Semiclassical analysis for the Kramers-Fokker-Planck equation

Frédéric Hérau
Laboratoire de Mathématiques
Université de Reims
Moulin de la Housse B.P. 1039
51687 Reims cedex 2
France
herau@univ-reims.fr

Johannes Sjöstrand
Centre de Mathématiques
Ecole Polytechnique
91128 Palaiseau Cédex
France
johannes@math.polytechnique.fr

Christiaan C. Stolk
Centre de Mathématiques
Ecole Polytechnique
91128 Palaiseau Cédex
France
stolk@math.polytechnique.fr

Résumé

On étudie des estimations semiclassiques sur la résolvente d'opérateurs qui ne sont ni elliptiques ni autoadjoints, que l'on utilise pour étudier le problème de Cauchy. En particulier on obtient une description précise du spectre pres de l'axe imaginaire, et des estimations de résolvente à l'intérieur du pseudo-spectre. On applique ensuite les résultats à l'opérateur de Kramers-Fokker-Planck.

Abstract

We study some accurate semiclassical resolvent estimates for operators that are neither selfadjoint nor elliptic, and applications to the Cauchy problem. In particular we get a precise description of the spectrum near the imaginary axis and precise resolvent estimates inside the pseudo-spectrum. We apply our results to the Kramers-Fokker-Planck operator.

Keywords and Phrases: Fokker-Planck, Kramers, pseudo-spectrum, semiclassical, Weyl Calculus, FBI-Bargmann transform

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Contents

1 Introduction

In certain applications one is interested in the long-time behavior of systems described by a linear partial differential equation. For example in kinetic equations one studies the decay to equilibrium of various linear and nonlinear systems. For the Kramers-Fokker-Planck equation, that will be studied here, exponential decay was shown in [18] and an explicit rate was given in [9], following earlier results of Desvillettes and Villani who established explicit decay of any polynomial order in t^{-1} [6]. In [9, 6] more general discussions on decay to equilibrium in kinetic equations can be found.

The study of the long-time behavior naturally leads one to study the spectrum, and, for non-selfadjoint problems that we study here, to study the growth of the resolvent. For the Kramers-Fokker-Planck equation this is complicated by the fact that the operator whose time evolution is to be computed is not elliptic, but only satisfies certain subellipticity conditions. To deal with this Hérau and Nier exploit the relation between the Kramers-Fokker-Planck operator and certain Witten Laplacians. They obtain estimates for the decay to equilibrium in terms of the first eigenvalue of this Witten Laplacian. More on the connection between the Kramers-Fokker-Planck equation and Witten Laplacians can be found in [8].

Resolvent estimates have been studied from a different perspective by a group of authors interested in the notion of pseudospectrum, i.e. the region in the complex spectral plane where the resolvent may be large. In recent years there has been a great interest in this area following work of Trefethen, Davies, Zworski and others [20, 2, 3, 21]. In [5] the authors studied the location of the spectrum inside the spectrum in the semiclassical limit, and adapted subelliptic estimates to this situation. In [10], M. Hitrik has obtained related results for operators in one dimension.

In the present work we apply such ideas to a class of pseudodifferential operators that includes the Kramers-Fokker-Planck operator. We obtain a number of higher eigenvalues, in the semiclassical limit, for the original operator, i.e. not the Witten Laplacian. We also obtain precise resolvent estimates. We use roughly the same estimates as [5] in one region of phase space, while in other regions we have to make important changes, and additions. This is then applied for the time evolution.

Evolution problems have also attracted recent interest [4, 19, 1], and here a difficulty is the generally quite wild growth of the resolvent inside the pseudospectrum. It is therefore of interest that for a concrete physically interesting model, we are able to control the resolvent sufficiently well to get quite precise results about the long-time evolution.

Spectral properties for some different Fokker-Planck equations (without the subellipticity property) have been discussed by Kolokoltsov [12]. In probability theory many other problems for equations with a small diffusive term have been studied, see for example the monograph [7].

Our main example, the Kramers-Fokker-Planck operator is given by

$$P = v \cdot h\partial_x - V'(x) \cdot h\partial_v + \frac{\gamma}{2}(-(h\partial_v)^2 + v^2 - hn) \tag{1.1}$$

on $\mathbb{R}^{2n}$, where V is a $\mathcal{C}^\infty$ potential, h is essentially the temperature, and $x, v \in \mathbb{R}^n$. The operator P is derived from the original equation, introduced in one-dimensional form by Kramers [13], in Section 13 below (see also [14], [9]). The time evolution problem is given by

$$(h\partial_t + P)u(t, x, v) = 0, \qquad u(0, \cdot, \cdot) = u_0.$$

As mentioned, we are interested in the low temperature limit

$$0 < h \ll 1,$$

and the equations are rescaled according to the standard convention in semiclassical analysis, where each derivative comes with an h. Our main result about the Kramers-Fokker-Planck equation is the following theorem.

Theorem 1.1 *Assume V is a Morse function and that outside a compact region, $|V'(x)| \geq c_0 > 0$. Assume also that the derivatives of V of order 2 or more are bounded. Then there exist constants $c, C' > 0$ such that for every $C > 1$:*

a) For any fixed neighborhood Ω of the eigenvalues of the quadratic approximation of $P_{|_{h=1}}$ at the critical points, there exist h_0, $C'' > 0$ such that for $0 < h \leq h_0$, $|z| \leq C$, $z \notin \Omega$,

$$h\|u\| \leq C''\|(P - hz)u\|, \qquad \forall u \in ß.$$

b) There exists $h_1 > 0$, such that for $0 < h \leq h_1$, $\mathrm{Re}\,(z) \leq c|z|^{1/3}h^{2/3}$ and $|z| \geq Ch$,

$$|z|^{1/3}h^{2/3}\|u\| \leq C'\|(P - z)u\|, \qquad \forall u \in ß.$$

In fact this theorem on the Kramers-Fokker-Planck operator is a consequence of a more general one. Let us first write the hypotheses that will be needed for the symbol p of the more general operator p^w that we shall study. We assume that $p = p_1 + ip_2$ is a smooth function on $\mathbb{R}^{2n}_{x,\xi}$ with $p_1 \geq 0$. (The previous space $\mathbb{R}^{2n}_{x,v}$ now becomes $\mathbb{R}^n_x$.)

Assumptions near the critical points: Assume that p has finitely many critical points $\rho_1, \rho_2, ..., \rho_N$ with $p(\rho_j) = 0$. Let $\delta(\rho) \geq 0$ be equivalent to the distance from ρ to $\mathcal{C} := \{\rho_1, \rho_2, ..., \rho_N\}$, with $\delta^2 \in C^\infty$. We assume in the following that in a fixed open ball $\mathcal{B}$ containing $\mathcal{C}$ we have

$$\textbf{(H1)} \qquad p_1 + \epsilon_0 H_{p_2}^2 p_1 \sim \delta^2 \tag{1.2}$$

for a sufficiently small $\epsilon_0 > 0$. The assumption that $p(\rho_j) = 0$ is for simplicity only. As we shall later, this implies that the critical points are non-degenerate.

Assumptions at infinity: In the following we use the notions of admissible metrics and weights in the sense of the Weyl-Hörmander calculus, that we review in Section 7. We first define an admissible metric on $\mathbb{R}^{2n}_\rho$ with $\rho = (x, \xi)$:

$$\Gamma_0 = dx^2 + \frac{d\xi^2}{\lambda^2},$$

where $\lambda = \lambda(\rho)$. There is no restriction to assume that $1 \leq \lambda \in \mathcal{C}^\infty$, and we suppose also that

$$\textbf{(H2)} \qquad \lambda \in S(\lambda, \Gamma_0), \qquad \partial\lambda \in S(1, \Gamma_0). \tag{1.3}$$

If m is an admissible weight, recall that $S(m, \Gamma_0)$ is the class of $\mathcal{C}^\infty$ symbols p satisfying $\partial_x^\alpha \partial_\xi^\beta p(\rho) = \mathcal{O}\left(m(\rho)\lambda(\rho)^{-|\beta|}\right)$. We suppose first that p is a symbol of order 2 but with the first and second derivatives better than what would be given by the symbolic calculus:

$$\textbf{(H3)} \qquad p \in S(\lambda^2, \Gamma_0), \quad \partial p \in S(\lambda, \Gamma_0), \quad \partial^2 p_1 \in S(1, \Gamma_0), \quad \partial H_{p_2} p_1 \in S(\lambda, \Gamma_0). \tag{1.4}$$

We now assume that outside any fixed neighborhood of $\mathcal{C}$ we have the following gain

$$\textbf{(H4)} \qquad p_1 + \epsilon_0 H_{p_2}^2 p_1 \sim \lambda^2. \tag{1.5}$$

Note that these assumptions are satisfied by the symbol of the Kramers-Fokker-Planck operator (see Section 13). In order to give a unique assumption on the whole space, we extend the function δ to $\mathbb{R}^{2n}$ to be a smooth function on $\mathbb{R}^{2n}$, strictly positive away from $\mathcal{C}$, and constant outside a fixed neighborhood of that set. There is no restriction to assume that $\lambda = 1$ inside the same neighborhood. Then (1.2–1.5) can be summarized in the following way

$$p_1 + \epsilon_0 H_{p_2}^2 p_1 \sim (\lambda\delta)^2. \tag{1.6}$$

We have the following theorem for $P = p^w$:

Theorem 1.2 *Suppose p satisfies* **(H1–H4)**. *Then there exist constants c, $C' > 0$ such that for every $C \geq 1$:*

a) For any fixed neighborhood Ω of the eigenvalues of the quadratic approximations of $P_{|_{h=1}}$ at the critical points, there exist $h_0, C'' > 0$ such that for $0 < h \leq h_0$, $|z| \leq C$, $z \notin \Omega$,

$$h\|u\| \leq C''\|(P - hz)u\|, \qquad \forall u \in ß.$$

b) There exists $h_1 > 0$, such that for $0 < h \leq h_1$, $\mathrm{Re}\,(z) \leq c|z|^{1/3}h^{2/3}$, $|z| \geq Ch$,

$$|z|^{1/3}h^{2/3}\|u\| \leq C'\|(P - z)u\|, \qquad \forall u \in ß.$$

Here, if ρ_0 is a critical point of p, we define the quadratic approximation P_0 of P, to be the $h = 1$ quantization of $\sum_{|\alpha+\beta|=2} \frac{1}{\alpha!\beta!}\partial_x^\alpha \partial_\xi^\beta p(\rho_0)x^\alpha \xi^\beta$. As we shall see, P_0 has discrete spectrum and compact resolvent in a weighted space and the eigenvalues can be computed explicitly. (In fact, the spectrum is discrete even without weights and this fact will be used in Section 11.)

Staying in the general case, we shall next give results about the spectrum and the associated heat equation. We then define P to be the closure of p^w with domain $\mathcal{S}$. In Section 7, we shall see that the Fefferman-Phong inequality implies that $\mathrm{Re}\,(Pu, u) \geq -Ch^2\|u\|^2$, $u \in \mathcal{S}$ and hence also for $u \in \mathcal{D}(P)$ (this is immediate in the KFP case). In other words, P is accretive and we shall assume

$$\textbf{(H5)} \quad P \text{ is m-accretive}, \tag{1.7}$$

i.e. P has no accretive strict extension. In the KFP-case this has recently been established in great generality by Helffer–Nier [8] and their result implies (H5) under our assumptions on V. In the general case, we shall see that the following assumption

$$\textbf{(H6)} \qquad \text{If } u \in L^2 \text{ and } (p^w + 1)u \in \mathcal{S}, \text{ then } u \in \mathcal{S},$$

implies for h sufficiently small, that $\mathcal{D}(P) = \{u \in L^2;\, Pu \in L^2\}$, and hence P is m-accretive.

Theorem 1.3 *Suppose P satisfies* **(H1–H5)** *and let $C > 0$. Then there exists $h_0 > 0$ such that for $0 < h \leq h_0$, the spectrum of P in the disc $D(0, Ch)$ is discrete, and the eigenvalues are of the form,*

$$\lambda_{j,k}(h) \sim h(\mu_{j,k} + h^{1/N_{j,k}}\mu_{j,k,1} + h^{2/N_{j,k}}\mu_{j,k,2} + \ldots), \tag{1.8}$$

where the $\mu_{j,k}$ are the eigenvalues in $D(0, C)$ (repeated with their multplicity) of the quadratic approximation of $P_{|_{h=1}}$ at the critical point ρ_k and $N_{j,k}$ is the dimension of the corresponding generalized eigenspace.

Here it is understood that C has been chosen, so that no quadratic approximation has any eigenvalues on the boundary of the disc $D(0, C)$. The explicit form of those eigenvalues will be given in Proposition 5.1 and in Section 13. Note that they are distributed in an angle in $\mathbb{R}^+ + i\mathbb{R}$ avoiding the imaginary axis (except in 0).

As a consequence of the resolvent estimates and the description of the eigenspaces we give the following theorem on the large time behavior of the semi-group associated to P :

Theorem 1.4 *Suppose P satisfies* **(H1)–(H5)**. *Consider the set $\{\mu_{jk}\}$ of eigenvalues of the quadratic approximation of $P|_{h=1}$ at the critical points (repeated with their multiplicities) defined in the preceding theorem. Let $b > 0$ be such that the line $\mathrm{Re}\, z = b$ avoids the set $\{\mu_{jk}\}$ and define* *the finite set* $J_b = \{\mu_{j,k};\ \mathrm{Re}\,(\mu_{j,k}) < b\}$*. Assume that the* $\mu_{j,k}$ *in* J_b *are simple and distinct. Then we have*

$$e^{-tP/h} = \sum_{\mu_{j,k}\in J_b} e^{-t\lambda_{j,k}/h}\Pi_{j,k} + \mathcal{O}(1)e^{-tb} \qquad in\ \ \mathcal{L}(L^2, L^2), \tag{1.9}$$

where $\lambda_{j,k}$ *is the eigenvalue of* P *associated to* $\mu_{j,k}$*, and* $\Pi_{j,k}$ *the associated (rank one) spectral projection. Here the term* $\mathcal{O}(1)$ *is with respect to* $t \geq 0$ *and* $h \to 0$*.*

We construct explicitly a global weight function G with controlled derivatives, satisfying in particular $G = \mathcal{O}(h)$, $G' = \mathcal{O}(h^{1/2})$ and $G'' = \mathcal{O}(1)$. The main idea (also used in many earlier works on resonances and non-selfadjoint operators) is that we get the new leading symbol $p \approx p + \frac{\epsilon}{i}\{p, G\}$ with an increased real part, where $\{.,.\}$ is the Poisson bracket, and ϵ is small and fixed. We will use it both near the critical points of p and at infinity. Contrary to the earlier works mentioned above (but similarly to [5]) we need resolvent and evolution estimates in the original L^2 space and this requires G/h to be bounded in order to have an equivalent norm on the weighted space. Consequently the estimates become more delicate. The technical realization of this idea can be made either by using the FBI-Bargmann transform and weighted spaces of holomorphic functions or using pseudodifferential calculus (since G/h is bounded). We found it convenient to use the first method near the critical points and the second one elsewhere. We choose the semiclassical variant of the Weyl-Hörmander calculus with a metric sufficiently general to cover the case of the KFP and related operators.

