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On a 1–1-Correspondence between Rooted Trees and Natural Numbers

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To each rooted tree, a natural number is assigned. It is shown that the correspondence is 1-to-1. An appendix gives the rooted trees corresponding to the numbers 1, 2,..., 45.

In the sequel, we suppose that $p(1) = 2$, $p(2)$, $p(3)$,... is the sequence of primes in ascending order.

Let T be a rooted tree, r its root. The connected components of $T - r$ are denoted by $T_1,..., T_v$, where v is the degree of r. The graphs T_j ($j = 1,..., v$) obviously are trees, which we transform into rooted trees by defining as the root of T_j the vertex of T_j which is adjacent to r in T. We now assign a natural number $\phi(T)$ to the rooted tree T.

DEFINITION. If T has only one vertex, then $\phi(T) = 1$. If the number of vertices of T is greater than 1, then

$$\phi(T) = \prod_{i=1}^{v} p(\phi(T_i)),$$

where $T_1,..., T_v$ are the rooted trees defined above.

EXAMPLE. In the figure below, we have $\phi(T) = p(\phi(T_1)) \cdot p(\phi(T_2))$, with $\phi(T_1) = p(1) = 2$, $\phi(T_2) = 2 \cdot \phi(2) = 6$, hence $\phi(T) = p(2) \cdot p(6) = 39$. Also see the Appendix.

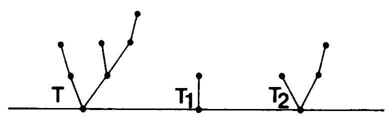

FIG. 1. Example for calculation of $\phi(T)$.

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PROPOSITION. *For each natural number f there is a unique rooted tree T such that $\phi(T)=f$.*

Proof. Suppose to the contrary that k is the smallest number which does not occur as a value of ϕ. If k is prime, then there is an n such that $p(n)=k$, and to n a rooted tree T' does correspond. Now we attach an edge to the root of T', and we define the other end of the edge as the new root. A tree T with $\phi(T)=k$ is obtained, and we have a contradiction. If k is composite, say the product of the primes $p, q,..., u$, then there exists a rooted tree for each of these primes, which trees can be joined at their roots to form a rooted tree T with $\phi(T)=k$.

The uniqueness of T can be proved in a similar manner. ∎

Applications

The mapping ϕ is well suited for the *identification* of rooted trees. For example, let a long sequence of rooted trees be given. Then the problem of detecting double occurrences can be conveniently reduced to detecting double occurrences in the transformed sequence of natural numbers. The mapping ϕ can also be applied to *generate* rooted trees, either systematically or stochastically (according to some probability distribution). When the question comes to generate rooted trees in a given class, e.g., with n points or with given height, the mapping ϕ is less suitable.

The Range of ϕ for Rooted Trees on n Points

Let α_n be the smallest $\phi(T)$ over all rooted trees on n points, and ω_n the largest. A few values of α_n and ω_n are given below.

n	α_n	ω_n
1	1	1
2	2	2
3	3	4
4	5	8
5	9	19
6	15	67
7	25	331
8	45	2221
9	75	19577

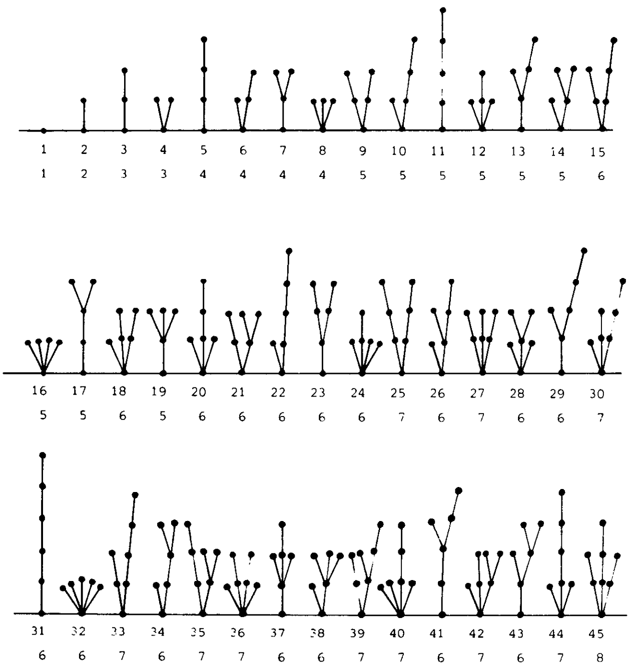

FIGURE 2

APPENDIX

Natural numbers ⩽45, corresponding rooted trees, and respective numbers of vertices (Fig. 2).

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