

## CONTRACTION THEOREMS IN HAMILTONIAN GRAPH THEORY

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We prove that a  $k$ -connected graph ( $k \geq 2$ ) is Hamiltonian if it is not contractible to one of a specified collection of graphs of order  $2k + 1$ . The theorem generalizes a previous result of the authors. The proof partly parallels that of the following, less general, result of Chvátal and Erdős: A  $k$ -connected graph containing no independent set of more than  $k$  points ( $k \geq 2$ ) is Hamiltonian (\*). Also stated in terms of contractibility are sufficient conditions for graphs to be traceable, Hamiltonian connected or 1-Hamiltonian, respectively. Conditions analogous to (\*) guaranteeing the same properties were found by Chvátal and Erdős and by Lesniak. For traceable and 1-Hamiltonian graphs the contraction theorems sharpen the corresponding analogues of (\*), while equivalence is conjectured for Hamiltonian connected graphs.

### 0. Terminology

Our basic terminology will be that of Harary's book [5]. We shall need a number of additional definitions.

(1) If a graph  $H$  is a subgraph of a graph  $G$ , then  $G - H$  is defined to be the subgraph of  $G$  induced by the points of  $V(G) - V(H)$ .

(2) By  $G_1 \leq G_2$  we denote the fact that  $G_1$  is a spanning subgraph of  $G_2$ .

(3)  $M_{2k+1}$  is the graph obtained from a wheel on  $2k$  spokes by deleting a set of  $k$  spokes incident to  $k$  mutually nonadjacent points.

(4) The  $2k + 1$  points of the path  $P_{2k+1}$  can be partitioned into two independent sets  $V_1$  and  $V_2$  with  $|V_1| = k + 1$  and  $|V_2| = k$ . By  $R_{2k+2}$  we denote the graph obtained from  $K_1 + P_{2k+1}$  by deleting all lines that connect  $K_1$  and  $V_1$ . Similarly, the graph  $S_{2k+2}$  is obtained from  $K_1 + P_{2k+1}$  by deleting the lines connecting  $K_1$  and  $V_2$ .

(5)  $T_{2k}$  denotes the graph obtained from  $M_{2k+1}$  by contracting a line incident with a point of degree two.

(6)  $U_{2k}$  is the graph obtained from  $M_{2k-1}$  by joining an additional point to the central point that has degree  $k - 1$  in  $M_{2k-1}$  and to a point of degree two in  $M_{2k-1}$ .

(7) If  $H$  is an oriented path or cycle and  $u$  and  $v$  are consecutive points on  $H$ ,  $u\vec{H}v$  and  $v\vec{H}u$  will denote, respectively, the segment of  $H$  from  $u$  to  $v$  and the reverse segment from  $v$  to  $u$ .

## 1. Hamiltonian graphs

In [6], the authors proved

**Theorem A** [6, Theorem 2]. *Every non-Hamiltonian 2-connected graph can be contracted to  $M_5$  or to  $K_2 + \bar{K}_3$ .*

Adopting the idea of proof of a theorem by Chvátal and Erdős [2] led to the following result, generalizing both the latter theorem (stated as Corollary 1.1) and Theorem A.

**Theorem 1.** *Let  $G$  be a  $k$ -connected graph ( $k \geq 2$ ). If  $G$  is not contractible to a graph  $H \in A_k$ , where  $A_k = \{H \mid M_{2k+1} \leq H \leq K_2 + \bar{K}_{k+1}\}$ , then  $G$  is Hamiltonian.*