The plan of the article is the following. The next section is devoted to the construction of G. In Sections 3 to 6 we work near the critical points by using the Fourier-Bros-Iagolnitzer transform in a modified L^2 space $L^2_{\Phi_\epsilon}$ associated to G. Here G will play the role of a local escape function. We recall in Section 3 some basic facts about the FBI transform and construct the spaces $L^2_{\Phi_\epsilon}$. In Section 4 we get local resolvent estimates for a truncated operator satisfying **(H1)** . In Section 5 we recall some facts on the quadratic differential operators from [15] and give a localized version of them. Then in Section 6 we compare the operator P to its quadratic approximations at the critical points to get precise local resolvent estimates near the critical points.

In Sections 7 to 9 we work away from the critical points of p in the real phase space using the semiclassical Weyl-Hörmander calculus. Here p satisfies hypothesis **(H2–H4)**. Section 7 is devoted to some basic facts about the semiclassical Weyl calculus and the construction of a metric adapted to the symbol p. In Sections 8 and 9 we get resolvent estimates using a multiplier method, where the symbol of the multiplier is essentially $1 + G/h$.

In Section 10 we combine all the resolvent estimates given in Sections 3 to 9 and we prove Theorem 1.2. Section 11 is devoted to the proof of Theorem 1.3, i.e. the asymptotic expansion of the eigenvalues of P: We solve a Grushin problem thanks to a slight variation of the resolvent estimates given in Section 10. In Section 12 we prove Theorem 1.4 about the large time behavior of the semigroup associated to P under hypothesis **(H5)**. Eventually in last section we check that all the hypotheses **(H1–H4)** are satisfied for the symbol of the KFP operator, which proves Theorem 1.1.

2 Bounded weight function

The aim of this section is to build a weight function G defined in the whole space, uniformly bounded by a multiple of h. Recall that $\mathcal{B}$ is the fixed open ball appearing in **(H1)**. The result is the following proposition:

Proposition 2.1 *Suppose p satisfies* **(H1–H4)**. *Then there exists a constant $C > 0$ and a function $G \in \mathcal{C}^\infty(\mathbb{R}^{2n}_\rho)$ such that uniformly in h, $\epsilon > 0$ sufficiently small, we have*

$$\begin{aligned}
&\partial^k G = \mathcal{O}\left(\delta^{(2-k)_+}\right) \quad \text{for } \delta\lambda \le h^{1/2},\\
&\partial^k G = \mathcal{O}\left(h(\delta\lambda h)^{-k/3}\right) \ \text{ in } \{\rho \in \mathcal{B}; \delta\lambda \ge h^{1/2}\},\\
&\partial_x^\alpha \partial_\xi^\beta G = \mathcal{O}\left(h^{1-k/3}\lambda^{-(\min(|\alpha|,1)+|\beta|)/3}\right) \quad \text{outside } \mathcal{B},\ |\alpha|+|\beta| = k.
\end{aligned} \tag{2.1}$$

Note that this implies $G = \mathcal{O}(h)$, $H_G = \mathcal{O}(h^{1/2})$ and $\partial^2 G = \mathcal{O}(1)$. Secondly G is such that

a) In $\mathcal{B}$, if we let p denote an almost analytic extension and if we put $\tilde{p}(\rho) \stackrel{\text{def}}{=} p(\rho + i\epsilon H_G(\rho)) = \tilde{p}_1(\rho) + i\tilde{p}_2(\rho)$ where $\rho \in \mathbb{R}^{2n}$, we have

$$\tilde{p}_1 \ge \frac{\epsilon}{C}\min\left((\delta\lambda)^2, (\delta\lambda h)^{2/3}\right), \qquad \tilde{p}_2 = \mathcal{O}((\delta\lambda)^2). \tag{2.2}$$

b) Outside $\mathcal{B}$, we have

$$p_1 + \epsilon H_{p_2} G \ge \frac{\epsilon}{C}(p_1 + (h\lambda)^{2/3}). \tag{2.3}$$

The construction near the critical points

Let $\rho_j \in \mathcal{C}$. Fix $T > 0$. In a neighborhood of ρ_j, we set

$$G_T = \int k_T(t)\, p_1 \circ \exp(tH_{p_2})dt, \tag{2.4}$$

where $k_T(t) = k(t/T)$ and $k \in \mathcal{C}(\mathbf{R} \setminus \{0\})$ is the odd function given by: $k(t) = 0$ for $|t| \ge 1/2$ and $k'(t) = -1$ for $0 < |t| < 1/2$. Notice that k and k_T have a jump of size 1 at the origin. G_T is a smooth function satisfying

$$H_{p_2} G_T = \langle p_1 \rangle_T - p_1,\ G_T = \mathcal{O}(\delta^2),\ \nabla G_T = \mathcal{O}(\delta),$$

where

$$\langle p_1 \rangle_T = \frac{1}{T}\int_{-T/2}^{T/2} p_1 \circ \exp(tH_{p_2})dt.$$

Consider the dilated symbol

$$\widetilde{p} = \widetilde{p}_\epsilon(\rho) = p(\rho + i\epsilon H_G(\rho)) = p(\rho) - i\epsilon H_p G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2),$$

with real and imaginary parts, given by

$$\begin{aligned}
\widetilde{p}_1 &= p_1(\rho) + \epsilon H_{p_2} G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2)\\
&= (1-\epsilon)p_1(\rho) + \epsilon\langle p_1 \rangle_T + \mathcal{O}_T(\epsilon^2\delta^2),\\
\widetilde{p}_2 &= p_2(\rho) - \epsilon H_{p_1} G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2).
\end{aligned} \tag{2.5}$$

Using (1.6) near $\mathcal{C}$, we see that if we fix $\epsilon > 0$ small enough, depending on T, then in an (ϵ, T) - dependent neighborhood of ρ_j, we have

$$\widetilde{p}_1 \geq \frac{\epsilon}{C}\delta^2, \ \widetilde{p}_2 = \mathcal{O}(\delta^2). \tag{2.6}$$

Note in particular that $\widetilde{p}$ takes its values in an angle around the positive real axis, $\widetilde{p}_1 \succ \widetilde{p}_2$. Note also that another choice of weight function near the critical point could have been $\tau H_{p_2} p_1$ for τ sufficiently small.

The construction away from the critical points

We work in a region

$$\{\rho; \, \delta\lambda(\rho) \geq h^{1/2}\}. \tag{2.7}$$

Let $\psi \in \mathcal{C}_0^\infty(]-2, 2[)$ be a cutoff function equal to 1 in $[-1, 1]$. Let M be a large constant to be fixed later. We choose the following function

$$G = h\frac{H_{p_2} p_1}{(\delta\lambda)^{4/3} h^{1/3}} \psi\left(\frac{M p_1}{(h\delta\lambda)^{2/3}}\right), \tag{2.8}$$

where we recall that $\delta = \delta(\rho)$ and $\lambda = \lambda(\rho)$.

We first check the bounds for the derivatives of G. Of course when $Mp_1 \geq 2(h\delta\lambda)^{2/3}$, $G = 0$ and we have only to study the derivatives in the region where $Mp_1 < 2(h\delta\lambda)^{2/3}$.

Observe that the estimates (2.1) for G in $\left\{\delta\lambda \geq h^{1/2}\right\}$ can be equivalently written using the following Riemannian metric

$$\Gamma_h = \frac{dx^2}{(\delta h)^{2/3}} + \frac{d\xi^2}{(\delta\lambda h)^{2/3}},$$

by saying (in the Hörmander terminology of spaces of symbols, see Section 7) that

Lemma 2.2 *$G \in S(h, \Gamma_h)$ and $\nabla G \in S(h(\delta\lambda h)^{-1/3}, \Gamma_h)$.*

Proof. For the following estimates of the derivatives we shall use this terminology and stay in the region $\left\{\delta\lambda \geq h^{1/2}\right\} \cap \left\{Mp_1 < 2(h\delta\lambda)^{2/3}\right\}$. We work step by step by studying the derivatives of each function entering in the composition of G.

Estimates of p. We know that $p \in S(\lambda^2, dx^2 + d\xi^2/\lambda^2)$ from the hypothesis. ¿From the fact that p is a Morse function we get that $p \in S((\delta\lambda)^2, dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2)$. For the same reason we have $\nabla p \in S(\delta\lambda, dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2)$. Besides we have on $\left\{\delta\lambda \geq h^{1/2}\right\}$

$$\Gamma_h \geq dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2 \geq C^{-1}\Gamma_0, \tag{2.9}$$

since $\delta^2 \geq (\delta h)^{2/3}$ and $(\delta\lambda)^2 \geq (\delta\lambda h)^{2/3}$ in this region. As a consequence we get that

$$\nabla p \in S(\delta\lambda, \Gamma_h). \tag{2.10}$$

Estimates for p_1. Since p_1 is nonnegative with bounded second derivatives, we can apply the well known inequality for $W^{2,\infty}$ functions

$$|\nabla f|^2 \leq 2f\|f''\|_\infty, \tag{2.11}$$

which yields $|\nabla p_1| \le C\sqrt{p_1}$. Since $p_1 \le 2(h\delta\lambda)^{2/3}$ we get that $\nabla p_1 = \mathcal{O}((h\delta\lambda)^{1/3})$. Together with the fact that p_1 has its second derivative bounded and (2.9) we get that

$$p_1 \in S((\delta\lambda h)^{2/3}, \Gamma_h), \quad \text{and} \quad \nabla p_1 \in S((\delta\lambda h)^{1/3}, \Gamma_h). \tag{2.12}$$

Here we used that $\nabla^2 p_1 \in S(1, \Gamma_0) \subset S(1, \Gamma_h)$.

Estimates for powers of $\delta\lambda$. Using (1.3), we first note that

$$\delta\lambda \in S((\delta\lambda), dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2)$$

Together with the fact that $\nabla(\delta\lambda) \in S(1, dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2)$, this gives for $\alpha \in \mathbb{R}$,

$$(\delta\lambda)^\alpha \in S((\delta\lambda)^\alpha, \Gamma_h), \quad \text{and} \quad \nabla(\delta\lambda)^\alpha \in S((\delta\lambda)^{\alpha-1}, \Gamma_h). \tag{2.13}$$

Estimates of $p_1/(h\delta\lambda)^{2/3}$. From (2.12) and (2.13) with $\alpha = -2/3$ we get immediately that

$$p_1/(h\delta\lambda)^{2/3} \in S(1, \Gamma_h).$$

Besides let us write

$$\nabla\left(p_1/(h\delta\lambda)^{-2/3}\right) = (\nabla p_1)(h\delta\lambda)^{-2/3} + p_1\nabla(h\delta\lambda)^{-2/3}.$$

From the same estimates for the derivatives we get

$$(\nabla p_1)(h\delta\lambda)^{-2/3} \in S((h\delta\lambda)^{-1/3}, \Gamma_h),$$

and

$$p_1\nabla(h\delta\lambda)^{-2/3} \in S((h\delta\lambda)^{2/3} \times h^{-2/3}(\delta\lambda)^{-5/3}, \Gamma_h) \subset S((h\delta\lambda)^{-1/3}, \Gamma_h),$$

where in the last inclusion we used the fact that $\delta\lambda \ge h^{1/2}$. Summing up we have proven that

$$p_1/(h\delta\lambda)^{2/3} \in S(1, \Gamma_h), \quad \text{and} \quad \nabla\left(p_1/(h\delta\lambda)^{2/3}\right) \in S((h\delta\lambda)^{-1/3}, \Gamma_h). \tag{2.14}$$

Estimates of $\psi(Mp_1/(h\delta\lambda)^{2/3})$. An immediate consequence of the first part of (2.14) is that

$$\psi(Mp_1/(h\delta\lambda)^{2/3}) \in S(1, \Gamma_h),$$

since ψ is $\mathcal{C}^\infty$ with compact support. We need to estimate the derivatives of this expression,

$$\nabla\psi(Mp_1/(h\delta\lambda)^{2/3}) = M\nabla\left(p_1/(h\delta\lambda)^{2/3}\right)\psi'(Mp_1/(h\delta\lambda)^{2/3}).$$

For the same reason as before we have

$$\psi'(Mp_1/(h\delta\lambda)^{2/3}) \in S(1, \Gamma_h).$$

Using the second part of (2.14), and summing up we have proven that

$$\psi(Mp_1/(h\delta\lambda)^{2/3}) \in S(1, \Gamma_h), \quad \text{and} \quad \nabla\psi(Mp_1/(h\delta\lambda)^{2/3}) \in S((h\delta\lambda)^{-1/3}, \Gamma_h). \tag{2.15}$$

Estimates for $H_{p_2}p_1$. We observe that $H_{p_2}p_1 = \sigma(\nabla p_2, \nabla p_1)$ where σ is the canonical symplectic form. Using (2.10) for p_2 and (2.12) for p_1 we get

$$H_{p_2}p_1 \in S((\delta\lambda)(h\delta\lambda)^{1/3}, \Gamma_h).$$

From the hypothesis (1.4) and the fact that p is a Morse function we can write

$$\nabla H_{p_2}p_1 \in S(\delta\lambda, dx^2/\delta^2 + d\xi^2/(\delta\lambda)^2) \subset S(\delta\lambda, \Gamma_h).$$

Summing up we have proven that

$$H_{p_2}p_1 \in S(h^{1/3}(\delta\lambda)^{4/3}, \Gamma_h), \quad \text{and} \quad \nabla H_{p_2}p_1 \in S(\delta\lambda, \Gamma_h). \tag{2.16}$$

Estimates for $H_{p_2}p_1/(h^{1/3}(\delta\lambda)^{4/3})$. From the first parts of (2.16) and (2.13) with $\alpha = -4/3$ we immediately get that $H_{p_2}p_1/(h^{1/3}(\delta\lambda)^{4/3}) \in S(1, \Gamma_h)$. Its derivative is given by

$$\nabla \frac{H_{p_2}p_1}{h^{1/3}(\delta\lambda)^{4/3}} = \frac{\nabla H_{p_2}p_1}{h^{1/3}(\delta\lambda)^{4/3}} + H_{p_2}p_1 \nabla(h^{-1/3}(\delta\lambda)^{-4/3}).$$

Using (2.16) and (2.13) we respectively get that

$$\frac{\nabla H_{p_2}p_1}{h^{1/3}(\delta\lambda)^{4/3}} \in S((h\delta\lambda)^{-1/3}, \Gamma_h),$$

and

$$H_{p_2}p_1 \nabla(h^{-1/3}(\delta\lambda)^{-4/3}) \in S(h^{1/3}(\delta\lambda)^{4/3} \times h^{-1/3}\delta\lambda)^{-7/3}, \Gamma_h) \subset S((\delta\lambda)^{-1}, \Gamma_h).$$

Using the fact that $\delta\lambda \geq (\delta\lambda h)^{1/3}$ in this formula gives

$$\frac{H_{p_2}p_1}{h^{1/3}(\delta\lambda)^{4/3}} \in S(1, \Gamma_h), \quad \text{and} \quad \nabla \frac{H_{p_2}p_1}{h^{1/3}(\delta\lambda)^{4/3}} \in S((h\delta\lambda)^{-1/3}, \Gamma_h). \tag{2.17}$$

Estimates for G and end of the proof of lemma 2.2. We can now prove the estimates for G. From the first parts of (2.15) and (2.17) and multiplying by h we get that

$$G \in S(h, \Gamma_h).$$

From the second part of the same expressions we also get immediately that

$$\nabla G \in S(h(h\delta\lambda)^{-1/3}, \Gamma_h).$$

This completes the proof of lemma 2.2 and therefore of the estimates (2.1) when $\delta\lambda \geq h^{1/2}$. $\square$

Proof of (2.2) in the intermediate region

We work here in the region $\{\rho \in \mathcal{B};\ h^{1/2} \le \delta\lambda\}$, but many of the estimates will be valid also near infinity and used later, so we indicate when the validity is restricted to a bounded region. Consider the function G defined in (2.8) :

$$G = h\frac{H_{p_2}p_1}{(\delta\lambda)^{4/3}h^{1/3}}\psi\left(\frac{Mp_1}{(h\delta\lambda)^{2/3}}\right).$$

For $\tilde{p}(\rho) \stackrel{\text{def}}{=} p(\rho + i\epsilon H_G(\rho)) = \tilde{p}_1(\rho) + i\tilde{p}_2(\rho)$ in $\mathcal{B}$, we have

$$\begin{aligned} \widetilde{p}_1 &= p_1 + \epsilon H_{p_2}G + \mathcal{O}(\epsilon^2|\nabla G|^2), \\ \widetilde{p}_2 &= p_2 - \epsilon H_{p_1}G + \mathcal{O}(\epsilon^2|\nabla G|^2). \end{aligned} \tag{2.18}$$

Let us estimate the remainders. ¿From (2.1) we know that $\nabla G = \mathcal{O}(h^{2/3}(\delta\lambda)^{-1/3})$. As a consequence

$$\mathcal{O}(\epsilon^2|\nabla G|^2) = \epsilon^2\mathcal{O}(h^{4/3}(\delta\lambda)^{-2/3}) \le \epsilon^2\mathcal{O}((h\delta\lambda)^{2/3}), \tag{2.19}$$

since $h^{4/3}(\delta\lambda)^{-2/3} = \mathcal{O}((h\delta\lambda)^{2/3})$ when $\delta\lambda \ge h^{1/2}$. Let us now study the first two terms of the expression of $\tilde{p}_1$ depending on the size of p_1.

