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A Generalization of a Result of Häggkvist and Nicoghossian

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Using a variation of the Bondy–Chvátal closure theorem the following result is proved: If G is a 2-connected graph with n vertices and connectivity κ such that $d(x)+d(y)+d(z) \geqslant n+\kappa$ for any triple of independent vertices x, y, z, then G is hamiltonian.

1. TERMINOLOGY

We consider only finite undirected graphs without loops or multiple edges. Our terminology is standard except as indicated. A good reference for any undefined terms is [4]. We begin by introducing some definitions and convenient notation. The set of vertices of a graph G is denoted by $V(G)$ or just V; the set of edges by $E(G)$ or just E. We use κ for the connectivity of a graph, and α to denote the cardinality of a maximum set of independent vertices in a graph. If C is a cycle in a graph G we call C a *dominating cycle* if every edge of G has at least one of its vertices on C. Given an (x, y)-path P of G we denote by $\mathbf{P}$ the path P with an orientation from x to y. If $u, v \in V(P)$, then $u\mathbf{P}v$ denotes the consecutive vertices on P from u to v in the direction specified by $\mathbf{P}$. If $u \neq y$, then u^+ denotes the

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successor of u on **P** and if $u \neq x$, then u^- denotes its predecessor. If $B \subseteq V(P)$, then $B^+ = \{v^+ \mid v \in B - \{y\}\}$ and $B^- = \{v^- \mid v \in B - \{x\}\}$. If C is a cycle in G and $B \subseteq V(C)$ then **C**, B^+, and B^- are analogously defined.

If $v \in V$ then $N(v)$ is the set of all vertices in V adjacent to v and $\mathrm{d}(v) = |N(v)|$. We let $\sigma_1 = \min\{d(v) \mid v \in V\}$ and more generally,

$$\sigma_k = \begin{cases} \min\{\sum_{i=1}^{k} d(v_i) \mid \{v_1, ..., v_k\} \text{ is an independent set} \\ \qquad \text{of vertices in } G\} \quad \text{if} \quad \alpha \geqslant k \\ \infty \quad \text{if} \quad \alpha < k. \end{cases}$$

2. Results

Our main purpose is to prove the following theorem.

THEOREM 1. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geqslant n + \kappa$. Then G is hamiltonian.*

Theorem 1 generalizes a result of Häggkvist and Nicoghossian [7].

THEOREM 2. *Let G be a 2-connected graph on n vertices with $\sigma_1 \geqslant \frac{1}{3}(n + \kappa)$. Then G is hamiltonian.*

An infinite class of graphs, similar to the class described in [7], shows that Theorem 1 is best possible. For $2 \leqslant \kappa < \frac{1}{2}n - 1$ let $m = \lfloor \frac{1}{3}(n + \kappa + 1) \rfloor$. Consider the graph G obtained from $K_{m,m}$ by joining each of κ vertices in one color class to every vertex in a complete graph on $n - 2m$ vertices. Then $\sigma_3 = n + \kappa - 1$ and G is not hamiltonian.

The proof of Theorem 2 above given in [7] relies on a theorem of Nash-Williams [8].

THEOREM 3. *Let G be a 2-connected graph on n vertices with $\sigma_1 \geqslant \max\{\frac{1}{3}(n + 2), \alpha\}$. Then G is hamiltonian.*

Our proof of Theorem 1 relies on the following generalization of Theorem 3.

THEOREM 4. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geqslant \max\{n + 2, 3\alpha\}$. Then G is hamiltonian.*

Theorem 4 is an immediate consequence of the following recent result [1].

THEOREM 5. *Let G be a 2-connected graph on n vertices with $\sigma_3 \geqslant s \geqslant n + 2$. Then G contains a cycle of length at least* $\min\{n, n + \frac{1}{3}s - \alpha\}$.

Our proof of Theorem 1 is facilitated by two lemmas. One lemma is simply a weaker version of our main result.

LEMMA 6. *Let G be a 2-connected graph on n vertices with* $\sigma_3 \geqslant n + \sigma_1$. *Then G is hamiltonian.*

The second lemma, Lemma 8, is a variation of the Bondy–Chvátal closure theorem below [3].

THEOREM 7. *Let G be a graph on n vertices with nonadjacent vertices u and v. If* $d(u) + d(v) \geqslant n$, *then G is hamiltonian if and only if* $G + uv$ *is hamiltonian.*

LEMMA 8. *Let G be a graph on n vertices and S a vertex cut of G. Suppose that some component of* $G - S$ *is complete and has vertex set A. If u and v are nonadjacent vertices in* $V - (S \cup A)$ *such that* $d(u) + d(v) \geqslant n - |A| + 1$, *then G is hamiltonian if and only if* $G + uv$ *is hamiltonian.*

Our proof of Theorem 1 also requires a number of known results, the first of which is due to Dirac [6].

