq 1$. So, without loss of generality, we may assume that P is nonextendable. By Theorem 9 this means that P is a Hamilton path, hence $p(G) = n$. Since G does not contain a cycle C with $|V(P) - V(C)| \leq 1$, we conclude $c(G) \leq n - 2$.

For a Hamilton path $Q = x_1\vec{Q}x_n = x_1x_2\dots x_n$, define

$$r(Q) = \max\{i \mid x_i \in N(x_1)\}$$

and

$$s(Q) = \min\{j \mid x_j \in N(x_n)\}.$$

Now suppose $P = x_1\vec{P}x_n = x_1x_2\dots x_n$ is chosen such that

(1) $r(P)$ is as large as possible, and
(2) $s(P)$ is as small as possible, subject to (1).

Let $r = r(P)$ and $s = s(P)$. Since $c(G) \leq n - 2$ and $n \geq 3$, we have $x_1x_n \notin E(G)$.

We consider four cases, depending on the relative values of r and s. In each case we obtain a contradiction, or we reach the conclusion $G \in \mathcal{F}(n)$.

Case 1. $r \leq s - 2$. Let $y \in x_r^+\vec{P}x_s^-$. By the definition of r and s we have $x_1y, x_ny \notin E(G)$, hence $\{x_1, x_n, y\}$ is an independent set. Furthermore, $N(x_1) \subseteq x_1^+\vec{P}x_r$ and $N(x_n) \subseteq x_s\vec{P}x_n^-$. Define

$$A = N(x_1)^-, \quad B = N(x_n)^+, \quad \text{and } D = N(y).$$

Since $r \leq s - 2$, we have $A \cap B = \emptyset$. If $x \in A \cap D$, then the path $P' = x\overleftarrow{P}x_1x^+\vec{P}x_n$ is a Hamilton path with $r(P') > r$, contradicting the choice of P in (1), while if $x \in B \cap D$, then the path $x_1\vec{P}x^-x_n\overleftarrow{P}x$ is a path that contradicts the choice of P in (2). So we have $A \cap D = B \cap D = \emptyset$. Since $\{x_1, x_n, y\}$ is an independent set and $y \notin A \cup B \cup D$, we reach a contradiction as in the proof of Theorem 9.

Case 2. $r = s - 1$. If x_rx_s is a cut edge, then $G \in \mathcal{F}_{1,1}(n)$. So we can assume there exists an edge y_1y_2 with $y_1 \in x_1\vec{P}x_r^-$ and $y_2 \in x_s\vec{P}x_n$, or $y_1 \in x_1\vec{P}x_r$ and $y_2 \in x_s^+\vec{P}x_n$. First suppose $y_1 \in x_1\vec{P}x_r^-$ and $y_2 \in x_s\vec{P}x_n$. Then $x_1y_1^+ \notin E(G)$, otherwise the path $y_1\overleftarrow{P}x_1y_1^+\vec{P}x_n$ contradicts the choice of P in (1); and $x_ny_1^+ \notin E(G)$, by the definition of s. So $\{x_1, x_n, y_1^+\}$ is an independent set. Set

$$A = N(x_1), \quad D_1 = (N(y_1^+) \cap x_1\vec{P}y_1)^+,$$
$$B = N(x_n), \quad D_2 = (N(y_1^+) \cap y_1^{++}\vec{P}x_n)^-.$$

Since $N(x_1) \subseteq x_1^+\vec{P}x_r$, $N(x_n) \subseteq x_s\vec{P}x_n^-$, $r = s - 1$, and $y_1^+ \in x_1\vec{P}x_r$, we have $A \cap B = \emptyset$ and $B \cap D_1 = \emptyset$. Furthermore, $D_1 \cap D_2 = \{y_1^+\}$. If $x \in A \cap D_1$, then the path $y_1\overleftarrow{P}xx_1\vec{P}x^-y_1^+\vec{P}x_n$ contradicts the choice of P in (1); and if $x \in A \cap D_2$, then $y_1\overleftarrow{P}x_1x\overleftarrow{P}y_1^+x^+\vec{P}x_n$ contradicts the choice of P in (1). This gives $A \cap D_1 = A \cap D_2 = \emptyset$. Finally, if $x \in B \cap D_2$, then the path $P' = x_1\vec{P}xx_n\overleftarrow{P}x^+$ is a Hamilton path with $r(P') = r$ and $s(P') < s$, contradicting the choice of P in (2). So $B \cap D_2 = \emptyset$. This shows $|A| + |B| + |D_1| + |D_2| = |A \cup B \cup D_1 \cup D_2| + 1$. Also, $x_1, x_n \notin A \cup B \cup D_1 \cup D_2$, hence it follows that