Estimates when p_1 is large. We work first in the *elliptic* region

$$\left\{\rho \in \mathbb{R}^{2n};\quad Mp_1 \ge (h\delta\lambda)^{2/3}\right\}.$$

From (2.1) and the fact that $H_{p_2} = \mathcal{O}(\delta\lambda)$, we get

$$H_{p_2}G = \mathcal{O}((\delta\lambda h)^{2/3}). \tag{2.20}$$

Restricting the attention to $\mathcal{B}$ we recall that the remainder in (2.19) is $\epsilon^2\mathcal{O}((\delta\lambda h)^{2/3})$ and that

$$\tilde{p}_1 = p_1 + \epsilon H_{p_2}G + \epsilon^2\mathcal{O}((\delta\lambda h)^{2/3}).$$

Choosing ϵ small enough yields

$$\tilde{p}_1 \ge \frac{(\delta\lambda h)^{2/3}}{CM}.$$

On the other hand we have using the bound on the remainder and of H_G that

$$\tilde{p}_2 = \mathcal{O}((\delta\lambda)^2).$$

Estimates when p_1 is small. In the region

$$\left\{\rho \in \mathbb{R}^{2n};\quad Mp_1 \le (h\delta\lambda)^{2/3}\right\}, \tag{2.21}$$

we can write $G = h\frac{H_{p_2}p_1}{(\delta\lambda)^{4/3}h^{1/3}}$. We have therefore

$$p_1 + \epsilon H_{p_2}G = p_1 + \epsilon h\frac{H_{p_2}^2 p_1}{(\delta\lambda)^{4/3}h^{1/3}} + \epsilon h(H_{p_2}p_1)H_{p_2}\left((\delta\lambda)^{-4/3}h^{-1/3}\right). \tag{2.22}$$

For the third term of (2.22) we use that $|\nabla p_1| \leq C\sqrt{p_1} \leq C(h\delta\lambda)^{1/3}/\sqrt{M}$, $|\nabla p_2| \leq C\delta\lambda$ and get using also (2.13)

$$\epsilon h(H_{p_2}p_1)H_{p_2}\left((\delta\lambda)^{-4/3}h^{-1/3}\right) = \frac{\epsilon}{\sqrt{M}}\mathcal{O}(h) = \frac{\epsilon}{\sqrt{M}}\mathcal{O}((\delta\lambda h)^{2/3}), \tag{2.23}$$

since $\delta\lambda \geq h^{1/2}$. We study next the sum of the first and the second term. We first observe that

$$\frac{h}{(\delta\lambda)^{4/3}h^{1/3}} \leq 1,$$

and from (1.6) provided $\epsilon < \epsilon_0$, we get

$$p_1 + \epsilon h \frac{H_{p_2}^2 p_1}{(\delta\lambda)^{4/3}h^{1/3}} \geq \frac{\epsilon}{\epsilon_0}\frac{h}{(\delta\lambda)^{4/3}h^{1/3}}(p_1 + \epsilon_0 H_{p_2}^2 p_1) \geq \frac{\epsilon\epsilon_1}{\epsilon_0}(\delta\lambda h)^{2/3}. \tag{2.24}$$

Therefore choosing M sufficiently large (and fixed from now on) gives

$$p_1 + \epsilon H_{p_2} G \geq \epsilon(\delta\lambda h)^{2/3}/C. \tag{2.25}$$

Since the remainder term in (2.19) is $\epsilon^2\mathcal{O}((\delta\lambda h)^{2/3})$, and choosing ϵ sufficiently small again, we get on $\mathcal{B}$:

$$\tilde{p}_1 \geq \epsilon(\delta\lambda h)^{2/3}/C \quad \text{for } \delta \geq h^{\frac{1}{2}}.$$

The global construction

We shall glue together the two weights constructed in the previous two subsections. Let us denote by G_{int} the interior weight G_T defined in (2.4) and G_{out} the one defined in (2.8) where we recall that the constants T, and M appearing in the definitions are fixed. Recall also the main properties of these weights:

$$\begin{cases} \partial^k G_{\text{int}} = \mathcal{O}(\delta^{(2-k)_+}), \\ p_1 + \epsilon H_{p_2} G_{\text{int}} \geq \frac{\epsilon}{C}(\delta\lambda)^2, \end{cases} \quad \text{in } \mathcal{B}, \tag{2.26}$$

and

$$\begin{cases} \partial^k G_{\text{out}} = \mathcal{O}(h(\delta\lambda h)^{-k/3}), \\ p_1 + \epsilon H_{p_2} G_{\text{out}} \geq \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, \end{cases} \quad \text{for } h^{1/2} \leq \delta\lambda. \tag{2.27}$$

We now build a function G defined everywhere and satisfying Proposition 2.1. In the following, we introduce an additional large real constant N to be fixed later. We first build modified functions G_{int} and G_{out}.

Construction of a modified G_{int}. Let us introduce the following function

$$\widehat{p_1} \stackrel{\text{def}}{=} \chi\left(\frac{\delta\lambda}{4Nh^{1/2}}\right)p_1,$$

where $\chi \in C_0^\infty(\mathbb{R};[0,1])$ is a standard cut-off near 0, equal to 1 on $[0,1]$ and to 0 on $[2,+\infty[$. Notice that $\widehat{p_1} = 0$ when $\delta\lambda \geq 8Nh^{1/2}$ and that $\widehat{p_1} = p_1$ when $\delta\lambda \leq 4Nh^{1/2}$. Define now $\widetilde{G}_{\text{int}}$ as in (2.4) but with p_1 there replaced by $\widehat{p_1}$. Then $\widetilde{G}_{\text{int}}$ has its support in $\{\rho;\, \delta\lambda \leq 16Nh^{1/2}\}$ and coincides with G_{int} when $\delta\lambda \le 2Nh^{1/2}$ (assuming that T has been fixed sufficiently small). As a consequence we get that

$$p_1 + \epsilon H_{p_2}\widetilde{G}_{\text{int}} \ge \begin{cases} \frac{\epsilon}{C}(\delta\lambda)^2, & \text{when } \delta\lambda \le 2Nh^{1/2}, \\ 0 & \text{everywhere.} \end{cases} \tag{2.28}$$

Note that this implies the following bounds :

$$p_1 + \epsilon H_{p_2}\widetilde{G}_{\text{int}} \ge \begin{cases} \frac{\epsilon}{C}(\delta\lambda)^2, & \text{when } \delta\lambda \le h^{1/2}, \\ \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{when } h^{1/2} \le \delta\lambda \le Nh^{1/2}, \\ \frac{\epsilon}{C}N^{4/3}(\delta\lambda h)^{2/3}, & \text{when } Nh^{1/2} \le \delta\lambda \le 2Nh^{1/2}, \\ 0, & \text{when } \delta\lambda \ge 2Nh^{1/2}, \end{cases} \tag{2.29}$$

where for the second bound we used the fact that $(\delta\lambda)^2 \ge (\delta\lambda h)^{2/3}$ when $\delta\lambda \ge h^{1/2}$, and for the third bound the fact that $(\delta\lambda)^2 \ge N^{4/3}(\delta\lambda h)^{2/3}$ when $\delta\lambda \ge Nh^{1/2}$.

Let us now study the derivatives of $\widetilde{G}_{\text{int}}$. Since $\widetilde{G}_{\text{int}} = G_{\text{int}}$ when $\delta\lambda \le 2Nh^{1/2}$ we get that

$$\partial^k \widetilde{G}_{\text{int}} = \mathcal{O}(\delta^{(2-k)_+}) \quad \text{when } \delta\lambda \le 2Nh^{1/2}. \tag{2.30}$$

When $2Nh^{1/2} \le \delta\lambda \le 16Nh^{1/2}$, $\widetilde{G}_{\text{int}}$ inherits the properties of $\widehat{p}_1$, i.e. $\partial^k\widetilde{G}_{\text{int}} = \mathcal{O}((Nh^{1/2})^{(2-k)})$ which yields

$$\partial^k \widetilde{G}_{\text{int}} = C_k(N)\mathcal{O}(h(h\delta\lambda)^{-k/3}) \quad \text{when } h^{1/2} \le \delta\lambda \le \mathcal{O}(1), \tag{2.31}$$

(of course this estimate is true when $\delta\lambda \ge 16Nh^{1/2}$ since $\widetilde{G}_{\text{int}}$ is zero there).

Construction of a modified G_{out}. For the same χ as before define

$$\tilde{G}_{\text{out}}(\rho) \stackrel{\text{def}}{=} G_{\text{out}}(\rho)\left(1 - \chi\left(\frac{\delta(\rho)\lambda(\rho)}{Nh^{1/2}}\right)\right).$$

We notice that $\tilde{G}_{\text{out}} = G_{\text{out}}$ when $\lambda\delta \ge 2Nh^{\frac{1}{2}}$ and $\tilde{G}_{\text{out}} = 0$ when $\lambda\delta \le Nh^{\frac{1}{2}}$. Therefore we have directly

$$p_1 + \epsilon H_{p_2}\tilde{G}_{\text{out}} \ge \begin{cases} \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{for } \lambda\delta \ge 2Nh^{\frac{1}{2}}, \\ 0, & \text{when } \delta\lambda \le Nh^{1/2}. \end{cases} \tag{2.32}$$

In the area $Nh^{\frac{1}{2}} \le \lambda\delta \le 2Nh^{\frac{1}{2}}$, we can write uniformly in $N \ge 1$ that

$$\begin{aligned} p_1 + \epsilon H_{p_2}\tilde{G}_{\text{out}} &\ge (p_1 + \epsilon H_{p_2}G_{\text{out}})(1-\chi) - \epsilon G_{\text{out}}H_{p_2}\chi \\ &\ge -\epsilon\,|G_{\text{out}}|\,.\,|H_{p_2}\chi|\,. \end{aligned} \tag{2.33}$$

We know that $G_{\text{out}} = \mathcal{O}(h)$ and that $|H_{p_2}\chi| \le |\partial p_2|\,|\partial\chi| = \mathcal{O}(\delta\lambda\frac{1}{Nh^{1/2}})$. This gives

$$|G_{\text{out}}|\,|H_{p_2}\chi| = \mathcal{O}\left(\frac{\delta\lambda h^{1/2}}{N}\right).$$

Now using the fact that $\delta\lambda = (\delta\lambda)^{2/3}(\delta\lambda)^{1/3} \le (\delta\lambda)^{2/3}(2N)^{1/3}h^{1/6}$, we deduce that uniformly in $N \ge 1$,

$$|G_{\text{out}}|\,|H_{p_2}\chi| = \mathcal{O}\left(\frac{(\delta\lambda h)^{2/3}}{N^{2/3}}\right) = \mathcal{O}\left((\delta\lambda h)^{2/3}\right).$$

Using this and (2.33) we find that

$$p_1 + \epsilon H_{p_2} \tilde{G}_{\text{out}} \geq -\epsilon \mathcal{O}\left((\delta\lambda h)^{2/3} \right), \quad \text{when } Nh^{\frac{1}{2}} \leq \lambda\delta \leq 2Nh^{\frac{1}{2}}.$$

Eventually we have the following bounds in the whole region $\delta\lambda \leq \mathcal{O}(1)$:

$$p_1 + \epsilon H_{p_2} \widetilde{G}_{\text{out}} \geq \begin{cases} 0, & \text{when } \delta\lambda \leq Nh^{1/2}, \\ -C\epsilon(\delta\lambda h)^{2/3}, & \text{when } Nh^{1/2} \leq \delta\lambda \leq 2Nh^{1/2}, \\ \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{when } \delta\lambda \geq 2Nh^{1/2}. \end{cases} \tag{2.34}$$

For the derivatives of $\widetilde{G}_{\text{out}}$ we can write immediately

$$\partial^k \widetilde{G}_{\text{out}} = 0 = \mathcal{O}(\delta^{(2-k)_+}), \quad \text{when } \delta\lambda \leq h^{1/2}, \tag{2.35}$$

since $\widetilde{G}_{\text{out}} = 0$ there. In the intermediate region we check that

$$\partial^k \left(\chi \left(\frac{\delta\lambda}{Nh^{1/2}} \right) \right) = C_k(N) \mathcal{O}(h^{-k/2}) = C'_k(N) \mathcal{O}((\delta\lambda h)^{-k/3})$$

since $\delta\lambda \leq 2Nh^{1/2}$. Of course the same estimate is true in the larger region $\{h^{1/2} \leq \delta\lambda\} \cap \mathcal{B}$, since χ is compactly supported. Now using the fact that $\partial^k G_{\text{out}} = \mathcal{O}(h(\delta\lambda h)^{-k/3})$, we get the same estimate for $\tilde{G}_{\text{out}}$

$$\partial^k \tilde{G}_{\text{out}} = C_k(N) \mathcal{O}(h(\delta\lambda h)^{-k/3}), \quad \text{when } h^{1/2}\delta\lambda \leq \mathcal{O}(1). \tag{2.36}$$

The construction of $\tilde{G}_{\text{out}}$ is complete.