THEOREM 9. *Let G be a graph on* $n \geqslant 3$ *vertices with* $\sigma_1 \geqslant \frac{1}{2} n$. *Then G is hamiltonian.*

The next two theorems are due to Chvátal and Erdös [5] and Bondy [2], respectively.

THEOREM 10. *Let G be a graph of order at least* 3 *with* $\alpha \leqslant \kappa$. *Then G is hamiltonian.*

THEOREM 11. *Let G be a 2-connected graph on n vertices with* $\sigma_3 \geqslant n + 2$. *Then every longest cycle in G is a dominating cycle.*

3. PROOFS

Proof of Lemma 6. The proof is by contradiction. Suppose there exists a non-hamiltonian 2-connected graph on n vertices for which $\sigma_3 \geqslant n + \sigma_1$. Let G be such a graph having the maximum number of edges. Let $u \in V$ have minimum degree, $S = N(u)$, and $T = V - (N(u) \cup \{u\})$. We first note that $T \neq \emptyset$, otherwise G would be complete. Furthermore T induces a complete subgraph of G. For if $v_1, v_2 \in T$ and $v_1 v_2 \notin E$ then $d(u) + d(v_1) + d(v_2) \geqslant n + \sigma_1$ and thus $d(v_1) + d(v_2) \geqslant n$. Hence by Theorem 7 $G + v_1 v_2$ is not hamiltonian, contradicting the maximality of G.

We next observe that $\alpha > \sigma_1$, otherwise G is hamiltonian by Theorem 4. Let R be any maximum independent set of vertices of G. Since $\alpha \geqslant \sigma_1 + 1 \geqslant 3$, the vertex u cannot be in R. It follows that $R = S \cup \{w\}$ for some $w \in T$ and $\alpha = \sigma_1 + 1$. Suppose $z_1, z_2, z_3 \in R$, where $d(z_1) = \sigma_1$. Then $d(z_1) + d(z_2) + d(z_3) \geqslant n + \sigma_1$ and thus $d(z_2) + d(z_3) \geqslant n$. Hence by Theorem 7, $G + z_2 z_3$ is not hamiltonian, again contradicting the maximality of G. Thus we conclude that

$$\text{no vertex of a maximum independent set has degree } \sigma_1. \qquad (*)$$

Now let C be a longest cycle in G. By Theorem 11, C is a dominating cycle. Let $x \in V - V(C)$ and $A = N(x)$. Then $R' = A^+ \cup \{x\}$ is an independent set with $|R'| = d(x) + 1 \geqslant \sigma_1 + 1$. Since $\alpha = \sigma_1 + 1$, it follows that R' is a maximum independent set and $d(x) = \sigma_1$, contradicting $(*)$. ▮

Proof of Lemma 8. If G is hamiltonian then clearly so is $G + uv$. Suppose $G + uv$ is hamiltonian and G is not hamiltonian. Then G contains a hamiltonian path $P = x_1 x_2 \cdots x_n$, where $x_1 = u$ and $x_n = v$. Let $\mathbf{P}$ be the path P with an orientation from u to v, $U = \{x \mid ux^+ \in E\}$, and $W = \{x \mid vx \in E\}$. Clearly $U \cap W = \emptyset$ or else G is hamiltonian. Set $Z_1 = \{x \mid x^- \in A \cap U\}$ and $Z_2 = (A^- \cap S) - W$. Since neither u nor v is adjacent to any vertex in A, we have $(A - Z_1^-) \cap (U \cup W) = \emptyset$. Also, $Z_2 \cap (U \cup W) = \emptyset$. Noting that $v \notin U \cup W$ while $A - Z_1^- \subseteq A$ and $Z_2 \subseteq S$ are disjoint, we obtain

$$\begin{aligned} n - |A| + 1 &\leqslant d(u) + d(v) = |U| + |W| = |U \cup W| \\ &\leqslant n - 1 - |A - Z_1^-| - |Z_2| = n - 1 - |A| + |Z_1| - |Z_2|. \end{aligned}$$

Hence $|Z_2| \leqslant |Z_1| - 2$. Consider subsets of $V(P)$ of the form $T = x_i \mathbf{P} x_j$ where $T \subseteq A$ and $x_i^-, x_j^+ \notin A$. Since S separates the induced subgraph $G[A]$ from the rest of G, $x_i^-, x_j^+ \in S$, $x_j \in Z_1^-$ if and only if $ux_j^+ \in E$, and $x_i^- \in Z_2$ if and only if $vx_i^- \notin E$. Since $|Z_2| \leqslant |Z_1| - 2$, there exist two such disjoint subsets $x_i \mathbf{P} x_j$ and $x_k \mathbf{P} x_r$ where $j < k$ and $ux_j^+, ux_r^+, vx_i^-, vx_k^- \in E$. Recalling that $G[A]$ is complete we conclude that G contains the hamiltonian cycle $C = ux_j^+ \cdots x_k^- vx_{n-1} \cdots x_k x_j \cdots u$, a contradiction. ▮