$$\begin{aligned} n &\leq \sigma_3(G) \leq d(x_1) + d(x_n) + d(y_1^+) = |A| + |B| + |D_1^-| + |D_2^+| \\ &= |A| + |B| + |D_1| + |D_2| = |A \cup B \cup D_1 \cup D_2| + 1 \\ &\leq n - 2 + 1 = n - 1, \end{aligned}$$

a contradiction.

If $y_1 \in x_1\vec{P}x_r$ and $y_2 \in x_s^+\vec{P}x_n$, then in a similar way a contradiction is reached by considering the vertices x_1, x_n, and y_2^-.

Case 3. $r = s$. If $x_r(=x_s)$ is a cut vertex, then $G \in \mathcal{F}_{1,2}(n)$. So we can assume there exists an edge y_1y_2 with $y_1 \in x_1\vec{P}x_r^-$ and $y_2 \in x_s^+\vec{P}x_n$. Assume y_1, y_2 are chosen such that $|y_1\vec{P}y_2|$ is minimum. Then $x_1y_1^+ \notin E(G)$, otherwise the path $y_1\overleftarrow{P}x_1y_1^+\vec{P}x_n$ contradicts the choice of P in (1), and, similarly, $x_ny_2^- \notin E(G)$. This also shows that $y_1^+ \neq x_r$ and $y_2^- \neq x_r$. Furthermore, $x_ny_1^+ \notin E(G)$, by the definition of s. So $\{x_1, x_n, y_1^+\}$ is an independent set. Define

$$\begin{aligned} A &= N(x_1), \quad D_1 = (N(y_1^+) \cap x_1\vec{P}y_1)^+, \\ B &= N(x_n), \quad D_2 = (N(y_1^+) \cap y_1^{++}\vec{P}x_n)^-. \end{aligned}$$

Since $N(x_1) \subseteq x_1^+\vec{P}x_r$, $N(x_n) \subseteq x_s\vec{P}x_n^-$, and $r = s$, we have $|A \cap B| = 1$. Similar to Case 2 we can prove $D_1 \cap D_2 = \{y_1^+\}$ and $A \cap D_1 = A \cap D_2 = B \cap D_1 = B \cap D_2 = \emptyset$. This shows $|A| + |B| + |D_1| + |D_2| = |A \cup B \cup D_1 \cup D_2| + 2$. Since $y_2^- \in x_r^+\vec{P}x_n$ and $x_ny_2^- \notin E(G)$, we have $y_2^- \notin A \cup B \cup D_1$. Finally, we have $y_1^+y_2 \notin E(G)$, by the choice of y_1 and y_2, so $y_2^- \notin D_2$. Thus we have $x_1, x_n, y_2^- \notin A \cup B \cup D_1 \cup D_2$. If follows that

$$\begin{aligned} n &\leq \sigma_3(G) \leq d(x_1) + d(x_n) + d(y_1^+) = |A| + |B| + |D_1^-| + |D_2^+| \\ &= |A| + |B| + |D_1| + |D_2| = |A \cup B \cup D_1 \cup D_2| + 2 \\ &\leq n - 3 + 2 = n - 1, \end{aligned}$$

the final contradiction in this case.

Case 4. $r \geq s + 1$. In this case we know that G is 2-connected. Let H be the path $x_r^+ \vec{P} x_n$. By the maximality of r we have $N(x_1) \cap V(H) = \emptyset$.

Let Q be the path $x_s^+ \vec{P} x_r x_1 \vec{P} x_s x_n \overleftarrow{P} x_r^+$. Then the path Q satisfies $r(Q) \geq r$, hence $r(Q) = r$, and $r(Q) \geq s(Q) + 1$. And we also have $N(x_s^+) \cap V(H) = \emptyset$. Note that the path Q satisfies (1), but does not necessarily satisfy (2). In the remainder of the proof we often reach a contradiction by the construction of a path that contradicts the choice of P in (1). In these cases the path Q can play a similar role as the path P.