Construction of the weight function G. We finally pose

$$G = (\widetilde{G}_{\text{in}} + \widetilde{G}_{\text{out}})/2. \tag{2.37}$$

Using the bounds (2.30, 2.31, 2.35, 2.36) for the derivatives of $\widetilde{G}_{\text{in}}$ and $\widetilde{G}_{\text{out}}$ we immediately get that

$$\partial^k G = \begin{cases} \mathcal{O}(\delta^{(2-k)_+}), & \text{when } \delta\lambda \leq h^{1/2}, \\ C'_k(N) \mathcal{O}(h(\delta\lambda h)^{-k/3}), & \text{in } \mathcal{B} \text{ when } h^{1/2} \leq \delta\lambda \leq \mathcal{O}(1), \end{cases} \tag{2.38}$$

i.e. the bounds given in the first two estimates of (2.1). On the other hand, combining (2.29) and (2.34) gives

$$2p_1 + 2\epsilon H_{p_2} G \geq \begin{cases} \frac{\epsilon}{C}(\delta\lambda)^2, & \text{when } \delta\lambda \leq h^{1/2}, \\ \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{when } h^{1/2} \leq \delta\lambda \leq Nh^{1/2}, \\ \left(\frac{\epsilon}{C} N^{4/3} - C\epsilon\right)(\delta\lambda h)^{2/3}, & \text{when } Nh^{1/2} \leq \delta\lambda \leq 2Nh^{1/2}, \\ \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{when } 2Nh^{1/2} \leq \delta\lambda \mathcal{O}(1). \end{cases}$$

Taking N sufficiently large and fixed from now on, and dividing by 2 gives with a new constant C

$$p_1 + \epsilon H_{p_2} G \geq \begin{cases} \frac{\epsilon}{C}(\delta\lambda)^2, & \text{when } \delta\lambda \leq h^{1/2}, \\ \frac{\epsilon}{C}(\delta\lambda h)^{2/3}, & \text{when } h^{1/2} \leq \delta\lambda \leq \mathcal{O}(1). \end{cases} \tag{2.39}$$

Let us now prove (2.2). This was already proven in (2.6) in the region $\delta\lambda \leq h^{1/2}$ since $G = G_T$ there. In the region $h^{1/2} \leq \delta\lambda \leq \mathcal{O}(1)$ we follow the same procedure. We write

$$\widetilde{p}(\rho) = p(\rho + i\epsilon H_G(\rho)) = p(\rho) - i\epsilon H_p G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2),$$

with real part given by

$$\widetilde{p}_1 = p_1(\rho) + \epsilon H_{p_2}G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2) = p_1(\rho) + \epsilon H_{p_2}G(\rho) + \epsilon^2\mathcal{O}((\delta\lambda h)^{2/3}),$$

since $\nabla G = \mathcal{O}((\delta\lambda h)^{1/3})$ by Lemma 2.2 and the fact that $\delta\lambda \geq h^{1/2}$. Using (2.39) and taking ϵ small enough yields

$$\widetilde{p}_1 \geq \frac{\epsilon}{C}(\delta\lambda h)^{2/3}.$$

For the imaginary part $\widetilde{p}_2$ we directly write

$$\widetilde{p}_2 = p_2(\rho) - \epsilon H_{p_1}G(\rho) + \mathcal{O}(\epsilon^2|\nabla G|^2) = \mathcal{O}(\delta^2).$$

This completes the proof of Proposition 2.1 in the region $\delta\lambda \leq \mathcal{O}(1)$.

End of the proof of Proposition 2.1. We now work outside $\mathcal{B}$. We first observe that the estimate (2.25) remains valid, therefore in the region $\{\rho;\quad Mp_1(\rho) \leq (h\delta(\rho)\lambda(\rho))^{2/3}\}$ we get (2.3) from (2.24) and (2.21). In the region $\{\rho;\quad Mp_1(\rho) \geq (h\delta(\rho)\lambda(\rho))^{2/3}\}$ we use (2.20) and for ϵ small enough we get

$$p_1 + \epsilon H_{p_2}G \geq \frac{\epsilon}{C}(p_1 + (h\delta\lambda)^{2/3}).$$

The proof of Proposition 2.1 is complete. □

3 Review of FBI tools

The aim of this section is to review the definitions about the FBI transform and the spaces associated to a function G satisfying the estimates of Proposition 2.1 in a bounded region and equal to 0 elsewhere. Note in particular that it has its second derivative bounded. The material here is essentially taken from [16]. In this section, and in Sections 4 and 6, we suppose that the symbol p satisfies hypothesis **(H1)** and is bounded with all its derivatives everywhere.

Definitions and main properties

Let T be a FBI-Bargmann transform:

$$Tu(x) = Ch^{-\frac{3n}{4}} \int e^{\frac{i}{h}\varphi(x,y)}u(y)dy, \tag{3.1}$$

where we may choose $\varphi(x,y) = \frac{i}{2}(x-y)^2$ as in the standard Bargmann transform. Other quadratic φ with the general properties reviewed in [17] are also possible. The associated canonical transformation is given by

$$\kappa_T : (y, -\partial_y\varphi(x,y)) \mapsto (x, \partial_x\varphi(x,y)). \tag{3.2}$$

We have the associated IR-space (see [17] for the terminology),

$$\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n}),\ \Phi_0(x) = -\mathrm{Im}\,\varphi(x, y_0(x)), \tag{3.3}$$

where y_0 is the point where $\mathbb{R}^n \ni y \mapsto -\mathrm{Im}\,\varphi(x, y)$ takes its non-degenerate maximum.

If $P = p^w$, then by the metaplectic invariance,

$$TP = \widehat{P}T,\ \widehat{P} = \widehat{p}^w, \tag{3.4}$$

we have the exact symbol relation:

$$\widehat{p} \circ \kappa_T = p. \tag{3.5}$$

Shortly, we will recall the definition of the Weyl quantization on the FBI-transform side.

From now on, we work entirely on the FBI-side, and we shall write P instead of $\widehat{P}$ and similarly for the symbols. We introduce the spaces $L^2_{\Phi_0} = L^2(\mathbb{C}^n; e^{-2\Phi_0/h}L(dx))$, where $L(dx)$ is the Lebesgue measure, and H_{Φ_0} the subspace of entire functions. The Weyl-quantization on H_{Φ_0} takes the form of a contour integral

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x+y}{2})} e^{i(x-y)\cdot\theta/h} p(\frac{x+y}{2}, \theta; h) u(y) dy d\theta. \tag{3.6}$$

By p, we also denote an almost holomorphic extension of p to a tubular neighborhood of Λ_{Φ_0}. If we introduce a $\mathcal{C}^\infty$ function ψ_0 equal to 1 near 0, we get for $u \in H_{\Phi_0}$:

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x+y}{2})} e^{i(x-y)\cdot\theta/h} \psi_0(x-y) p(\frac{x+y}{2}, \theta; h) u(y) dy d\theta + R_1 u(x),$$

where $R_1 = \mathcal{O}(h^\infty) : L^2_{\Phi_0} \to L^2_{\Phi_0}$. We make a contour deformation:

$$\Gamma_t \stackrel{\mathrm{def}}{=} \left\{ \theta = \frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x+y}{2}) + it(\overline{x-y}) \right\},\ 0 \le t \le t_0,\ t_0 > 0.$$

Stokes' formula gives,

$$\begin{aligned} Pu(x) = &\frac{1}{(2\pi h)^n} \iint_{\Gamma_{t_0}} e^{\frac{i}{h}(x-y)\cdot\theta} \psi_0(x-y) p(\frac{x+y}{2}, \theta; h) u(y) dy d\theta \\ &+ \frac{1}{(2\pi h)^n} \iiint_{\Gamma_{[0,t_0]}} e^{\frac{i}{h}(x-y)\cdot\theta} u(y) \overline{\partial}_{y,\theta}(\psi_0(x-y) p(\frac{x+y}{2}, \theta; h)) \wedge dy \wedge d\theta + R_1 u(x), \end{aligned}$$

where $\Gamma_{[0,t_0]}$ is the naturally defined union of all the Γ_t for $t \in [0, t_0]$. The effective kernel of the first integral, viewed as an operator on $L^2_{\Phi_0}$, is $\mathcal{O}(h^{-n})e^{-\frac{t_0}{h}|x-y|^2}$, which implies that this integral does indeed define a uniformly bounded operator: $L^2_{\Phi_0} \to L^2_{\Phi_0}$. The effective kernel of the second integral can be estimated by a constant times

$$\begin{aligned} &\int_0^{t_0} h^{-n} e^{-\frac{t}{h}|x-y|^2} \mathrm{dist}\,((\frac{x+y}{2}, \theta), \Lambda_{\Phi_0})^\infty dt \\ &\quad = \mathcal{O}(1) \int_0^{t_0} h^{-n} e^{-\frac{t}{h}|x-y|^2} (t|x-y|)^\infty dt = \mathcal{O}(h^\infty). \end{aligned}$$

We conclude that

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint_{\theta=\frac{2}{i}\frac{\partial\Phi_0}{\partial x}(\frac{x+y}{2})+it_0\overline{(x-y)}} e^{\frac{i}{h}(x-y)\cdot\theta}\psi_0(x-y)p(\frac{x+y}{2},\theta;h)u(y)dyd\theta + R_2u, \quad (3.7)$$

for $u \in H_{\Phi_0}$, where $R_2 = \mathcal{O}(h^\infty) : L^2_{\Phi_0} \to L^2_{\Phi_0}$.

The aim of the next subsections is to introduce and study a new strictly subharmonic function Φ_ϵ related to G. As for Φ_0, the function Φ_ϵ is associated to the space $L^2_{\Phi_\epsilon} = L^2(\mathbb{C}^n; e^{-2\Phi_\epsilon/h}L(dx))$ and its subspace of entire functions H_{Φ_ϵ}. These spaces will be used later to get local resolvent estimates.

Definition and derivative estimates of Φ_ϵ

Recall that our weight function $G(\rho)$, $\rho = (y,\eta)$ defined in Proposition 2.1 satisfies the estimates in the region $\delta\lambda \le \mathcal{O}(1)$

$$\nabla^k G = \mathcal{O}(\delta^{(2-k)_+}),\ \delta(\rho) \le \sqrt{h}, \quad (3.8)$$

$$\nabla^k G = \mathcal{O}(h(h\delta)^{-\frac{k}{3}}),\ \delta(\rho) \ge \sqrt{h}. \quad (3.9)$$

It follows that in the same region

$$\nabla^k G = \mathcal{O}(hr^{-k}), \quad (3.10)$$

where

$$r(\rho) := h^{\frac{1}{3}}(h^{\frac{1}{2}} + \delta(\rho))^{\frac{1}{3}}. \quad (3.11)$$

Notice that

$$h^{\frac{1}{2}} \le r \le h^{\frac{1}{2}} + \delta, \quad (3.12)$$

so that $h^{\frac{1}{2}} + \delta(\rho)$ is uniformly of constant order of magnitude in $B(\rho_0, \frac{1}{C_0}r(\rho_0))$ if $C_0 > 0$ is large enough and independent of ρ_0.

In $B(\rho_0, \frac{1}{C_0}r(\rho_0))$ we introduce the scaled variables $\widetilde{\rho}$, by

$$\rho = \rho_0 + r_0\widetilde{\rho},\ r_0 = r(\delta_0). \quad (3.13)$$

Then the scaled function $G(\rho_0 + r_0\widetilde{\rho})$ satisfies

$$\nabla^k_{\widetilde{\rho}}(G(\rho_0 + r_0\widetilde{\rho})) = \mathcal{O}(h),\ |\widetilde{\rho}| < \frac{1}{C_0}. \quad (3.14)$$

Let

$$\mathrm{Im}\,(y,\eta) = \epsilon H_G(\mathrm{Re}\,(y,\eta)). \quad (3.15)$$

Then for $\epsilon > 0$ small enough, we have

$$\kappa_T(\Lambda_{\epsilon G}) = \Lambda_{\Phi_\epsilon} \overset{\text{def}}{=} \left\{(x,\xi) \in \mathbb{C}^{2n};\ \ \xi = \frac{2}{i}\frac{\partial\Phi_\epsilon}{\partial x}(x)\right\}, \quad (3.16)$$

where $\Phi_\epsilon(x)$ is a critical value w.r.t. (y,η),

$$\Phi_\epsilon(x) = \text{v.c.}_{(y,\eta)\in\mathbb{C}^n\times\mathbb{R}^n}(-\mathrm{Im}\,\varphi(x,y) - (\mathrm{Im}\,y)\cdot\eta + \epsilon G(\mathrm{Re}\,y,\eta)). \quad (3.17)$$

We note that, when $\epsilon = 0$, the unique critical point is non-degenerate.

We are in the presence of the following general problem (where we change and simplify the notation), namely to study the critical value

$$\Phi_\epsilon(x) = \text{v.c.}_y F_\epsilon(x,y),\ x \in \mathbb{R}^n,\ y \in \mathbb{R}^n, \tag{3.18}$$

where $F_\epsilon(x,y)$ is a smooth real-valued function such that

$$y \mapsto F_0(x,y) \text{ has a unique non-degenerate critical point } y_0(x), \tag{3.19}$$

$$\partial_\epsilon^2 F_\epsilon(x,y) = 0, \tag{3.20}$$

$$\partial_x^\alpha \partial_y^\beta \partial_\epsilon F_\epsilon(x,y) = \mathcal{O}(hr^{-|\beta|}), \tag{3.21}$$

where $r = h^{\frac{1}{3}}(h^{\frac{1}{2}} + \delta(y))^{\frac{1}{3}}$ and $\delta(y) \geq 0$ is a Lipschitz function. From (3.20)–(3.21) we see that

$$\partial_y F_\epsilon - \partial_y F_0 = \mathcal{O}(\frac{h\epsilon}{r}) \ll \epsilon,\ \partial_y^2 F_\epsilon - \partial_y^2 F_0 = \mathcal{O}(\frac{h}{r^2}\epsilon) \ll 1,$$

for $\epsilon \ll 1$. So, for $0 \leq \epsilon \leq \epsilon_0 \ll 1$, we see that $y \mapsto F_\epsilon(x,y)$ has a unique critical point $y_\epsilon(x)$, depending smoothly on (x,ϵ).