Proof of Theorem 1. The proof is by contradiction. Suppose there exists a non-hamiltonian 2-connected graph on n vertices with $\sigma_3 \geqslant n + \kappa$. Let G be such a graph having the maximum number of edges. By Theorem 4 we may assume

$$3\alpha > \sigma_3 \geqslant n + \kappa. \qquad (1)$$

We first show that $\alpha \geqslant \kappa + 2$. By Theorem 10 we may assume $\alpha \geqslant \kappa + 1$. Suppose $\alpha = \kappa + 1$. Then $\kappa = \alpha - 1 > \frac{1}{3}(n + \kappa) - 1$, implying that $\kappa \geqslant \frac{1}{2} n - 1$.

If $\sigma_1 \geqslant \frac{1}{2}n$ then G is hamiltonian by Theorem 9 and if $\sigma_1 = \kappa$ then G is hamiltonian by Lemma 6. Otherwise we conclude $\frac{1}{2}n > \sigma_1 > \kappa \geqslant \frac{1}{2}n - 1$. This contradiction shows that $\alpha \geqslant \kappa + 2$.

Let T be an independent set of vertices of size α, let S be a vertex cut of size κ, and let $G_1, G_2, ..., G_s$ be the components of $G - S$. Choose $w \in T$ such that $d(w) \leqslant d(x)$ for all $x \in T$. Consider any pair v_1, v_2 of distinct vertices in $T - \{w\}$. Then

$$d(v_1) + d(v_2) \geqslant \tfrac{2}{3}(d(v_1) + d(v_2) + d(w)) \geqslant \tfrac{2}{3}(n + \kappa).$$

Since $|N(v_1) \cap N(v_2)| = d(v_1) + d(v_2) - |N(v_1) \cup N(v_2)|$ and $|N(v_1) \cup N(v_2)| \leqslant n - \alpha$ we obtain

$$|N(v_1) \cap N(v_2)| \geqslant \tfrac{2}{3}(n + \kappa) - (n - \alpha) > \tfrac{2}{3}(n + \kappa) - n + \tfrac{1}{3}(n + \kappa) = \kappa.$$

It follows that two vertices in $T - \{w\}$ cannot be in different components of $G - S$. Assume without loss of generality that $T - \{w\} \subseteq S \cup V(G_1)$. From the fact that $\alpha \geqslant \kappa + 2$ we deduce that $|T \cap V(G_1)| \geqslant 1$. In fact, as we show next, $|T \cap V(G_1)| \geqslant 2$. Let $A = V - (S \cup V(G_1))$, $n_1 = |V(G_1)|$, and $n_2 = |A|$, so that $n = n_1 + n_2 + \kappa$.

Suppose $|T \cap V(G_1)| = 1$, say that $T \cap V(G_1) = \{u\}$. Then $\alpha = \kappa + 2$, $T = S \cup \{u, w\}$, and $w \in A$. Hence $N(u) \subseteq V(G_1)$. Recalling that by Lemma 6 we may assume $\sigma_1 > \kappa$ we have

$$n_1 \geqslant d(u) + 1 \geqslant \sigma_1 + 1 \geqslant \kappa + 2.$$

Similarly $n_2 \geqslant \kappa + 2$. Thus, using (1),

$$n = n_1 + n_2 + \kappa \geqslant 3\kappa + 4 = 3\alpha - 2 > n + \kappa - 2 \geqslant n.$$

This contradiction shows that $|T \cap V(G_1)| \geqslant 2$. Let u_1 and u_2 be two distinct vertices in $T \cap V(G_1)$.

We now prove that $G[A]$ is complete. Suppose $x_1, x_2 \in A$ and $x_1 x_2 \notin E$. Then

$$\begin{aligned} n + \kappa &\leqslant d(u_1) + d(x_1) + d(x_2) \\ &\leqslant |V(G_1) \cup S| - |T - \{w\}| + 2(n_2 + \kappa - 2) \\ &= n_1 + \kappa - (\alpha - 1) + 2(n_2 + \kappa - 2). \end{aligned}$$

Since $n = n_1 + n_2 + \kappa$ we conclude that

$$n_2 \geqslant \alpha - \kappa + 3. \tag{2}$$

On the other hand,

$$n+\kappa \leqslant d(u_1)+d(u_2)+d(x_1) \leqslant 2(n_1+\kappa-\alpha+1)+n_2+\kappa-1,$$

implying that

$$n_1 \geqslant 2\alpha-\kappa-1. \tag{3}$$

Adding (2) and (3) we obtain $n_1+n_2 \geqslant 3\alpha-2\kappa+2$. Using (1) we reach the contradiction $n_1+n_2>n-\kappa+2$ and hence $G[A]$ is complete.