For $x \in V(H)$, let $A_x = N(x) \cap x_1 \vec{P} x_r$, and set $A = \bigcup_{x \in V(H)} A_x$.

Claim 1. If $a \in A$, then $a^+ \notin N(x_1) \cup N(x_s^+)$.

Proof of Claim 1. Suppose $a \in A_x$ with $a^+ \in N(x_1) \cup N(x_s^+)$ for some $x \in V(H)$. First suppose $a \in x_1 \vec{P} x_s$. Assume $a = x_s$. Then $a^+ = x_s^+$ and $a^+ \notin N(x_s^+)$. If $x_s^+ = a^+ \in N(x_1)$, then $x_1 \vec{P} x_s x_n \overleftarrow{P} x_s^+ x_1$ is a Hamilton cycle, a contradiction. Hence we may assume $a \neq x_s$, thus $\{a, a^+\} \subseteq x_1 \vec{P} x_s$. If $a^+ \in N(x_1)$, then the path $a \overleftarrow{P} x_1 a^+ \vec{P} x_n$ contradicts the choice of P in (1). If $a^+ \in N(x_s^+)$, then the path $x_1 \vec{P} a x \vec{P} x_n x_s \overleftarrow{P} a^+ x_s^+ \vec{P} x^-$ contradicts the choice of P in (1). We conclude $a^+ \notin N(x_1) \cup N(x_s^+)$.

If $a \in x_s^+ \vec{P} x_r$, then we reach the same conclusion by considering the path Q instead of P. ∎

Now set

$$R_1 = (N(x_1) \cap x_1 \vec{P} x_s)^-, \qquad S_1 = N(x_s^+) \cap x_1 \vec{P} x_s\,,$$
$$R_2 = N(x_1) \cap x_s^+ \vec{P} x_r, \qquad S_2 = (N(x_s^+) \cap x_s^+ \vec{P} x_r)^-\,,$$

and

$$R = R_1 \cup R_2, \qquad S = S_1 \cup S_2\,.$$

Claim 2. $d(x_1) = |R|$ and $d(x_s^+) = |S|$.

Proof of Claim 2. We have $N(x_1) = R_1^+ \cup R_2$ and $R_1^+ \cap R_2 = \emptyset$. So $d(x_1) = |R_1^+| + |R_2| = |R_1| + |R_2| = |R|$.

The claim $d(x_s^+) = |S|$ is proved in the same way. ∎

Claim 3. $R \cap S = \emptyset$.

Proof of Claim 3. Assume $R \cap S \neq \emptyset$ and let $a \in R \cap S$. First suppose $a \in x_1 \vec{P} x_s$. By definition, $x_s \notin R$, so $\{a, a^+\} \subseteq x_1 \vec{P} x_s$. This means $a \in R_1 \cap S_1$, so $a^+ \in N(x_1)$ and $a \in N(x_s^+)$. Then $x_1 \vec{P} a x_s^+ \vec{P} x_n x_s \overleftarrow{P} a^+ x_1$ is a Hamilton cycle, a contradiction.

If $a \in x_s^+\vec{P}x_r$, then the same conclusion is obtained by considering the path Q. ∎

Claim 4. If $a \in A - \{x_r, x_s\}$, then $\{a, a^+\} \not\subseteq R \cup S$.

Proof of Claim 4. Let $a \in A - \{x_r, x_s\}$ and assume $\{a, a^+\} \subseteq R \cup S$. Again, we only consider the case $a \in x_1\vec{P}x_s$; the case $a \in x_s^+\vec{P}x_r$ is proved similarly by considering the path Q. Since $a \neq x_s$, $\{a, a^+\} \subseteq x_1\vec{P}x_s$, hence $\{a, a^+\} \subseteq R_1 \cup S_1$. By Claim 1, $a \notin R_1$ and $a^+ \notin S_1$. Therefore, we have $a \in S_1$ and $a^+ \in R_1$. This means $a \in N(x_s^+)$ and $a^{++} \in N(x_1) \cap x_1\vec{P}x_s$. We can construct the cycle $x_1\vec{P}ax_s^+\vec{P}x_nx_s\overleftarrow{P}a^{++}x_1$ of length $n - 1$, a contradiction. ∎

Claim 5. $A_{x_n}^+ \cap A = \emptyset$, $A_{x_n}^- \cap A = \emptyset$, $A_{x_r^+}^+ \cap A = \emptyset$, and $A_{x_r^+}^- \cap A = \emptyset$.