In order to estimate the derivatives of $y_\epsilon(x)$ we work in an r_0-neighborhood of a variable point $(x_0,y_0) = (x_0, y_0(x_0))$, $r_0 = r(\delta(y_0))$, and put $x = x_0 + r_0\widetilde{x}$, $y_\epsilon(x) = y_0(x_0 + r_0\widetilde{x}) + r_0\widetilde{y}_\epsilon(\widetilde{x})$, with $\widetilde{y}_0(\widetilde{x}) = 0$ where we hope that $\widetilde{y}_\epsilon = \mathcal{O}(\epsilon)$. Then $\widetilde{y}_\epsilon(\widetilde{x})$ is the critical point of

$$\widetilde{y} \mapsto \frac{1}{r_0^2}\big(F_\epsilon(x_0 + r_0\widetilde{x}, y_0(x_0 + r_0\widetilde{x}) + r_0\widetilde{y}) - F_0(x_0 + r_0\widetilde{x}, y_0(x_0 + r_0\widetilde{x}))\big) =: G_\epsilon(\widetilde{x},\widetilde{y}), \tag{3.22}$$

with

$$\partial_{\widetilde{x}}^\alpha \partial_{\widetilde{y}}^\beta G_\epsilon = \mathcal{O}(1),\ \partial_{\widetilde{x}}^\alpha \partial_{\widetilde{y}}^\beta \partial_\epsilon G_\epsilon(\widetilde{x},\widetilde{y}) = \mathcal{O}(\frac{h}{r_0^2}), \tag{3.23}$$

$$\partial_{\widetilde{y}} G_0(\widetilde{x}, 0) = 0,\ |\det \partial_{\widetilde{y}}^2 G_\epsilon| \geq 1/C. \tag{3.24}$$

Introducing the rescaled parameter $\widetilde{\epsilon}$ by $\epsilon = \frac{r_0^2}{h}\widetilde{\epsilon}$, $\partial_{\widetilde{\epsilon}} = \frac{r_0^2}{h}\partial_\epsilon$, we have uniform bounds on all the derivatives $\partial_{\widetilde{x}}^\alpha \partial_{\widetilde{y}}^\beta \partial_{\widetilde{\epsilon}}^\gamma G_\epsilon$ while $\partial_{\widetilde{y}}^2 G_\epsilon$ is uniformly non-degenerate, and the same is therefore true about $\partial_{\widetilde{x}}^\alpha \partial_{\widetilde{\epsilon}}^\gamma \widetilde{y}_\epsilon(\widetilde{x})$, so

$$\partial_{\widetilde{x}}^\alpha \partial_\epsilon^\gamma \widetilde{y}_\epsilon(\widetilde{x}) = \mathcal{O}((\frac{h}{r_0^2})^\gamma),$$

$$\partial_x^\alpha \partial_\epsilon^\gamma (y_\epsilon(x) - y_0(x)) = \mathcal{O}(r(\frac{h}{r^2})^\gamma r^{-|\alpha|}),\ r = r(x) = h^{\frac{1}{3}}(h^{\frac{1}{2}} + \delta(y_0(x)))^{\frac{1}{3}}). \tag{3.25}$$

The critical value $G_\epsilon(\widetilde{x}, \widetilde{y}_\epsilon(x))$ also satisfies $\partial_{\widetilde{x}}^\alpha \partial_{\widetilde{\epsilon}}^\gamma (G_\epsilon(\widetilde{x}, \widetilde{y}_\epsilon(x))) = \mathcal{O}(1)$, so

$$\partial_x^\alpha \partial_\epsilon^\gamma (F_\epsilon(x, y_\epsilon(x)) - F_0(x, y_0(x))) = \mathcal{O}(r^2(\frac{h}{r^2})^\gamma r^{-|\alpha|}). \tag{3.26}$$

We can Taylor expand this with respect to ϵ and get

$$F_\epsilon(x, y_\epsilon(x)) = F_0(x, y_0(x)) + F_1(x)\epsilon + F_2(x)\epsilon^2 + ... + F_{N-1}(x)\epsilon^{N-1} + R_N(x,\epsilon)\epsilon^N,$$

where

$$F_1(x) = ((\partial_\epsilon)_{\epsilon=0} F_\epsilon)(x, y_0(x)),$$
$$\partial_x^\alpha F_k(x) = \mathcal{O}(r^2 (\frac{h}{r^2})^k r^{-|\alpha|}),\ k \geq 1,$$
$$\partial_x^\alpha \partial_\epsilon^\gamma R_N(x,\epsilon) = \mathcal{O}(r^2 (\frac{h}{r^2})^{N+\gamma} r^{-|\alpha|}).$$

Returning to (3.17), we get

$$\Phi_\epsilon(x) = \Phi_0(x) + \Phi_1(x)\epsilon + ... + \Phi_{N-1}(x)\epsilon^{N-1} + R_N(x,\epsilon)\epsilon^N, \tag{3.27}$$

where $\Phi_1, ..., \Phi_{N-1}, R_N$ satisfy the same estimates and

$$\Phi_1(x) = G(y(x), \eta(x)),\ (y(x), \eta(x)) = \kappa_T^{-1}(x, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x)).$$

Study of P as an operator on H_{Φ_ϵ}

Recall that H_{Φ_ϵ} is the subspace of entire functions of $L^2_{\Phi_\epsilon} = L^2(\mathbb{C}^n; e^{-2\Phi_\epsilon/h} L(dx))$. Since $\Phi_\epsilon - \Phi_0 = \mathcal{O}(\epsilon h)$, we first notice that

$$e^{-C\epsilon} \leq \|u\|_{\Phi_\epsilon} / \|u\|_{\Phi_0} \leq e^{C\epsilon},$$

and hence for instance

$$R_1 = \mathcal{O}(1) e^{C\epsilon} h^\infty : L^2_{\Phi_\epsilon} \to L^2_{\Phi_\epsilon}.$$

Similarly, the effective kernel of the integral in (3.7) as an operator: $L^2_{\Phi_\epsilon} \to L^2_{\Phi_\epsilon}$ can be estimated by

$$\mathcal{O}(h^{-n}) e^{-\frac{t_0}{h}|x-y|^2 + 2C\epsilon},$$

corresponding to an operator of norm $\mathcal{O}(1)e^{2\epsilon C} : L^2_{\Phi_\epsilon} \to L^2_{\Phi_\epsilon}$.

With the previous t_0 fixed, we now make the new contour deformation:

$$\Gamma_t \stackrel{\text{def}}{=} \left\{ \theta = \frac{2}{i}\frac{\partial}{\partial x}((1-t)\Phi_0 + t\Phi_\epsilon)(\frac{x+y}{2}) + it_0\overline{(x-y)} \right\}, \quad 0 \leq t \leq 1.$$

Along this contour we have, using (3.26), (3.12):

$$\overline{\partial}_{y,\theta} \psi_0(x-y) p(\frac{x+y}{2}, \theta; h) = \mathcal{O}(1)(|x-y| + \epsilon \frac{h}{r(\frac{x+y}{2})})^\infty \leq \mathcal{O}(1)(|x-y| + \epsilon h^{1/2})^\infty.$$

By Stokes' formula, we see that

$$Pu(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}(\frac{x+y}{2}) + it_0\overline{(x-y)}} e^{\frac{i}{h}(x-y)\cdot\theta} \psi_0(x-y) p(\frac{x+y}{2}, \theta; h) u(y) dy d\theta + R_\epsilon u, \tag{3.28}$$

for $u \in H_{\Phi_\epsilon}$, where

$$R_\epsilon = \mathcal{O}(1)(e^{C\epsilon} h^\infty + h^\infty) :\ L^2_{\Phi_\epsilon} \to L^2_{\Phi_\epsilon}. \tag{3.29}$$

Quantization vs. multiplication

The aim of this short subsection is to check formula (3.30) below i.e. the equivalent of [16, formula 1.6] for the Weyl quantization. Recall that the I-lagrangian manifold $\Lambda_{\epsilon G}$ is defined by $\Lambda_{\epsilon G} = \left\{\rho + i\epsilon H_G(\rho); \;\; \rho \in \mathbb{R}^{2n}\right\}$, and that G has bounded second derivatives. We also have $\kappa_T(\Lambda_{\epsilon G}) = \Lambda_{\Phi_\epsilon} \stackrel{\text{def}}{=} \left\{\xi = \xi_\epsilon(x) \stackrel{\text{def}}{=} \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}(x)\right\}$. Notice that the second derivatives of Φ_ϵ, and the first ones of $\xi_\epsilon(x)$ are bounded. Recall that p is (an almost analytic extension of) a $\mathcal{C}^\infty$ symbol with all its derivatives bounded. We get for $u \in H_{\Phi_\epsilon}$

$$Pu(x) = \frac{1}{(2\pi h)^n}\iint_{\Gamma_\epsilon} e^{\frac{i}{h}(x-y)\cdot\theta}\psi_0(x-y)p(\frac{x+y}{2},\theta)u(y)dyd\theta + R_\epsilon u,$$

where $\Gamma_\epsilon = \left\{\theta = \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}(\frac{x+y}{2}) + it_0\overline{(x-y)}\right\}$ is the contour of integration and $R_\epsilon = \mathcal{O}(h^\infty) : L^2_{\Phi_\epsilon} \longmapsto L^2_{\Phi_\epsilon}$. Sometimes we omit the subscript ϵ.

We want to prove that for h sufficiently small

$$(\chi Pu|u)_{H_{\Phi_\epsilon}} = \int p_\epsilon |u|^2 \chi(x) e^{-2\Phi_\epsilon(x)/h} L(dx) + \mathcal{O}(h)\|u\|^2, \tag{3.30}$$

where χ has bounded derivatives (for example $\chi = 1$) and we define $p_\epsilon = p(x, \frac{2}{i}\frac{\partial\Phi_\epsilon}{\partial x}(x))$ to be the restriction of p to Λ_{Φ_ϵ}. The proof is a simple adaptation of the proof given in [16]. We first make the Taylor expansion of p,

$$p(\frac{x+y}{2},\theta) = p(x,\xi(x)) + \sum p^{(j)}(x,\xi(x))(\theta_j - \xi_j(x)) + \sum p_{(j)}(x,\xi(x))\left(\frac{y_j - x_j}{2}\right) + r(x,y,\theta). \tag{3.31}$$

On $\Gamma_\epsilon(x)$ we have $(\theta_j - \xi_j(x)) = iC\overline{(x-y)}$, and $r(x,y,\theta) = \mathcal{O}(|x-y|^2 + h^\infty)$. The effective kernel of the operator R corresponding to r is therefore of the form

$$\begin{aligned} |R(x,y)| &= \mathcal{O}\left(h^{-n}e^{-C|x-y|^2/2h}|x-y|^2\tilde{\psi}(x-y)\right) \\ &= \mathcal{O}\left(h^{-n}e^{-C|x-y|^2/2h}|x-y|^2/h\right)h, \end{aligned} \tag{3.32}$$

for C sufficiently large (since the second derivative of Φ is bounded). As a consequence

$$\operatorname{Re}(Ru|u)_{\Phi_\epsilon} = \mathcal{O}(h)\|u\|^2_{\Phi_\epsilon}. \tag{3.33}$$

For the contribution to (3.30) from the second term of (3.31) we integrate by part as in [16] and we see that this term is $\mathcal{O}(h)\|u\|^2_{\Phi_\epsilon}$. For the third term we simply write

$$\begin{aligned} &(2\pi h)^{-n}\iint_\Gamma e^{\frac{i}{h}(x-y)\cdot\theta}\left(\sum p_{(j)}(x,\xi(x))\frac{y_j - x_j}{2}\right)u(y)dyd\theta \\ &= \sum p_{(j)}(x,\xi(x))\,(x_j/2 - x_j/2)\,u(x) = 0. \end{aligned}$$

It follows that we have (3.30).

Notice that we can take χ to be Lipschitz in (3.30), and hence that relation can be iteratated to give:

$$(\chi Pu|Pu)_{\Phi_\epsilon} = \int |p_\epsilon|^2|u|^2\chi(x)e^{-2\Phi_\epsilon(x)/h}L(dx) + \mathcal{O}(h)\|u\|^2. \tag{3.34}$$

4 Local resolvent estimates for large z.

Again in this section we suppose that p satisfies the hypothesis **(H1)** and that it is bounded with all its derivatives outside a large compact set. The aim of this section is to get resolvent estimates for functions localized near the critical points on the FBI side, and for

$$h \ll |z|. \tag{4.1}$$

We realize P as an operator with leading symbol $p_\epsilon = p_{|_{\Lambda_{\epsilon G}}}$ as $TPT^{-1} : H_{\Phi_\epsilon} \to H_{\Phi_\epsilon}$, with $\Lambda_{\Phi_\epsilon} = \kappa_T(\Lambda_{\epsilon G})$, and in the following we identify P with TPT^{-1}. We have seen that $\nabla^2\Phi_\epsilon$ is uniformly bounded and consequently (see (3.30)) we have with $\Phi = \Phi_\epsilon$ and scalar products and norms in $L^2_{\Phi_\epsilon}$:

$$(\chi P u|u) = \int p_\epsilon |u|^2 \chi(x) e^{-2\Phi(x)/h} L(dx) + \mathcal{O}(h)\|u\|^2, \tag{4.2}$$

where $p_\epsilon = p_{|_{\Lambda_{\epsilon G}}}$ is viewed as a function on Λ_{Φ_ϵ}, and $\chi(x) \in C_0^\infty(\mathbb{C}^n)$. We replace in this section the small parameter h in the construction of the function G by Ah where A is some large constant. As a consequence for ϵ fixed we get from Proposition 2.1 that $p_\epsilon(\rho)$ satisfies the estimates

$$\operatorname{Re} p_\epsilon(\rho) \geq \frac{1}{C_0} \min(\delta(\rho)^2, (Ah)^{\frac{2}{3}} \delta(\rho)^{\frac{2}{3}}), \tag{4.3}$$

inside a large compact set K containing the support of χ. From now on the inequalities we give are to be understood in K. Note that $C_0 > 0$ and the uniform estimate on $\nabla^2\Phi_\epsilon$ do not depend on A.

Let $\chi_0 \in C_0^\infty(\mathbb{R}, [0,1])$ be a standard cutoff to a neighborhood of $0 \in \mathbb{R}$ and consider

$$\widetilde{p_\epsilon}(\rho) = p_\epsilon(\rho) + \frac{1}{C_0} \min(|z|, (Ah)^{\frac{2}{3}} |z|^{\frac{1}{3}}) \chi_0(\frac{\delta(\rho)^2}{|z|}). \tag{4.4}$$

Then there exists a $C_1 > 0$ such that

$$\operatorname{Re} \widetilde{p_\epsilon}(\rho) \geq \frac{1}{C_1} (\min(\delta(\rho)^2, (Ah)^{\frac{2}{3}} \delta(\rho)^{\frac{2}{3}})) + \min(|z|, (Ah)^{\frac{2}{3}} |z|^{\frac{1}{3}})). \tag{4.5}$$

Let us mention for further use that we can choose the support of χ_0 to be contained in a sufficiently small neighborhood of 0, so that

$$|p_\epsilon(\rho) - z| \geq |z|/C_2, \text{ when } \chi_0(\frac{\delta(\rho)^2}{|z|}) \neq 0. \tag{4.6}$$

Write

$$\Lambda^2 \stackrel{\text{def}}{=} \min(\delta(\rho)^2, (Ah)^{\frac{2}{3}} \delta(\rho)^{\frac{2}{3}}), \quad \text{ and } \quad Z \stackrel{\text{def}}{=} \min(|z|, (Ah)^{\frac{2}{3}} |z|^{\frac{1}{3}}).$$

and denote

$$\chi_{|z|}(\rho) \stackrel{\text{def}}{=} \chi_0(\frac{\delta(\rho)^2}{|z|}),$$

then (4.4–4.5) can be written as

$$p_\epsilon + \frac{Z}{C_0} \chi_{|z|} \geq \frac{1}{C_1} (\Lambda^2 + Z).$$

Considering $\chi_{|z|}$ as a function of x on the FBI-transform side, we get from (4.2)

$$\mathrm{Re}\,(\chi(P+\frac{Z}{C_0}\chi_{|z|}-z)u|u)+\mathcal{O}(h)\|u\|^2 \geq \frac{1}{C_3}\left(\int \chi\Lambda^2|u|^2e^{-2\Phi/h}L(dx)+Z(\chi u,u)\right), \tag{4.7}$$

provided that χ is nonnegative and (in addition to (4.1)):

$$\mathrm{Re}\, z \leq Z/C_3. \tag{4.8}$$

Here $C_3>0$ is some sufficiently large constant which is independent of A, and $\chi_{|z|}$ in (4.7) denotes the natural multiplication operator on the FBI-side.

We shall combine (4.7) with an estimate for $(\chi_{|z|}u|u)$, that we shall obtain using the ellipticity property (4.6). This will be obtained using an estimate analogous to (4.2) (that can also be found in [16]) but since the support of $\chi_{|z|}$ may be very small we shall use a rescaling which also dilates the Planck constant.

Proposition 4.1 *Under the assumptions (4.1,4.8) we have*

$$\|\chi_{|z|}u\| \leq C(\frac{1}{|z|}\|(P-z)u\|+\sqrt{\frac{h}{\min(1,|z|)}}\|u\|), \tag{4.9}$$

for all $u\in H_{\Phi_\epsilon}$.