Now let v be an arbitrary vertex of A. Then

$$d(u_1)+d(u_2) \geqslant n+\kappa-d(v) \geqslant n+\kappa-(n_2-1+\kappa)=n-n_2+1.$$

Hence by Lemma 8, $G+u_1u_2$ is non-hamiltonian. This contradiction with the maximality of G completes the proof. ∎

4. Final Remarks

We close with a few remarks concerning Lemma 8. To begin we note that it is best possible in the sense that the quantity $n-|A|+1$ cannot be reduced. To see this let G_m be the graph formed from $K_{m,m}$ by joining each of m vertices in one color class to a complete graph on m vertices with vertex set A, where $m \geqslant 2$. The graph G_m contains nonadjacent vertices u and v such that $d(u)=d(v)=m$. However, $d(u)+d(v)=2m=n-|A|$, G_m+uv is hamiltonian, and G_m is not hamiltonian.

It is possible to establish yet another variation of the closure theorem (Theorem 7). Noting that $|Z_1| \leqslant |S|-1$ in the proof of Lemma 8 (since $Z_1 \subseteq S$ and the first vertex of S along P is not in Z_1), we may easily prove the next theorem.

Theorem 12. *Let G be a graph on n vertices and S a vertex cut of G. Let A be the union of the vertex sets of a number of components of $G-S$. If u and v are nonadjacent vertices of $V-(S\cup A)$ such that $d(u)+d(v) \geqslant \min\{n, n-|A|+|S|-1\}$, then G is hamiltonian if and only if $G+uv$ is hamiltonian.*

Theorem 12 is also best possible. For $s, t \geqslant 1$ let $G_{s,t}$ be the join $K_s \vee (s+1)K_t$, S the vertex set of K_s, and A the union of the vertex sets of $s-1$ copies of K_t. If u and v are nonadjacent vertices of $G_{s,t}-(S\cup A)$ then $d(u)=d(v)=s+t-1$. But now $d(u)+d(v)=2s+2t-2=n-|A|+|S|-2$, $G_{s,t}+uv$ is hamiltonian, and $G_{s,t}$ is not hamiltonian.

Let $\mathscr{H}$ be the class of hamiltonian graphs which do not have a complete closure (in the sense of Theorem 7). The following class of graphs belongs to $\mathscr{H}$. For $2 \leqslant p \leqslant q$ and $q \equiv 1 \pmod 4$ let $H_{p,q} = K_p \vee (H_1 \cup H_2)$, where H_1 and H_2 are disjoint $(\frac{1}{2}(q-1))$-regular graphs of order q. It is possible, however, to deduce that the graphs $H_{p,q}$ are hamiltonian by first applying Theorem 12 and then using Theorem 7. Let $S = V(K_p)$ and $A = V(H_2)$. If u and v are any nonadjacent vertices of H_1 then $d(u) + d(v) = 2p + q - 1 = n - |A| + |S| - 1$. Hence we may assume H_1 is complete and similarly we may assume H_2 is complete. Now the closure of the resulting graph is complete.

The above example is an indication that Theorem 12 may find future applications in hamiltonian graph theory.

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Note Added in Proof. We recently observed that our Lemma 8 is less general than Corollary 1 in [A. Ainouche and N. Christofides, Strong sufficient conditions for the existence of hamiltonian circuits in undirected graphs, *J. Combin. Theory Ser. B* **31** (1981), 339–343]. The latter result does not imply our Theorem 12.

References

1. D. Bauer, A. Morgana, E. Schmeichel, and H. J. Veldman, Long cycles in graphs with large degree sums, *Discrete Math.*, in press.
2. J. A. Bondy, "Longest Paths and Cycles in Graphs of High Degree," Research Report CORR 80-16, University of Waterloo, Waterloo, Ontario, 1980.
3. J. A. Bondy and V. Chvátal, A method in graph theory, *Discrete Math.* **15** (1976), 111–135.
4. J. A. Bondy and U. S. R. Murty, "Graph Theory with Applications," MacMillan & Co., London and Amer. Elsevier, New York, 1976.
5. V. Chvátal and P. Erdös, A note on hamiltonian circuits, *Discrete Math.* **2** (1972), 111–113.
6. G. A. Dirac, Some theorems on abstract graphs, *Proc. London Math. Soc.* **2** (1952), 69–81.
7. R. Häggkvist and G. G. Nicoghossian, A remark on hamiltonian cycles, *J. Combin. Theory Ser. B* **30** (1981), 118–120.
8. C. St. J. A. Nash-Williams, Edge-disjoint hamiltonian circuits in graphs with vertices of large valency, *in* "Studies in Pure Mathematics," pp. 157–183, Academic Press, San Diego/London, 1971.