Proof of Claim 5. First we assume $A_{x_n}^+ \cap A \neq \emptyset$, say $a^+ \in A_{x_n}^+ \cap A$. Then $a \in N(x_n) \cap x_1\vec{P}x_r$ and $a^+ \in N(x) \cap x_1\vec{P}x_r$ for some $x \in V(H)$. This implies $a^+ \neq x_1$. The path $x_1\vec{P}ax_n\overleftarrow{P}xa^+\vec{P}x^-$ contradicts the choice of P in (1).

Next, assume $A_{x_n}^- \cap A \neq \emptyset$, say $a^- \in A_{x_n}^- \cap A$. Then $a \in N(x_n) \cap x_1\vec{P}x_r$ and $a^- \in N(x) \cap x_1\vec{P}x_r$ for some $x \in V(H)$. This implies $a \neq x_1$, and the path $x_1\vec{P}a^-x\vec{P}x_na\vec{P}x^-$ contradicts the choice of P in (1) again.

We can prove $A_{x_r^+}^+ \cap A = \emptyset$ and $A_{x_r^+}^- \cap A = \emptyset$ in the same way, by considering the path Q instead of P ∎

We call a vertex $x \in V(H)$ *good* if $A_x^+ \cap A_x = \emptyset$. Then x_n is a good vertex since $A_{x_n}^+ \cap A_{x_n} \subseteq A_{x_n}^+ \cap A = \emptyset$.

Suppose $x \in V(H)$ is a good vertex. Let $A_x - \{x_r, x_s\} = \{a_1, \ldots, a_{k(x)}\}$. For $i = 1, \ldots, k(x)$, there exists a vertex $b_i \in \{a_i, a_i^+\}$ such that $b_i \notin R \cup S$, by Claim 4. Since $A_x^+ \cap A_x = \emptyset$, we have $b_i \neq b_j$ if $i \neq j$. Let

$$\ell(x) = |x_1\vec{P}x_r - (R \cup S) - \{b_1, \ldots, b_{k(x)}\}|.$$

Then $|R \cup S| = r - k(x) - \ell(x)$. Let $\delta(x) = |A_x \cap \{x_r, x_s\}|$. Then we have $0 \leq \delta(x) \leq 2$ and $|A_x| = k(x) + \delta(x)$. Therefore,

$$k(x) = |A_x| - \delta(x) = d_G(x) - d_H(x) - \delta(x).$$

Let $\alpha(x) = |V(H)| - 1 - d_H(x) = n - r - 1 - d_H(x)$ $(0 \leq \alpha(x) \leq n - r - 1)$. Then $k(x) = d_G(x) - n + r + 1 + \alpha(x) - \delta(x)$, and so we have

$$|R \cup S| = r - \ell(x) - d_G(x) + n - r - 1 - \alpha(x) + \delta(x)$$
$$= n - d_G(x) - \ell(x) - 1 - \alpha(x) + \delta(x).$$

On the other hand, $|R \cup S| = |R| + |S| = d(x_1) + d(x_s^+)$, by Claims 2 and 3, and this means

$$d(x_1) + d(x_s^+) + d(x) = n - \ell(x) - 1 - \alpha(x) + \delta(x).$$

Since $\sigma_3(G) \geq n$ and $\{x_1, x_s^+, x\}$ is an independent set, we have $n - \ell(x) - 1 - \alpha(x) + \delta(x) \geq n$, or $\delta(x) \geq 1 + \alpha(x) + \ell(x)$. Thus we have proved the following claim.

Claim 6. If $x \in V(H)$ is a good vertex, then $2 \geq \delta(x) \geq 1 + \alpha(x) + \ell(x)$. In particular, $2 \geq \delta(x_n) \geq 1 + \alpha(x_n) + \ell(x_n)$

Claim 7. $A \cap A^+ = \emptyset$.