Proof. First assume $|z|\leq 1$. Make the change of variables on the FBI-transform side

$$x=|z|^{\frac{1}{2}}\widetilde{x},\ hD_x=|z|^{\frac{1}{2}}\widetilde{h}D_{\widetilde{x}},\ \widetilde{h}=\frac{h}{|z|}. \tag{4.10}$$

Then,

$$P(x,hD_x;h)-z=|z|(\widetilde{P}(\widetilde{x},\widetilde{h}D_{\widetilde{x}};\widetilde{h})-\widetilde{z}), \tag{4.11}$$

$$\widetilde{z}=\frac{z}{|z|},\ \widetilde{P}(\widetilde{x},\widetilde{\xi};\widetilde{h})=\frac{1}{|z|}P(x,\xi;h),\ (x,\xi)=|z|^{\frac{1}{2}}(\widetilde{x},\widetilde{\xi}). \tag{4.12}$$

If $P(x,\xi;h)=p(x,\xi)+hp_1(x,\xi)+h^2p_2(x,\xi)+...$, (where we now consider the symbols in the complex domain), we see that

$$\widetilde{P}(\widetilde{x},\widetilde{\xi};\widetilde{h})\sim\sum_0^\infty \widetilde{p}_j(\widetilde{x},\widetilde{\xi})\widetilde{h}^j,$$

where $\widetilde{p}=\widetilde{p}_0=\frac{1}{|z|}p(|z|^{\frac{1}{2}}(\widetilde{x},\widetilde{\xi}))$, $\widetilde{p}_j(\widetilde{x},\widetilde{\xi})=|z|^{j-1}p_j(|z|^{\frac{1}{2}}(\widetilde{x},\widetilde{\xi}))$ are nice bounded symbols, since $p(x,\xi)=\mathcal{O}((x,\xi)^2)$. Then using (4.11)

$$\frac{1}{|z|}(P(x,hD_x;h)-z)=(\widetilde{P}(\widetilde{x},\widetilde{h}D_{\widetilde{x}};\widetilde{h})-\widetilde{z}),\ \widetilde{h}=\frac{h}{|z|}. \tag{4.13}$$

L^2_Φ transforms into

$$L^2_{\widetilde{\Phi}}=\{\widetilde{u};\int|\widetilde{u}(\widetilde{x})|^2e^{-2\widetilde{\Phi}(\widetilde{x})/\widetilde{h}}L(d\widetilde{x})<\infty\},$$

with the naturally associated norm and with $\widetilde{\Phi}(\widetilde{x})/\widetilde{h} = \Phi(x)/h$, so that

$$\widetilde{\Phi}(\widetilde{x}) = \Phi(|z|^{\frac{1}{2}}\widetilde{x})/|z|,$$

has a uniformly bounded Hessian. Further, $\chi_{|z|}(x) = \chi_1(\widetilde{x})$.

We have (omitting the Jacobians)

$$\begin{aligned}
\|\frac{1}{|z|}(P-z)u\|_\Phi^2 &= \|(\widetilde{P}-\widetilde{z})\widetilde{u}\|_{\widetilde{\Phi}}^2 \geq \|\chi_1(\widetilde{P}-\widetilde{z})\widetilde{u}\|_{\widetilde{\Phi}}^2 \\
&= \int |\chi_1(\widetilde{x})|^2 |\widetilde{p_\epsilon} - \widetilde{z}|^2 |\widetilde{u}|^2 e^{-2\widetilde{\Phi}/\widetilde{h}} L(d\widetilde{x}) - \mathcal{O}(\widetilde{h})\|\widetilde{u}\|_{\widetilde{\Phi}}^2 \\
&= \int |\chi_{|z|}(x)|^2 \frac{1}{|z|^2} |p_\epsilon - z|^2 |u|^2 e^{-2\Phi/h} L(dx) - \mathcal{O}(\frac{h}{|z|})\|u\|_\Phi^2 \\
&\geq \frac{1}{C}\|\chi_{|z|}u\|_\Phi^2 - \mathcal{O}(\frac{h}{|z|})\|u\|_\Phi^2.
\end{aligned}$$

Here we used (3.34) to obtain the second equality and (4.6) to get the last estimate.

In the case $|z| \geq 1$, we get more directly

$$\begin{aligned}
\|\frac{1}{|z|}(P-z)u\|_\Phi^2 &= \int |\chi_{|z|}(x)|^2 \frac{1}{|z|^2} |p_\epsilon - z|^2 |u|^2 e^{-2\Phi/h} L(dx) + \mathcal{O}(h)\|u\|_\Phi^2 \\
&\geq \frac{1}{C}\|\chi_{|z|}u\|_\Phi^2 - \mathcal{O}(h)\|u\|_\Phi^2.
\end{aligned} \tag{4.14}$$

This completes the proof of Proposition 4.1. □

We can therefore write

$$\begin{aligned}
&\operatorname{Re}(\chi(P-z)u|u) + \frac{1}{C_0} Z(\chi\chi_{|z|}u|u) \\
&\quad \leq \|(P-z)u\|\|u\| + \frac{C}{C_0} Z \left(\frac{1}{|z|}\|(P-z)u\| + \sqrt{\frac{h}{\min(1,|z|)}}\|u\| \right) \|u\| + \mathcal{O}(h)\|u\|^2.
\end{aligned}$$

Combining this with (4.7), we get

$$\frac{Z}{C_2}(\chi u|u) \leq (1 + \frac{C}{C_0})\|(P-z)u\|\|u\| + \frac{C}{C_0}\sqrt{\frac{h}{\min(1,|z|)}} Z\|u\| + \mathcal{O}(h)\|u\|^2.$$

and writing $\chi = 1 + (\chi - 1)$ yields

$$\frac{Z}{C_2}\|u\|^2 \leq (1 + \frac{C}{C_0})\|(P-z)u\|\|u\| + \frac{C}{C_0}\sqrt{\frac{h}{\min(1,|z|)}} Z\|u\| + \mathcal{O}(h)\|u\|^2 + CZ\|(1-\chi)u\|\|u\|$$

Assuming $h/\min(1,|z|)$ sufficiently small independently of A, we get the main result of this section:

$$Z\|u\| \leq \mathcal{O}(1)\left(\|(P-z)u\| + Z\|(1-\chi)u\|\right), \tag{4.15}$$

where we recall the assumptions (4.1) and (4.8) on z.

5 The quadratic case

The main purpose of this first section is to get resolvent estimates for operators with quadratic symbol. The main reference for this is [15], and all the computations are explicit. In the special case of the quadratic Kramers-Fokker-Planck operator, the form of the spectrum is well known (see for example [14]) and we compute it explicitly in section 13.

Sectorial property in a linear weighted space and applications

Let P_0 be a quadratic operator in the sense that the symbol $p = p_1 + ip_2$ is a complex-valued quadratic form and assume that the symbol satisfies $p_1 \geq 0$ and a subelliptic estimate

$$p_1 + \epsilon_0 H_{p_2}^2 p_1 \geq \frac{\epsilon_0}{C} d_0^2, \tag{5.1}$$

where $d_0(\rho) = |\rho|^2$. Note that this implies that p has 0 as unique critical point.

Now we use the weight

$$G^0 = G_T,$$

introduced in (2.4) near the critical point, and use the definition there in the whole space. Since p is quadratic, so is G^0, and we have for $0 < \epsilon \leq \epsilon_0$

$$p_1 + \epsilon H_{p_2} G^0 \geq \frac{\epsilon}{C} d_0^2, \tag{5.2}$$

As in section 3 we use the global FBI transform with quadratic phase φ

$$Tu(x) = Ch^{-\frac{3n}{4}} \int e^{\frac{i}{h}\varphi(x,y)} u(y) dy.$$

The canonical transformation associated with the FBI transform T is given by $\kappa_T : (y, -\partial_y \varphi(x,y)) \mapsto (x, \partial_x \varphi(x,y))$ and we define $\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n})$ and $\Phi_0(x) = -\mathrm{Im}\,\varphi(x, y_0(x))$, where $y_0(x)$ is the point where $\mathbb{R}^n \ni y \mapsto -\mathrm{Im}\,\varphi(x,y)$ takes its non-degenerate maximum.

We define

$$\mathbb{C}^{2n} \supset \Lambda_{\epsilon G^0} \stackrel{\text{def}}{=} \{(y,\eta);\ \mathrm{Im}\,(y,\eta) = \epsilon H_{G^0}(\mathrm{Re}\,(y,\eta))\} \tag{5.3}$$

and for ϵ small enough we check that

$$\kappa_T(\Lambda_{\epsilon G^0}) = \Lambda_{\Phi_\epsilon^0} \stackrel{\text{def}}{=} \left\{ (x,\xi)\ \xi = \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}(x) \right\},$$

where Φ_ϵ^0 is defined using the following procedure: the function

$$F_\epsilon^0(x,y,\eta) = -\mathrm{Im}\,\varphi(x,y) - (\mathrm{Im}\,y)\cdot\eta + \epsilon G^0(\mathrm{Re}\,y, \eta)$$

is quadratic and when $\epsilon = 0$ it has a unique non-degenerate critical point for x fixed. By homogeneity, this is also the case for F_ϵ^0. The unique critical point $(y_\epsilon(x), \eta_\epsilon(x))$ depends linearly on x and smoothly on ϵ. We finally write

$$\Phi_\epsilon^0(x) = \mathrm{v.c.}_{(y,\eta)\in\mathbf{C}^n\times\mathbb{R}^n}(-\mathrm{Im}\,\varphi(x,y) - (\mathrm{Im}\,y)\cdot\eta + \epsilon G^0(\mathrm{Re}\,y,\eta))$$

From now on we work entirely on the FBI side, denoting by u a function on the FBI side (instead of Tu), and by the same letter P_0 the (unbounded) operator on $L^2_{\Phi_0}$

$$P_0 u(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x+y}{2})} e^{\frac{i}{h}(x-y)\cdot\theta} p(\frac{x+y}{2}, \theta) u(y) dy d\theta.$$

Since the symbol of P_0 is quadratic, it is holomorphic and we also have the following formula for P_0 as an unbounded operator on $L^2_{\Phi_0}$:

$$P_0 u(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(\frac{x+y}{2}) + it_0\overline{(x-y)}} e^{\frac{i}{h}(x-y)\cdot\theta} p(\frac{x+y}{2}, \theta) u(y) dy d\theta.$$

We can now make a new contour deformation, and we get an unbounded operator again denoted P_0 on the space $L^2_{\Phi^0_\epsilon}$ naturally associated to Φ^0_ϵ:

$$P_0 u(x) = \frac{1}{(2\pi h)^n} \iint_{\theta = \frac{2}{i}\frac{\partial \Phi^0_\epsilon}{\partial x}(\frac{x+y}{2}) + it_0\overline{(x-y)}} e^{\frac{i}{h}(x-y)\cdot\theta} p(\frac{x+y}{2}, \theta) u(y) dy d\theta.$$

Of course coming back to the real side by the FBI transform, P_0 can be viewed as an unbounded operator on $L^2(\mathbb{R}^n)$ with symbol

$$\tilde{p} = p(\rho + i\epsilon H_{G^0}),$$

and here the symbol of P_0 is quadratic and satisfies

$$\begin{aligned} \widetilde{p}_1 &= p_1(\rho) + \epsilon H_{p_2} G^0(\rho) + \mathcal{O}(\epsilon^2 |\nabla G^0|^2), \\ \widetilde{p}_2 &= p_2(\rho) - \epsilon H_{p_1} G^0(\rho) + \mathcal{O}(\epsilon^2 |\nabla G^0|^2). \end{aligned} \tag{5.4}$$

Now each term is quadratic therefore using the homogeneity, (5.2), and choosing $\epsilon > 0$ small enough yields

$$\widetilde{p}_1 \geq \frac{\epsilon}{C} d_0^2, \ \widetilde{p}_2 = \mathcal{O}(d_0^2).$$

In particular $\widetilde{p}$ takes its values in an angle around the positive real axis, $\widetilde{p}_1 \geq \epsilon |\widetilde{p}_2|/C$. As a consequence we can apply to P_0 as an unbounded operator on $L^2_{\Phi^0_\epsilon}$ the result of [15, Theorem 3.5], which gives with $d(x) = |x|$:

Proposition 5.1 *Consider P_0 as an operator on $H_{\Phi^0_\epsilon}$. Then*

a) the spectrum of P_0 is a set $\{\mu_l\}$ given by

$$\left\{ \frac{h}{i} \sum_{\mathrm{Im}\,\lambda_j > 0} \left(\frac{1}{2} + k_j \right) \lambda_j; \quad \lambda_j \in Sp(F), \quad k_j \in \mathbb{N} \right\},$$

where the λ_js are the eigenvalues, repeated with their multiplicities, of the fundamental matrix F of $\mathrm{Hess}\, p$.

b) Let z vary in a compact set $K \subset \mathbb{C}$ disjoint from the union of the μ_js, then

$$\|(h + d^2)u\| \leq C\|(P_0 - hz)u\|, \ \|(h + d^2)^{\frac{1}{2}} u\| \leq C\|(h + d^2)^{-\frac{1}{2}}(P_0 - hz)u\| \tag{5.5}$$

where $d(x) = |x|$ (essentially equal to $|\rho|^2$ if we lift it to Λ_Φ), and for u holomorphic with $(h + d^2)u \in L^2_{\Phi^0_\epsilon}$ and $(h + d^2)^{\frac{1}{2}} u \in L^2_{\Phi^0_\epsilon}$ respectively.

Recall that the fundamental matrix of the quadratic form p is the matrix of the (linearized) Hamilton flow and is given by

$$F = \begin{pmatrix} p''_{\xi,x} & p''_{\xi,\xi} \\ -p''_{x,x} & -p''_{x,\xi} \end{pmatrix}$$

Proof. This follows from [15, Theorem 3.5] and some remarks: First we note that the presence of the small parameter h is easy to deal with since P_0 is linearly conjugated with $h(P_0|_{h=1})$ by the symplectic change of coordinates $(x,\xi) \to (h^{\frac{1}{2}}x, h^{-\frac{1}{2}}\xi)$. We also notice that the eigenvalues of the fundamental matrix F of p are the same as the ones of the fundamental matrix $\tilde{F}$ of $\tilde{p}$, also by a symplectic change of variables. Point $b)$ of the proposition is a direct consequence of [15, Theorem 3.5] and the change of symplectic coordinates $(x,\xi) \to (h^{\frac{1}{2}}x, h^{-\frac{1}{2}}\xi)$. □

In Section 13 we shall explicitly compute the eigenvalues in the case of the Kramers-Fokker-Planck operator. In the next subsection we shall compare an operator P with its quadratic approximation near its critical points: In order to get a global a priori estimate for $P - hz$, we will need a truncated version of (5.5).

Localized resolvent estimates

Let $\chi_0 \in C_0^\infty(\mathbb{C}^n)$, $\chi_0 = 1$ near $x = 0$. We fix $\epsilon > 0$ small and write in this subsection Φ^0 instead of Φ^0_ϵ. The simple idea is to apply (5.5) with u replaced by $\chi_0 u$ and then try to estimate the commutator $[P_0, \chi_0]u$. However, $\chi_0 u$ is not holomorphic, so we will replace $\chi_0 u$ by $\Pi\chi_0 u$, where $\Pi : L^2_{\Phi^0} \to H_{\Phi^0}$ is the orthogonal projection.

The main result of this subsection is

Proposition 5.2 *Let $\chi_0 \in C_0^\infty(\mathbb{C}^n)$ be fixed and equal to 1 near 0, and fix $k \in \mathbb{R}$. Then for z varying in a compact set which does not contain any eigenvalues of $P_0|_{h=1}$, we have*

$$\|(h + d^2)^{1-k}\chi_0 u\| \le C\|(h + d^2)^{-k}\chi_0(P_0 - hz)u\| + \mathcal{O}(h^{\frac{1}{2}})\|1_K u\|, \tag{5.6}$$

where K is any fixed neighborhood of $\mathrm{supp}(\nabla\chi_0)$.