**Proof.** By contraposition. Let  $C$  be a cycle of maximal length in a  $k$ -connected non-Hamiltonian graph  $G$  and fix an orientation for  $C$ . Since  $G$  is  $k$ -connected,  $C$  has length at least  $k$  and a point  $u_0 \in V(G - C)$  is joined to at least  $k$  points  $v_1, v_2, \dots, v_k$  of  $C$  (indices according to the order of occurrence on  $C$ ) by paths  $P_1, P_2, \dots, P_k$  that have only  $u_0$  in common. The paths  $P_i$  are regarded to be oriented from  $u_0$  to  $v_i$ . Let  $u_i$  be the point immediately following  $v_i$  on  $C$ ,  $i = 1, 2, \dots, k$ , and let  $K$  denote the subgraph induced by the points of

$$\bigcup_{i=1}^k V(P_i) \cup V(C).$$

Before describing the process of contracting  $G$  to an element of  $A_k$  we prove that  $I = \{u_0, u_1, \dots, u_k\}$  is an independent set of points. Assuming the adjacency of  $u_0$  and  $u_i$  for some  $i \in \{1, 2, \dots, k\}$ , the cycle  $u_0 u_i \bar{C} v_i \bar{P}_i u_0$  contains  $u_0$  and all points of  $C$ , contradicting the maximality of  $C$ .  $u_i u_j$  ( $i, j \in \{1, 2, \dots, k\}$ ,  $i < j$ ) cannot be a line of  $G$  either, since in that case  $u_0 \bar{P}_j v_j \bar{C} u_i \bar{C} v_i \bar{P}_i u_0$  would be a cycle longer than  $C$ . By the same reasoning two points of  $I$  cannot be joined by a path exclusively consisting of points outside  $K$  (except for the endpoints).

Let  $m$  be the number of components of  $G - K$ . If  $m > 0$ , the first step is contracting these components to single points  $w_1, w_2, \dots, w_m$ . Since  $G$  is (at least) 2-connected, each point of  $W = \{w_1, w_2, \dots, w_m\}$  has degree  $\geq 2$  in the resulting graph  $L$ . No point of  $W$  can be adjacent in  $L$  to two points of  $I$ , otherwise in the original graph there would exist a path outside  $K$  joining two points of  $I$ . Thus each point of  $W$  can be contracted to a point of  $K$  not in  $I$ . In the resulting graph  $I$  is still an independent set.

As a second step, all points of the path  $P_i$  except  $u_0$  are contracted to  $v_i$ , using lines of  $P_i$  ( $i = 1, 2, \dots, k$ ). Again, no two points of  $I$  become adjacent.

Finally, using lines of  $C$ , the points of  $C$  lying between  $u_i$  and  $v_{i+1}$  (indices modulo  $k$ ) are contracted to  $v_{i+1}$  for  $i = 1, 2, \dots, k$ . In the resulting graph  $H$ ,  $I$  has

remained an independent set, so that  $H \in K_k - \bar{K}_{k+1}$ . The proof is completed by noting that  $M_{2k+1}$  is a spanning subgraph of  $H$ .  $\square$

Clearly, for  $k = 2$  we have Theorem A.

**Corollary 1.1** [2, Theorem 1]. *If  $G$  is a  $k$ -connected graph ( $k \geq 2$ ) and  $\beta_0(G) \leq k$ , then  $G$  is Hamiltonian.*

**Proof.** Let  $G$  be a  $k$ -connected non-Hamiltonian graph, so that  $G$  is contractible to a graph  $H \in A_k$ . Carrying out an elementary contraction does not increase the point independence number  $\beta_0$ . Thus  $\beta_0(G) \geq \beta_0(H) = k + 1 > k$ .  $\square$

Theorem 1 is also more general than the following result of Goodman and Hedetniemi.

**Corollary 1.2** [4, Theorem 4]. *If a 2-connected graph  $G$  contains no induced subgraph isomorphic to either  $K_{1,3}$  or  $K_{1,3} + x$ , then  $G$  is Hamiltonian.*

**Proof.** If a graph  $G$  is contractible to a graph  $H$  that contains  $K_{1,3}$  or  $K_{1,3} + x$  as an induced subgraph, then  $G$  itself contains  $K_{1,3}$  or  $K_{1,3} + x$  as an induced subgraph. This is easily seen using induction on the number of points of  $G$  [8]. Since any 2-connected non-Hamiltonian graph is contractible to an element of  $A_2$  by Theorem 1, the result follows from the fact that both graphs in  $A_2$  contain an induced  $K_{1,3}$ .  $\square$

As an example, the graph  $K_{4,4} - x$  is 3-connected, not contractible to a graph in  $A_3$  and thus Hamiltonian by Theorem 1. This conclusion can neither be drawn from Theorem A nor from one of the Corollaries 1.1 and 1.2.