Proof of Claim 7. Assume $A \cap A^+ \neq \emptyset$, say $a^+ \in A \cap A^+$. First suppose $a \in x_1\vec{P}x_s$. Let $a \in A_x$ and $a^+ \in A_y$ for $x, y \in V(H)$. By Claim 5, $x, y \neq x_n$. We consider two cases.

Case 7.1. $y \in x\vec{P}x_n$. If $x^- \in N(x_n)$, then the existence of the path $x_1\vec{P}ax\vec{P}ya^+\overleftarrow{P}x^-x_n\overleftarrow{P}y^+$ contradicts the choice of P in (1). Therefore, we have $x^- \notin N(x_n)$. If $x \neq x_r^+$, then $\alpha(x_n) \geq 1$. This implies $\alpha(x_n) = 1$, $\ell(x_n) = 0$, and $\delta(x_n) = 2$, by Claim 6. If $x = x_r^+$, then $x_r = x^- \in N(x_n)$, hence we have $\delta(x_n) \leq 1$.. This implies $\delta(x_n) = 1$ and $\alpha(x_n) = \ell(x_n) = 0$, again by Claim 6. Therefore, we have $\ell(x_n) = 0$ in either case. By Claim 5, $\{a, a^+\} \cap (A_{x_n}^+ \cup A_{x_n}) = \emptyset$. Since $\ell(x_n) = 0$, this implies $\{a, a^+\} \subseteq R \cup S$. By Claim 4, this means $a \notin A - \{x_r, x_s\}$, hence $a = x_s$. So $a^+ \in A_{x_n}^+ \cap A$, which contradicts Claim 5.

Case 7.2. $x \in y^+\vec{P}x_n$. If $y^- \in N(x_n)$, then the path $x_1\vec{P}ax \overleftarrow{P}ya^+\vec{P}y^-x_n\overleftarrow{P}x^+$ contradicts the choice of P in (1). Hence we have $y^- \notin N(x_n)$. This implies $\ell(x_n) = 0$, and by the same argument as in Case 7.1 we obtain a contradiction.

If $a \in x_s^+\vec{P}x_r$, we consider the path Q to reach similar contradictions. ∎

By Claim 7, all vertices in H are good.

Claim 8. $|A - A_{x_n} - \{x_r, x_s\}| \leq 1$ and $|A - A_{x_r^+} - \{x_r, x_s\}| \leq 1$.

Proof of Claim 8. Assume $|A - A_{x_n} - \{x_r, x_s\}| \geq 2$, say $a, b \in A - A_{x_n} - \{x_r, x_s\}$, $a \neq b$. By Claim 4 we have $\{a, a^+\} \not\subseteq R \cup S$ and $\{b, b^+\} \not\subseteq R \cup S$. Let $a' \in \{a, a^+\}$ and $b' \in \{b, b^+\}$ such that $\{a', b'\} \cap (R \cup S) = \emptyset$. By Claim 7 we have $a' \neq b'$. By Claim 5 and the assumption that $a \notin A_{x_n}$ we have $\{a, a^+\} \cap (A_{x_n}^+ \cup A_{x_n}) = \emptyset$, and hence $a' \notin A_{x_n}^+ \cup A_{x_n}$. This means $a' \in x_1\vec{P}x_r - (R \cup S) - (A_{x_n}^+ \cup A_{x_n})$. Similarly, we obtain $b' \in x_1\vec{P}x_r - (R \cup S) - (A_{x_n}^+ \cup A_{x_n})$ and hence $\ell(x_n) \geq 2$. By Claim 6 this gives $\delta(x_n) \geq 1 + \alpha(x_n) + \ell(x_n) \geq 3$, a contradiction.

By the same arguments we can show $|A - A_{x_r^+} - \{x_r, x_s\}| \leq 1$. ∎

Claim 9. For any distinct $a, b \in A$, there exists a Hamilton path $P_{ab} = y_1\vec{P}_{ab}y_{n-r}$ in H with $a \in N(y_1)$ and $b \in N(y_{n-r})$.

Proof of Claim 9. We consider three cases, one of which is trivial.

