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NOTE ON COMPLETELY MONOTONE DENSITIES

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1. Introduction and summary. In [2] it is proved that mixtures of exponential distributions are infinitely divisible (id). In [3] it is proved that the same holds for the discrete analogue, i.e. for mixtures of geometric distributions. In this note we show that these results imply that a density function $f(x)$ (or distribution $\{p_n\}$ on the integers) is id if the function $f(x)$ (or the sequence $\{p_n\}$) is completely monotone (cm). For the definition and properties of cm functions and sequences we refer to [1].

2. Completely monotone densities. Characteristic functions of the form

$$\int_0^\infty \lambda(\lambda - it)^{-1}\, dF(\lambda),$$

with $F(0+) = 0$, are id (cf. [2]). It follows that densities of the form

$$\int_0^\infty \lambda e^{-\lambda x}\, dF(\lambda), \tag{1}$$

with $F(0+) = 0$ are id.

Clearly, (1) is cm. On the other hand, if $g(x)$ is a cm density function then ([1], p. 416) $g(x)$ can be written as

$$g(x) = \int_0^\infty e^{-\lambda x}\, d\mu(\lambda),$$

with

$$\int_0^\infty g(x)\, dx = \int_0^\infty \lambda^{-1}\, d\mu(\lambda) = \int_{0+}^\infty \lambda^{-1}\, d\mu(\lambda) = 1.$$

Therefore $g(x)$ has the form (1), with $dF(\lambda) = \lambda^{-1}\, d\mu(\lambda)$ and $F(0+) = 0$, i.e. $g(x)$ is a mixture of exponential densities, which is id. We therefore have

THEOREM 1. *All completely monotone densities are infinitely divisible.*

REMARK. We may restrict ourselves to distributions on $[0, \infty)$, as the monotonicity condition implies that the support of the distribution must be of the form $[a, \infty)$ with $a > -\infty$, and a change of location does not affect the infinite divisibility.

The cm criterion is useful, because it is much easier to verify that a function is cm than to prove (directly) that it is a mixture of exponential densities.

Examples of densities satisfying this criterion are the densities proportional to the following functions: $(1 + x)^{-k}$, $x^{-2} \exp(x^{-1})$, $x^{\alpha-1}e^{-x} (0 < \alpha \leqq 1)$ and $\exp(-x^\alpha)$ $(0 < \alpha \leqq 1)$. It further follows that arbitrary mixtures of cm densities are id (see also [1], p. 417).

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3. Completely monotone sequences. In [3] it is proved that characteristic functions of the form

$$\int_0^\infty \lambda(\lambda + 1 - e^{it})\, dF(\lambda),$$

with $F(0+) = 0$, are id. Equivalently, integer valued random variables with $p_n = \text{Prob}\,(X = n)$ given by

$$p_n = \int_0^1 (1 - p)p^n\, dG(p), \tag{2}$$

with $G(1-) = 1$, are id. The p_n defined by (2) form a cm sequence. On the other hand, if $\{p_n\}$ is a cm probability distribution on the non-negative (see Remark following Theorem 1) integers, then by Hausdorff's theorem ([1], p. 223) the p_n are the moments of a finite measure μ on [0, 1] with $\mu[0, 1] = p_0$.

We have

$$p_n = \int_0^1 p^n\, d\mu(p) = \int_0^1 (1 - p)p^n\, dF(p),$$

with $F(1-) = 1$, as

$$\sum_0^\infty p_n = \int_0^1 (1 - p)^{-1}\, d\mu(p) = \int_0^{1-} (1 - p)^{-1}\, d\mu(p) = 1.$$

Therefore $\{p_n\}$ is a mixture of geometric probability distributions and hence we have

THEOREM 2. *All completely monotone lattice distributions are infinitely divisible.*

REMARK. It is easily seen that a completely monotone distribution $\{p_n\}$ on an arbitrary, ordered point set $\{x_n\}$ need not be id.

Formally we may restate Theorems 1 and 2 as follows:

THEOREM 3. *If* $F(x) = \int_{-\infty}^x f(x)\, dx$ *(or* $F(x) = \sum_{n \leq x} p_n$*) where* $f(x)$ *(or* $\{p_n\}$*) is completely monotone, then*

$$-(d/d\tau) \log \int_{-\infty}^\infty e^{-\tau x}\, dF(x)$$

is completely monotone.

See [1], p. 425 seq.

REFERENCES

[1] FELLER, W. (1966). *An Introduction to Probability Theory and Its Applications* **2.** Wiley, New York.

[2] STEUTEL, F. W. (1967). Note on the infinite divisibility of exponential mixtures. *Ann. Math. Statist.* **38** 1303–1305.

[3] STEUTEL, F. W. (1968). A class of infinitely divisible mixtures. *Ann. Math. Statist.* **39** 1153–1157.