We need a series of technical preparations.

Estimates for $[P_0, \chi_0]$. We have

$$[P_0, \chi] = \sum_{|\alpha+\beta|=1} h\chi_{\alpha,\beta}(x)x^\alpha(hD_x)^\beta + h^2\chi_{0,0}(x),$$

where $\chi_{\alpha,\beta} \in C_0^\infty(\mathbb{C}^n)$, $\mathrm{supp}\,(\chi_{\alpha,\beta}) \subset \mathrm{supp}\,\nabla\chi_0$. We can conclude that

$$\|[P_0, \chi]u\| \le Ch\|1_K u\|, \tag{5.7}$$

where C depends on χ and K is an arbitrarily small neighborhood of $\mathrm{supp}\,(\nabla\chi)$. Here we also use that $\|1_{\mathrm{supp}\,\chi}(hD)^\alpha u\| \le C\|1_K u\|$, if u is holomorphic near K.

Estimates for $[P_0, \Pi]$. Recall (from e.g. [17]) that Π is given by

$$\Pi u(x) = Ch^{-n}\int e^{\frac{2}{h}(\Psi^0(x,\overline{y}) - \Phi^0(y))}u(y)L(dy), \tag{5.8}$$

where $\Psi^0(x,z)$ is the unique (second order) holomorphic polynomial on $\mathbb{C}^{2n}$ with $\Psi^0(x,\overline{x}) = \Phi^0(x)$. Notice that

$$(\partial_x \Psi^0)(x,\overline{x}) = \partial_x \Phi^0(x), \tag{5.9}$$

and recall the well known fact that

$$2\mathrm{Re}\, \Psi^0(x,\overline{y}) - \Phi^0(x) - \Phi^0(y) \sim -|x-y|^2. \tag{5.10}$$

For $|\alpha+\beta| \le 2$, we get by integration by parts,

$$\begin{aligned}[x^\alpha (hD_x)^\beta, \Pi]u(x) &= Ch^{-n} \int (x^\alpha (hD_x)^\beta - (-hD_y)^\beta y^\alpha) e^{\frac{2}{h}(\Psi^0(x,\overline{y}) - \Phi^0(y))} u(y) L(dy) \\ &= Ch^{-n} \int e^{\frac{2}{h}(\Psi^0(x,\overline{y}) - \Phi^0(y))} a_{\alpha,\beta}(x,y;h) u(y) L(dy),\end{aligned}$$

where

$$a_{\alpha,\beta} = (x^\alpha (hD_x + \frac{2}{i}\partial_x \Psi^0(x,\overline{y}))^\beta - (-hD_y + \frac{2}{i}\partial_y \Phi^0(y))^\beta \circ y^\alpha)(1).$$

Using (5.9), we see that

$$a_{\alpha,\beta} = \begin{cases} 0, & |\alpha+\beta| = 0, \\ b_1(x,y), & |\alpha+\beta| = 1, \\ b_2(x,y) + hb_0, & |\alpha+\beta| = 2, \end{cases} \tag{5.11}$$

where b_j is a homogeneous polynomial of degree j, vanishing on the diagonal when $j = 1, 2$.

Relation (5.10) implies that the effective kernel of $\Pi : L^2_{\Phi^0} \to L^2_{\Phi^0}$ is $\mathcal{O}(h^{-n} e^{-|x-y|^2/(Ch)})$, so

$$\|[x^\alpha (hD)^\beta, \Pi]u\| \le \mathcal{O}(h^{\frac{1}{2}}) \times \begin{cases} \|u\|, & |\alpha+\beta| = 1, \\ \|(h+d^2)^{\frac{1}{2}} u\|, & |\alpha+\beta| = 2. \end{cases}$$

It follows that

$$\|[P_0, \Pi]u\| \le \mathcal{O}(h^{\frac{1}{2}}) \|(h+d^2)^{\frac{1}{2}} u\|, \tag{5.12}$$

since in the case of P_0, we do not have to consider any commutators with $x^\alpha (hD)^\beta$ with $|\alpha+\beta| = 1$. The standard inequality

$$\frac{1+|x|}{1+|y|} \le 1 + |x-y|,$$

implies that

$$\frac{h^{\frac{1}{2}} + d(x)}{h^{\frac{1}{2}} + d(y)} \le 1 + \frac{|x-y|}{\sqrt{h}} \le C_\epsilon e^{\epsilon |x-y|^2/h},$$

for every $\epsilon > 0$. It is therefore clear that we can conjugate $[P_0, \Pi]$ in (5.12) by any power of $h^{\frac{1}{2}} + d$. Indeed, the proof there shows that the effective kernel of $[P_0, \Pi](h^{\frac{1}{2}} + d)^{-1}$ is $\mathcal{O}(1) h^{\frac{1}{2}-n} e^{-|x-y|^2/(Ch)}$. Hence

$$\|(h^{\frac{1}{2}} + d)^{-k} [P_0, \Pi]u\| \le \mathcal{O}(h^{\frac{1}{2}}) \|(h^{\frac{1}{2}} + d)^{1-k} u\|, \tag{5.13}$$

for every $k \in \mathbb{R}$.

Estimates for $1 - \Pi$. We briefly recall Hörmander's L^2-method for the $h\overline{\partial}$-complex, following [17],

$$C_0^\infty(\mathbb{C}^n) \to C_0^\infty(\mathbb{C}^n; \wedge^{0,1}\mathbb{C}^n) \to C_0^\infty(\mathbb{C}^n; \wedge^{0,2}\mathbb{C}^n) \to \dots \to C_0^\infty(\mathbb{C}^n; \wedge^{0,n}\mathbb{C}^n).$$

We have here the natural Hilbert space norms induced by the weight $e^{-2\Phi^0/h}L(dx)$. Equivalently, we consider the conjugated complex $e^{-\Phi^0/h}h\overline{\partial}e^{\Phi^0/h} = h\overline{\partial}+\overline{\partial}(\Phi^0)^\wedge$ in the standard L^2-spaces. The adjoint of the last complex is then given by $h\overline{\partial}^* + \partial(\Phi^0)^\rfloor$. More explicitly,

$$h\overline{\partial} + \overline{\partial}(\Phi^0)^\wedge = \sum Z_j d\overline{z}_j^\wedge, \quad h\overline{\partial}^* + \partial(\Phi^0)^\rfloor = \sum Z_j^* dz_j^\rfloor,$$

where $Z_j = h\partial_{\overline{z}_j} + \partial_{\overline{z}_j}\Phi^0$. The corresponding Hodge Laplacian is then

$$\begin{aligned}&(h\overline{\partial} + \overline{\partial}(\Phi^0)^\wedge)(h\overline{\partial}^* + \partial(\Phi^0)^\rfloor) + (h\overline{\partial}^* + \partial(\Phi^0)^\rfloor)(h\overline{\partial} + \overline{\partial}(\Phi^0)^\wedge)\\ &= \sum_{j,k}(Z_j Z_k^* \otimes d\overline{z}_j^\wedge dz_k^\rfloor + Z_k^* Z_j \otimes dz_k^\rfloor d\overline{z}_j^\wedge) = (\sum_j Z_j^* Z_j) \otimes 1 + h\sum_{j,k} 2\partial_{\overline{z}_j}\partial_{z_k}\Phi^0 d\overline{z}_j^\wedge dz_k^\rfloor,\end{aligned}$$

where we used that $[Z_j, Z_k^*] = 2h\partial_{\overline{z}_j}\partial_{z_k}\Phi^0$ and the standard identity, $d\overline{z}_j^\wedge dz_k^\rfloor + dz_k^\rfloor d\overline{z}_j^\wedge = \langle dz_k | d\overline{z}_j\rangle = \delta_{j,k}$. In particular, the Hodge Laplacian

$$\Delta_1 = h\overline{\partial}^* h\overline{\partial} + h\overline{\partial}\, h\overline{\partial}^*$$

on $(0,1)$-forms can be identified with

$$\widetilde{\Delta}_1 = (\sum Z_j^* Z_j) \otimes 1_{\mathbf{C}^n} + 2h(\partial_{\overline{z}_j}\partial_{z_k}\Phi^0), \tag{5.14}$$

acting on $L^2(\mathbb{C}^n;\mathbb{C}^n)$. The strict plurisubharmonicity of Φ^0 means that the Hermitian matrix appearing in the last term in (5.14) is $\geq 1/C$, and hence we get the apriori estimate (now using for a while ordinary L^2-norms):

$$\frac{h}{C}\|u\|^2 + \sum \|Z_j u\|^2 \leq (\widetilde{\Delta}_1 u|u), \tag{5.15}$$

leading first to

$$h\|u\| \leq C\|\widetilde{\Delta}_1 u\|, \tag{5.16}$$

and then to

$$h^{\frac{1}{2}}\|Z_j u\| \leq C\|\widetilde{\Delta}_1 u\|. \tag{5.17}$$

We can also write $\sum Z_j^* Z_j = \sum Z_j Z_j^* + \mathcal{O}(h)$, so $\widetilde{\Delta}_1 = \sum Z_j Z_j^* + \mathcal{O}(h)$, and hence

$$(\widetilde{\Delta}_1 u|u) \geq \sum \|Z_j^* u\|^2 - Ch\|u\|^2,$$

which together with (5.15) implies first

$$h\|u\|^2 + \sum \|Z_j u\|^2 + \sum \|Z_j^* u\|^2 \leq C(\widetilde{\Delta}_1 u|u), \tag{5.18}$$

and then

$$h^{\frac{1}{2}}\|Z_j^* u\| \leq C\|\widetilde{\Delta}_1 u\|. \tag{5.19}$$

We also need to check that these estimates remain valid after conjugation of $\widetilde{\Delta}_1$ by any power of $h + d^2$ or equivalently by any power of $\lambda h + d^2$, where $\lambda \gg 1$ is independent of h. This will follow from the following observations:

1)

$$(\lambda h+d^2)^{-k}Z_j(\lambda h+d^2)^k=Z_j+k\frac{h\partial_{z_j}(d^2)}{\lambda h+d^2},$$

and

$$\left|\frac{h\partial_{z_j}(d^2)}{\lambda h+d^2}\right|\le\frac{Chd}{\lambda h+d^2}\le\frac{C}{\lambda^{\frac{1}{2}}}\frac{(\lambda h)^{\frac{1}{2}}d}{\lambda h+d^2}h^{\frac{1}{2}}\le\alpha(\lambda)h^{\frac{1}{2}},$$

where $\alpha(\lambda)\to 0$ when $\lambda\to\infty$. A similar remark holds for $(\lambda h+d^2)^{-k}Z_j^*(\lambda h+d^2)^k$.

2) We have

$$\begin{aligned}\widehat{\Delta}_1&:=(\lambda h+d^2)^{-k}\widetilde{\Delta}_1(\lambda h+d^2)^k\\&=\sum(Z_j^*+o(1)h^{\frac{1}{2}})(Z_j+o(1)h^{\frac{1}{2}})+2h(\partial_{\overline{z}_j}\partial_{z_k}\Phi^0),\end{aligned}$$

where $o(1)$ refers to the limit $\lambda\to\infty$. Thus

$$\begin{aligned}\mathrm{Re}\,(\widehat{\Delta}_1u|u)&\ge\frac{h}{C}\|u\|^2+\sum\|Z_ju\|^2-o(1)h^{\frac{1}{2}}\|u\|\|Z_ju\|-o(1)(hu|u)\\&\ge\frac{h}{2C}\|u\|^2+\frac{1}{2}\sum\|Z_ju\|^2,\end{aligned}$$

$$h\|u\|^2+\sum\|Z_ju\|^2\le C\mathrm{Re}\,((\widehat{\Delta}_1)u|u).\tag{5.20}$$

Then do as before with $\widetilde{\Delta}_1$ replaced by $\mathrm{Re}\,\widehat{\Delta}_1$, to get

$$h\|u\|^2+\sum\|Z_ju\|^2+\sum\|Z_j^*u\|^2\le C\mathrm{Re}\,(\widehat{\Delta}_1u|u).\tag{5.21}$$

Back to the original Δ_1 we thus have (with the norms now being those of $L^2_{\Phi^0}$):

$$\|\Delta_1^{-1}\|\le\mathcal{O}(\frac{1}{h}),\ \|h\overline{\partial}^*\Delta_1^{-1}\|\le\mathcal{O}(\frac{1}{\sqrt{h}}),\tag{5.22}$$

as well as the same estimates for

$$(h+d^2)^k\Delta_1^{-1}(h+d^2)^{-k},\ (h+d^2)^kh\overline{\partial}^*\Delta_1^{-1}(h+d^2)^{-k}.$$

Now use the fact, that

$$1-\Pi=h\overline{\partial}^*\Delta_1^{-1}h\overline{\partial},\tag{5.23}$$

to conclude that if $\chi_0\in C_0^\infty(\mathbb{C}^n)$ is fixed and equal to 1 near 0, and u is holomorphic near $\mathrm{supp}\,\chi_0$, then

$$(1-\Pi)(u\chi_0)=h\overline{\partial}^*\Delta_1^{-1}(u(h\overline{\partial}\chi_0))$$

satisfies

$$\|(h+d^2)^k(1-\Pi)(u\chi_0)\|\le C_kh^{\frac{1}{2}}\|u\overline{\partial}\chi_0\|.\tag{5.24}$$

Recall that here and until further notice the norms are those of $L^2_{\Phi^0}$.

Let $\chi_0\in C_0^\infty(\mathbb{C}^n)$ be fixed and $=1$ near 0. Recall that z varies in a compact set which does not contain any eigenvalues of $(P_0)_{h=1}$.

Proof of Proposition 5.2. We start from (5.5):

$$\|(h+d^2)^{1-k}u\| \le C\|(h+d^2)^{-k}(P_0-hz)u\|, \tag{5.25}$$

for u holomorphic with $(h+d^2)^{1-k}u \in L^2_{\Phi^0}$. Replace u by $\Pi\chi_0 u$:

$$\|(h+d^2)^{1-k}\Pi\chi_0 u\| \le C\|(h+d^2)^{-k}(P_0-hz)\Pi\chi_0 u\|.$$

It follows that

$$\begin{aligned}\|(h+d^2)^{1-k}\chi_0 u\| &\le \|(h+d^2)^{1-k}\Pi\chi_0 u\| + \|(h+d^2)^{1-k}(1-\Pi)\chi_0 u\| \\ &\le C\|(h+d^2)^{-k}(P_0-hz)\Pi\chi_0 u\| + \mathcal{O}(h^{\frac{1}{2}})\|u\overline{\partial}\chi_0\|.\end{aligned} \tag{5.26}$$

where we used (5.24) and the fact that $h+d^2 \sim 1$ on $\operatorname{supp}\overline{\partial}\chi_0$.