Case 9.1. $\{a, b\} \cap \{x_r, x_s\} = \emptyset$. By Claim 8, $\{a, b\} \cap A_{x_n} \neq \emptyset$ and $\{a, b\} \cap A_{x_r^+} \neq \emptyset$. We may assume $a \in A_{x_n}$. If $b \in A_{x_r^+}$, then $x_n\overleftarrow{P}x_r^+$ is a required Hamilton path in H. So we may assume $\{a, b\} \cap A_{x_r^+} = \{a\}$ and hence $\{a, b\} \cap A_{x_n} = \{a\}$.

Let $b \in A_x$ $(x \neq x_r^+, x_n)$. Since $b \in A - A_{x_n} - \{x_r, x_s\}$, it follows that $\{b, b^+\} \not\subseteq R \cup S$, by Claim 4, and $\{b, b^+\} \cap (A_{x_n}^+ \cup A_{x_n}) = \emptyset$. This implies $\ell(x_n) \geq 1$. Therefore, we have $\delta(x_n) = 2$, $\alpha(x_n) = 0$, and $\ell(x_n) = 1$ by Claim 6. Since $\alpha(x_n) = 0$, we have $x^- \in N(x_n)$. Then $x_r^+\vec{P}x^-x_n\overleftarrow{P}x$ is a required Hamilton path in H.

Case 9.2. $|\{a, b\} \cap \{x_r, x_s\}| = 1$. We may assume $a = x_r$ and $b \neq x_s$. If $b \in A_{x_n}$, then $x_r^+\vec{P}x_n$ is a required Hamilton path. Thus we may assume $b \notin A_{x_n}$. Then we have $\ell(x_n) \geq 1$ and hence $\alpha(x_n) = 0$ and $\delta(x_n) = 2$. Let $b \in A_x - \{x_r, x_s\}$. If $x \neq x_r^+$, then $x^- \in N(x_n)$ since $\alpha(x_n) = 0$, and $x_r^+\vec{P}x^-x_n\overleftarrow{P}x$ is a required Hamilton path. If $x = x_r^+$, then we have $a = x_r \in N(x_n)$ since $\delta(x_n) = 2$, and $x_n\overleftarrow{P}x_r^+$ is a required path.

Case 9.3. $\{a, b\} = \{x_r, x_s\}$. This case is trivial.

This completes the proof of Claim 9. ∎

By Claim 9 we can describe the structure of G as follows: G contains a cycle $C = x_1\vec{P}x_rx_1$ and a path $H = x_r^+\vec{P}x_n$ satisfying $V(H) = G - V(C)$ and $|V(H)| \geq 2$, such that, if we set $A = N_G(H)$, then for any distinct $a, b \in A$, there exists a Hamilton path $P_{ab} = y_1\vec{P}_{ab}y_{n-r}$ in H with $a \in N(y_1)$ and $b \in N(y_{n-r})$. This means that we can use the following result, established implicitly in the proof of Bauer et al. [1, Theorem 5] (already cited in Theorem 2).

Theorem 10. (Bauer et al. [1]). Let G be a 2-connected graph on n vertices with $\sigma_3(G) \geq n$. Suppose G contains a cycle C and a nontrivial component H (i.e., $|V(H)| \geq 2$) of $G - V(C)$. Then one of the following holds.

(a) there is a cycle C' in G such that $|V(C') \cap V(H)| \geq 1$, $|V(C') \cap V(C)| \geq |V(C)| - 1$, and the vertices in $V(C') \cap V(H)$ form a path in C', or

(b) there exists a nonempty subset $S \subseteq V(G)$ such that $G - S$ contains more than $|S|$ nontrivial components.

If we apply Theorem 10 to our situation, then we observe that possibility (a) cannot occur. If there exists a cycle C' with $|V(C') \cap V(C)| \geq |V(C)| - 1$, $|V(C') \cap V(H)| \geq 1$, and the vertices in $V(C') \cap V(H)$ form a path in C', then, by Claim 9, we can form a cycle C_0 with $|V(C_0) \cap V(C)| \geq |V(C)| - 1$ and $V(H) \subseteq V(C_0)$, hence $|V(C_0)| \geq n - 1$, a contradiction. This proves the following claim.