No graph in  $A_k$  is 1-tough, i.e. for each  $H \in A_k$  there exists a set  $S \subset V(H)$  such that  $k(H - S) > |S|$ . Demanding 1-toughness, Bigalke and Jung recently proved another generalisation of Corollary 1.1.

**Theorem B** [1, Satz 3]. *Let  $G$  be a 1-tough,  $k$ -connected graph. If  $G$  is not the Petersen graph and  $\beta_0(G) \leq k + 1$ , then  $G$  is Hamiltonian.*

Comparing Theorem 1 and Theorem B leads to the conclusion that no one of them is more general than the other. The graph obtained from  $K_{k+1, k+1} - x$  by addition of a line between a point of degree  $k$  and one of degree  $k + 1$  ( $k \geq 3$ ) is  $k$ -connected and satisfies the conditions of Theorem B; it is however contractible to  $K_{k, k+1} \in A_k$ . On the other hand, the wheel  $W_{2m+1}$  has  $\kappa(W_{2m+1}) = 3$  and  $\beta_0(W_{2m+1}) = m$ , so that, for  $m \geq 5$ ,  $\beta_0(W_{2m+1}) > \kappa(W_{2m+1}) + 1$ .  $W_{2m+1}$  is not contractible to a graph in  $A_3$  and hence Hamiltonian according to Theorem 1.

Only for  $k = 2$  all graphs in  $A_k$  are  $k$ -connected. One might hope for the possibility to sharpen Theorem 1 to the proposition: "Every  $k$ -connected

non-Hamiltonian graph is contractible to a graph in  $\{H \in B_k \mid \kappa(H) = k\}$  ( $= \{H \mid K_{k,k+1} \leq H \leq K_k + \bar{K}_{k+1}\}$ ). Suppose, however, this were true. Then, considering  $k = 3$ , every 3-connected non-Hamiltonian graph would be contractible to a graph  $H \geq K_{3,4}$  and hence, carrying out one additional elementary contraction, to a graph  $M \geq K_{3,3}$ . Since there exist 3-connected non-Hamiltonian planar graphs, this would yield a contradiction with the fact that no planar graph contains a subgraph contractible to  $K_{3,3}$ . This proves that the proposition mentioned above is not true for arbitrary  $k$ .

**2. Traceable graphs**

By applying basically the same line of thought as used in the proof of Theorem 1 contraction theorems analogous to Theorem 1 can be derived concerning other Hamiltonlike properties. Traceability (possession of a Hamiltonian path) is one such property. Henceforth omitting proofs, we have

**Theorem 2.** *Let  $G$  be a  $k$ -connected graph ( $k \geq 1$ ). If  $G$  is not contractible to a graph  $H \in B_k$ , where  $B_k = \{H \mid R_{2k+2} \leq H \leq K_k + \bar{K}_{k+2}\}$ , then  $G$  is traceable.*

Note that  $B_1$  consists of  $K_{1,3}$  only.

**Corollary 2** [2, Theorem 2]. *If  $G$  is a  $k$ -connected graph ( $k \geq 1$ ) and  $\beta_0(G) \leq k + 1$ , then  $G$  is traceable.*

The graph  $G_0$ , depicted in Fig. 1, does not satisfy the condition of Corollary 2:  $3 = \beta_0(G_0) > \kappa(G_0) + 1 = 2$ . However, it is not contractible to  $K_{1,3}$ , hence traceable by Theorem 2.