Claim 10. There exists a nonempty subset $S \subseteq V(G)$ such that $G - S$ contains more than $|S|$ nontrivial components.

The 2-connected graphs G on n vertices with $\sigma_3(G) \geq n$ that satisfy the condition in Claim 10 are characterized in Bauer, Schmeichel, and Veldman [2]. Since the proof is rather short, we reproduce it here.

Let $S \subseteq V(G)$ be a nonempty cut set such that $G - S$ contains at least $|S| + 1$ nontrivial components. Let $s = |S|$. Since G is 2-connected, we have $s \geq 2$. Let $H_1, H_2, \ldots, H_{s+1+j}$ $(j \geq 0)$ be the nontrivial components of $G - S$ and let $n_i = |V(H_i)|$ $(i = 1, \ldots, s + 1 + j)$, where we assume $2 \leq n_1 \leq n_2 \leq \cdots \leq n_{s+1+j}$. Let t be the number of trivial components of $G - S$.

By counting the number of neighbors of vertices in the three smallest components of $G - S$ we see

$$n \leq \sigma_3(G) \leq n_1 - 1 + n_2 - 1 + n_3 - 1 - \min\{3, t\} + 3s\,.$$

Since $n = n_1 + n_2 + \ldots + n_{s+1+j} + s + t$, we obtain

$$2(s + j - 2) \leq n_4 + \cdots + n_{s+1+j} \leq 2s - t - \min\{3, t\} - 3\,, \qquad (3)$$

implying that $2j \leq 1 - t - \min\{3, t\}$. We conclude that $j = t = 0$.

If $s = 2$, then we have $G \in \mathcal{F}_{2,1}(n)$.

If $s = 3$, then (3) implies $n_4 \leq 3$, hence $G \in \mathcal{F}_{2,2}(n)$.

Finally, if $s \geq 4$, then (3) implies $n_s = 2$ and $n_{s+1} \leq 3$. If $n_{s+1} = 3$, then $G \in \mathcal{F}_{2,3}(n)$, and if $n_{s+1} = 2$, then $G \in \mathcal{F}_{2,4}(n)$.

This settles Case 4 and hence completes the proof of Theorem 6. ∎

Proof of Corollary 8. Let G, C, and A be as defined in the statement of the corollary. Since none of the graphs in $\mathcal{F}(n)$ contains a dominating cycle, we have $c(G) \geq p(G) - 1$, by Theorem 5. Suppose $a, b \in (V(G) - V(C)) \cup A^+$ such that $ab \in E(G)$. Since C is a dominating cycle, we can assume that $a \in A^+$ and $b \in V(G) - V(C)$, or $a, b \in A^+$.

First suppose $a \in A^+$ and $b \in V(G) - V(C)$. By the definition of A, there exist an $x_a \in V(G) - V(C)$ such that $a^- \in N(x_a)$. If $b = x_a$, then $ba\vec{C}a^-b$ is a cycle of length $|V(C)| + 1$ contradicting the choice of C as a longest cycle. And if $b \neq x_a$, then $ba\vec{C}a^-x_a$ is a path of length $|V(C)| + 2$, contradicting $p(G) \leq c(G) + 1$.

Next suppose $a, b \in A^+$, hence $a^-, b^- \in A$. There exist $x_a, x_b \in V(G) - V(C)$ such that $a^- \in N(x_a)$ and $b^- \in N(x_b)$. Suppose a and b are neighbors on the cycle, say $b^+ = a$. Then $x_a a^- \overleftarrow{C} b^- x_b$ is a cycle of length $|V(C)| + 1$ (if $x_a = x_b$), or a path of length $|V(C)| + 2$ (if $x_a \neq x_b$). In both cases we have a contradiction. So we can assume that a and b are not neighbors on the cycle. Then $x_a a^- \overleftarrow{C} b a \vec{C} b^- x_b$ is a cycle of length $|V(C)| + 1$ (if $x_a = x_b$), or a path of length $|V(C)| + 2$ (if $x_a \neq x_b$). So also in this last case, we always obtain a contradiction.

This shows that there exists no pair $a, b \in (V(G) - V(C)) \cup A^+$ such that $ab \in E(G)$, thus proving Corollary 8. ∎

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