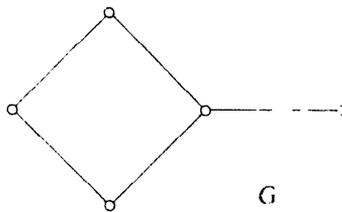


Fig. 1.

**3. Hamiltonian connected graphs**

A graph  $G$  is called *Hamiltonian connected* if every pair of points is connected by a Hamiltonian path. By considering two adjacent points of  $G$  it is obvious that every Hamiltonian connected graph with more than two points is Hamiltonian and consequently 2-connected. If  $G$  has connectivity two and  $G$  is not complete,

then there are two points  $u$  and  $v$  in  $G$  the removal of which results in a disconnected graph. Any path connecting  $u$  and  $v$  can only contain points of one component of  $G - \{u, v\}$ , so  $u$  and  $v$  are not connected by a Hamiltonian path. Hence the only Hamiltonian connected graph of connectivity two is the complete graph  $K_3$ . This is why in deriving a contraction theorem for Hamiltonian connected graphs analogous to Theorems 1 and 2 we restrict ourselves to the class of 3-connected graphs.

**Theorem 3.** *Let  $G$  be a  $k$ -connected graph ( $k \geq 3$ ). If  $G$  is not contractible to a graph  $H \in D_k$ , where  $D_k = \{H \mid S_{2k} \leq H \leq K_k + \bar{K}_k\}$ , then  $G$  is Hamiltonian connected.*

**Corollary 3** [2, Theorem 3]. *If  $G$  is a  $k$ -connected graph ( $k \geq 3$ ) and  $\beta_0(G) \leq k - 1$ , then  $G$  is Hamiltonian connected.*

So far no  $k$ -connected graph has been found that satisfies the condition of Theorem 3 but not that of Corollary 3, for any  $k \geq 3$ . It is conjectured here, that Theorem 3 and Corollary 3 are equivalent.

**Conjecture 1.** *A  $k$ -connected graph  $G$  is not contractible to a graph  $H \in D_k$  if and only if  $\beta_0(G) \leq k - 1$  ( $k \geq 3$ ).*

Without proof we mention that Conjecture 1 is true for  $k = 3$ .

#### 4. 1-Hamiltonian graphs

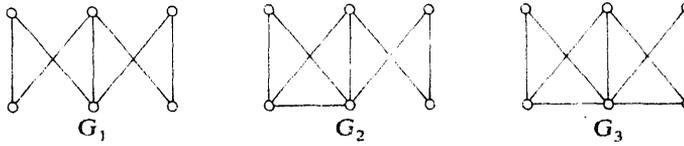
A graph  $G$  is  $s$ -Hamiltonian ( $s \geq 0$ ) if the deletion of any  $t$  points of  $G$ , where  $0 \leq t \leq s$ , results in a Hamiltonian graph. We shall state a contraction theorem for 1-Hamiltonian graphs. The removal of an arbitrary point of a 1-Hamiltonian graph results in a Hamiltonian, hence 2-connected graph, so that a 1-Hamiltonian graph is necessarily 3-connected.

**Theorem 4.** *Let  $G$  be a  $k$ -connected graph ( $k \geq 3$ ). If  $G$  is not contractible to a graph  $H \in E_k = E_{k_1} \cup E_{k_2}$ , where  $E_{k_1} = \{H \mid T_{2k} \leq H \leq K_k + \bar{K}_k\}$  and  $E_{k_2} = \{H \mid U_{2k} \leq H \leq K_k + \bar{K}_k\}$ , then  $G$  is 1-Hamiltonian.*

**Corollary 4.** *If  $G$  is a  $k$ -connected graph ( $k \geq 3$ ) and  $\beta_0(G) \leq k - 1$ , then  $G$  is 1-Hamiltonian.*

Corollary 4 is the case  $s = 1$  of the following result of Lesniak.

**Theorem C** [7, Theorem C]. *Let  $G$  be a graph with  $p$  points,  $p \geq 3$ , and let  $0 \leq s \leq p - 3$ . If  $\beta_0(G) \leq \kappa(G) - s$ , then  $G$  is  $s$ -Hamiltonian.*

Fig. 2.  $D_3 - E_3$ .

The set  $E_k$  is properly contained in the set  $D_k$  defined in Theorem 3.  $E_3$ , for example, consists of 13 of the 16 graphs in  $D_3$ ; the elements of  $D_3$  not in  $E_3$  are depicted in Fig. 2.

The wheel  $W_7$  does not satisfy the condition of Corollary 4, since  $\beta_0(W_7) = \kappa(W_7) = 3$ . However,  $W_7$  cannot be contracted to an element of  $E_3$ , so it is 1-Hamiltonian by Theorem 4. This shows that for 1-Hamiltonian graphs the contraction theorem again is more general than its corollary in terms of  $\beta_0$  and  $\kappa$ , in contrast with the conjectured equivalence of the corresponding theorems for Hamiltonian connected graphs. Note in this context that  $W_7$  is contractible to the graph  $G_3 \in D_3$  (see Fig. 2).

The proof of Theorem 4 does not generalize to proofs of analogous contraction theorems concerning  $k$ -connected  $s$ -Hamiltonian graphs with  $k \geq s+2$ . One would expect the opposite, since Corollary 4 generalizes to Theorem C.

## 5. Discussion

In Section 1 it was shown that in stating Theorem 1 the set  $A_k$  cannot be reduced to contain only  $k$ -connected graphs. Analogously one proves that Theorems 2, 3 and 4 cannot be sharpened in this sense either. In other words, the sets  $B_k$ ,  $D_k$  and  $E_k$  generally must contain graphs of connectivity less than  $k$  too. Less extreme reductions of the sets  $A_k$ ,  $B_k$ ,  $D_k$  and  $E_k$  have not been investigated, but may well be possible. As a consequence a sharpened version of Theorem 3 might be proved to be actually more powerful than Corollary 2, while in its present form it presumably is not (Conjecture 1).

We have stated four contraction theorems and proved one of them. As a matter of fact, the proof of Theorem 1 was already given in 1976 by Cohen and Hoede [3]. The proofs of the other three are available in a more elaborate version of this paper [9]. The list of Hamiltonlike notions for which contraction theorems analogous to Theorem 1 can be derived is unlikely to be exhausted with the four we treated.

## References

- [1] A. Bigalke and H.A. Jung, Über Hamiltonsche Kreise und unabhängige Ecken in Graphen, *Monatsh. Math.* 88 (1979) 195–210.
- [2] V. Chvátal and P. Erdős, A note on Hamiltonian circuits, *Discrete Math.* 2 (1972) 111–113.

- [3] A.M. Cohen and C. Hoede, On contractions and an inequality for nonhamiltonian graphs, Memorandum nr. 148, Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands (1976).
- [4] S. Goodman and S. Hedetniemi, Sufficient conditions for a graph to be Hamiltonian, *J. Combinatorial Theory (B)* 16 (1974) 175–180.
- [5] F. Harary, *Graph Theory* (Addison-Wesley, Reading, MA, 1969).
- [6] C. Hoede and H.J. Veldman, On characterization of Hamiltonian graphs, *J. Combinatorial Theory (B)* 25 (1978) 47–53.
- [7] L. Lesniak, On  $n$ -Hamiltonian graphs, *Discrete Math.* 14 (1976) 165–169.
- [8] H.J. Veldman, *Het Hamiltoncykei probleem*, Bachelor's thesis, Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands (1976). In Dutch.
- [9] H.J. Veldman and C. Hoede, Contraction theorems in Hamiltonian graph theory, Memorandum nr. 303, Department of Applied Mathematics, Twente University of Technology, Enschede, The Netherlands (1980).