Here

$$\begin{aligned}&\|(h+d^2)^{-k}(P_0-hz)\Pi\chi_0 u\| \\ &\le \|(h+d^2)^{-k}\Pi\chi_0(P_0-hz)u\| + \|(h+d^2)^{-k}[P_0,\Pi\chi_0]u\| \\ &\le C\|(h+d^2)^{-k}\chi_0(P_0-hz)u\| + \|(h+d^2)^{-k}[P_0,\Pi\chi_0]u\|.\end{aligned} \tag{5.27}$$

Now

$$\begin{aligned}[P_0,\Pi\chi_0]u &= [P_0,\Pi]\chi_0 u + \Pi[P_0,\chi_0]u \\ &= [P_0,\Pi]\Pi\chi_0 u + [P_0,\Pi](1-\Pi)\chi_0 u + \Pi[P_0,\chi_0]u \\ &= [P_0,\Pi](1-\Pi)\chi_0 u + \Pi[P_0,\chi_0]u,\end{aligned} \tag{5.28}$$

where we used that $[P_0,\Pi]\Pi = 0$, since P_0 conserves holomorphic functions. Combining (5.28), (5.13), (5.24), (5.7), we see that

$$\|(h+d^2)^{-k}[P_0,\Pi\chi_0]u\| \le \mathcal{O}(h)\|1_K u\|. \tag{5.29}$$

Combining this with (5.26), (5.27), (5.29), we get (5.6). □

Remark 5.3 In Proposition 5.2 we can replace the norm $L^2_{\Phi^0_\epsilon}$ by $L^2_{\Phi_0}$ or any other norm which is equivalent to the $L^2_{\Phi^0_\epsilon}$ norm for functions with support near K.

6 Local resolvent estimate for small z.

Again in this section we suppose that p satisfies the hypothesis **(H1)** and that it is bounded with all its derivatives outside a large compact set K. We also replace for a while the small parameter h by Ah in the construction of G, where A is some large constant, and work in K.

Recall that $G = G_{Ah}$ satisfies the estimates:

$$\nabla G = \mathcal{O}(\delta^{(2-k)_+}),\ \delta(\rho) \le \sqrt{Ah}, \tag{6.1}$$

$$\nabla^k G = \mathcal{O}(Ah(Ah\delta)^{-k/3}),\ \delta(\rho) \ge \sqrt{Ah}, \tag{6.2}$$

implying,

$$\nabla^k G = \mathcal{O}(Ah((Ah)^{\frac{1}{3}}(Ah+\delta^2)^{\frac{1}{6}})^{-k}) = \mathcal{O}(Ahr^{-k}), \tag{6.3}$$

$$r(\rho) := (Ah)^{\frac{1}{3}}(Ah+\delta^2)^{\frac{1}{6}}. \tag{6.4}$$

Writing

$$p_{|_{\Lambda_{\epsilon G}}} = p_\epsilon = p_1 + ip_2,$$

we recall that in K

$$p_1 \geq \frac{\epsilon}{C}\min(\delta(\rho)^2, (\delta Ah)^{\frac{2}{3}}), \tag{6.5}$$

$$p_2 = \mathcal{O}(\delta^2). \tag{6.6}$$

We represent $\Lambda_{\epsilon G}$ on the FBI-transform side by

$$\xi = \frac{2}{i}\frac{\partial \Phi_\epsilon}{\partial x}(x),\ \ \Phi_\epsilon = \Phi_0 + \epsilon\widetilde{G}(x;h),$$

where $\widetilde{G}$ has the same properties as G (cf (3.27)). We also know that $\widetilde{G}$ and Φ_ϵ are independent of h in the region $|x| \leq \sqrt{Ah}$. From now on $\epsilon > 0$ will be small and fixed.

Assume for simplicity that $\mathcal{C}$ consists of just one point, corresponding to $x = 0$. Let

$$p_0(x,\xi) = \sum_{|\alpha+\beta|=2} \frac{\partial_x^\alpha \partial_\xi^\beta p(0,0)}{\alpha!\beta!} x^\alpha \xi^\beta \tag{6.7}$$

be the quadratic approximation of p, so that

$$p - p_0 = \mathcal{O}((x,\xi)^3) = \mathcal{O}((h+(x,\xi)^2)^{\frac{3}{2}}). \tag{6.8}$$

We may assume that $\Phi = \Phi_\epsilon(x)$ is a quadratic function Φ^0 in the region $|x| \leq \sqrt{Ah}$ and for x in that region, we realize $p_0(x, hD_x)u$ with a contour as in (3.28). The difference between the corresponding effective kernels of $P = p^w$ and $P_0 = p_0(x, hD_x)u$ is then

$$\mathcal{O}(1)h^{-n}e^{-\frac{t_0}{h}|x-y|^2}(h+|x|^2+|y|^2)^{\frac{3}{2}} = \mathcal{O}(1)h^{-n}e^{-\frac{t_0}{h}|x-y|^2}(h^{\frac{3}{2}}+|x|^3+|x-y|^3).$$

We conclude that

$$\|Pu - P_0u\|_{H_\Phi(|x|\leq\sqrt{Ah})} \leq \mathcal{O}((Ah)^{\frac{3}{2}})\|u\|_{H_\Phi}. \tag{6.9}$$

Here both P and P_0 are realized with a contour as in (3.28). However, p_0 is a polynomial and we check that if we replace P_0u by the corresponding differential expression

$$P_0 u = \left(\sum_{|\alpha+\beta|=2} \frac{\partial_x^\alpha \partial_\xi^\beta p(0,0)}{\alpha!\beta!}(x^\alpha (hD)^\beta)^w\right) u(x),$$

then we commit an error w, satisfying

$$\|w\|_{H_\Phi(|x|\leq\sqrt{Ah})} \leq e^{-\frac{1}{Ch}}\|u\|_{H_\Phi}. \tag{6.10}$$

Now for P_0 we can apply Proposition 5.2 and Remark 5.3. We get that for every fixed $k \in \mathbb{R}$ and for z in a fixed compact set avoiding the eigenvalues of $P_0|_{h=1}$:

$$\|(h+d^2)^{1-k}\chi_0 u\| \le C\|(h+d^2)^{-k}\chi_0(P_0-hz)u\| + \mathcal{O}(h^{\frac{1}{2}})\|1_K u\|, \tag{6.11}$$

where K is any fixed neighborhood of $\mathrm{supp}(\nabla\chi_0)$.

Notice that we can write the last term in (6.11) as $\mathcal{O}(h^{\frac{1}{2}})\|(h+d^2)^{1-k}1_K u\|$. We now want to replace the fixed cutoff χ_0 in (6.11) by $\chi_0(x/\sqrt{Ah})$ for $A \gg 1$ independent of h. Consider the change of variables, $x = \sqrt{Ah}\widetilde{x}$, $hD_x = \sqrt{Ah}\widetilde{h}D_{\widetilde{x}}$, $\widetilde{h} = 1/A$. Then

$$p_0(x, hD_x) = \frac{h}{\widetilde{h}} p_0(\widetilde{x}, \widetilde{h}D_{\widetilde{x}}) =: \frac{h}{\widetilde{h}}\widetilde{P}_0,$$

and with $d = d(x)$, $\widetilde{d} = d(\widetilde{x})$:

$$h + d^2 = \frac{h}{\widetilde{h}}(\widetilde{h} + \widetilde{d}^2), \quad e^{-2\Phi^0(x)/h} = e^{-2\Phi^0(\widetilde{x})/\widetilde{h}}.$$

Start from (6.11) with x, h replaced by $\widetilde{x}, \widetilde{h}$:

$$\|(\widetilde{h}+\widetilde{d}^2)^{1-k}\chi_0(\widetilde{x})u\| \le C\|(\widetilde{h}+\widetilde{d}^2)^{-k}\chi_0(\widetilde{x})(\widetilde{P}_0-\widetilde{h}z)u\| + C\widetilde{h}^{\frac{1}{2}}\|(\widetilde{h}+\widetilde{d}^2)^{1-k}1_K u\|,$$

$$\begin{aligned}&\|\left(\frac{\widetilde{h}}{h}\right)^{1-k}(h+d^2)^{1-k}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)u\| \le \\ &C\|\left(\frac{\widetilde{h}}{h}\right)^{1-k}(h+d^2)^{-k}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)(P_0-hz)u\| + C\widetilde{h}^{\frac{1}{2}}\|\left(\frac{\widetilde{h}}{h}\right)^{1-k}(h+d^2)^{1-k}1_K\left(\frac{x}{\sqrt{Ah}}\right)u\|,\end{aligned}$$

$$\begin{aligned}&\|(h+d^2)^{1-k}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)u\| \le \\ &C\|(h+d^2)^{-k}\chi_0\left(\frac{x}{\sqrt{Ah}}\right)(P_0-hz)u\| + \frac{C}{\sqrt{A}}\|(h+d^2)^{1-k}1_K\left(\frac{x}{\sqrt{Ah}}\right)u\|.\end{aligned} \tag{6.12}$$

This estimate will be applied with $k = 1/2$.

We now return to the full operator P (on the FBI-side) and the norms and scalar products will now be with respect to $e^{-2\Phi/h}$, $\Phi = \Phi_\epsilon$, $\epsilon > 0$ small and fixed. Recall however that $\Phi = \Phi^0$ in $|x| \le \sqrt{Ah}$. Let χ be a cutoff function equal to 1 in a fixed neighboorhood of the critical points, but recall however the simplifying assumption that we only have one critical point corresponding to $x = 0$. Let us denote

$$\Lambda^2 = h + \min(d^2, (dAh)^{\frac{2}{3}}). \tag{6.13}$$

Using (3.30) as in Section 4, we get for $z = \mathcal{O}(1)$,

$$\|\Lambda u\|^2 \le C(\mathrm{Re}\,(\chi(x)(P-hz)u|u) + C^2(\chi_0^2\left(\frac{x}{\sqrt{Ah}}\right)\Lambda u|\Lambda u)) + C'\|(1-\chi)\Lambda u\|\|\Lambda u\|. \tag{6.14}$$

Then using (6.12) we get for $\tau > 0$:

$$\begin{aligned}\|\Lambda u\|^2 &\leq C\|\Lambda^{-1}(P-hz)u\|\|\Lambda u\| + C\|\Lambda\chi_0(\frac{x}{\sqrt{Ah}})u\|^2 + C'\|(1-\chi)\Lambda u\|\|\Lambda u\| \\ &\leq \frac{C}{\tau}\|\Lambda^{-1}(P-hz)u\|^2 + C\tau\|\Lambda u\|^2 + \widetilde{C}\|\Lambda^{-1}\chi_0(\frac{x}{\sqrt{Ah}})(P_0-hz)u\|^2 \\ &+ \frac{\widetilde{C}}{A}\|\Lambda 1_K(\frac{x}{\sqrt{Ah}})u\|^2 + C'\|(1-\chi)\Lambda u\|\|\Lambda u\|.\end{aligned} \tag{6.15}$$

Here we also need (and we can clearly generalize (6.9) for that purpose)

$$\|\chi_0(\frac{x}{\sqrt{Ah}})\Lambda^{-1}(P-P_0)u\| \leq C(A)h^{\frac{1}{2}}\|\Lambda u\|. \tag{6.16}$$

Insertion in (6.15) gives

$$\begin{aligned}\|\Lambda u\|^2 \leq &\frac{C}{\tau}\|\Lambda^{-1}(P-hz)u\|^2 + C\tau\|\Lambda u\|^2 \\ &+ 2\widetilde{C}\|\Lambda^{-1}\chi_0(\frac{x}{\sqrt{Ah}})(P-hz)u\|^2 + \widetilde{C}(A)h\|\Lambda u\|^2 + \frac{\widetilde{C}}{A}\|\Lambda u\|^2 \\ &+ C'\|(1-\chi)\Lambda u\|\|\Lambda u\|.\end{aligned}$$

Choosing first τ, $1/A$ small enough and then h small enough, we get

$$\|\Lambda u\| \leq C\|\Lambda^{-1}(P-hz)u\| + C''\|(1-\chi)\Lambda u\|. \tag{6.17}$$

and noticing that $h \leq \Lambda^2 \leq Ch^{2/3}$, we get the main result of this section

$$h\|u\| \leq C\|(P-hz)u\| + C''h^{5/6}\|(1-\chi)u\|. \tag{6.18}$$

7 Review of semiclassical Weyl Calculus

In this section we introduce some tools and make some remarks about the translation into the semiclassical point of view of some basic facts on the classical Weyl-Hörmander Calculus.

Weyl-Hörmander calculus

First recall the framework of the Weyl-Hörmander Calculus, which can be found in [11, Chapter 18]. We put a subscript cl everywhere here to emphasize the fact the we are in the original (opposite to semiclassical) framework of the calculus. Recall that the classical Weyl quantization is given for an admissible symbol p_{cl} (to be defined below) by

$$(p_{cl}^{w_{cl}}u)(x) = \frac{1}{(2\pi)^n}\iint e^{i\langle x-y,\xi\rangle}p_{cl}(\frac{x+y}{2},\xi)u(y)dyd\xi. \tag{7.1}$$

Consider the symplectic space $\mathbb{R}^{2n}$ equipped with the symplectic form $\sigma = \sum_{i=1}^n d\xi_i \wedge dx_i$. If g is a positive definite quadratic form, we define

$$g_{cl}^\sigma(T) = \sup_{g_{cl}(Y)=1} \sigma(T,Y)^2, \tag{7.2}$$

which is also a positive definite quadratic form. We say that g_{cl} is a cl-admissible metric if

$$\begin{array}{ll}\forall X \in \mathbb{R}^{2n}, \ \ g_{cl,X} \leq g^{\sigma}_{cl,X} & (cl\text{-Uncertainty Principle}),\\ \exists C_0 > 0 \text{ such that } g_{cl,X}(X-Y) \leq C_0^{-1} \Longrightarrow (g_{cl,X}/g_{cl,Y})^{\pm 1} \leq C_0 & (cl\text{-slowness}),\\ \exists C_1, N_1 > 0 \text{ such that } g_{cl,X}/g_{cl,Y} \leq C_1 \left(1 + g^{\sigma}_{cl,X}(X-Y)\right)^{N_1} & (cl\text{-temperance}),\end{array} \tag{7.3}$$

for positive constants C_0, C_1, N_1. Let us note that if the metric g_{cl} depends on a parameter (for example h), we call it cl-admissible if (7.3) occurs uniformly in this parameter. The same is true for the cl-admissible weights we introduce now. A cl-admissible weight is a positive function m_{cl} on the phase space $\mathbb{R}^{2n}$, for which there exists $\tilde{C}_0, \tilde{C}_1, \tilde{N}_1 > 0$ such that

$$\begin{array}{ll} g_{cl,X}(X-Y) \leq \tilde{C}_0 \Longrightarrow (m_{cl}(Y)/m_{cl}(X))^{\pm 1} \leq \tilde{C}_0 & (cl\text{-slowness}),\\ m_{cl}(Y)/m_{cl}(X) \leq \tilde{C}_1 \left(1 + g^{\sigma}_{cl,X}(X-Y)\right)^{\tilde{N}_1} & (cl\text{-temperance}).\end{array} \tag{7.4}$$

We define next the *cl-uncertainty parameter* λ_{cl}, which is a special admissible weight for g,

$$\lambda_{cl}(X) = \inf_{T \in \mathbb{R}^{2n}/\{0\}} \left(g^{\sigma}_{cl,X}(T)/g_{cl,X}(T)\right)^{1/2} \geq 1. \tag{7.5}$$

Let us now introduce some spaces of symbols. We say that a function p_{cl} is a symbol in $S(m_{cl}, g_{cl})$ if $p_{cl} \in \mathcal{C}^{\infty}(\mathbb{R}^{2n})$, and if the following semi-norms are finite

$$\sup_{X \in \mathbb{R}^{2n}, g_{cl,X}(T_j) \leq 1} \left| \left\langle p^{(k)}_{cl}(X), T_1 \otimes \ldots \otimes T_l \right\rangle \right| m^{-1}_{cl}(X). \tag{7.6}$$

If m_{cl} is of the form λ^{μ}_{cl}, we say that p_{cl} is of order μ. For good symbols (in $S(m_{cl}, g_{cl})$ classes for instance), we define the composition law $\sharp_{cl}$ such that $(p_{cl}\sharp_{cl}q_{cl})^{w_{cl}} = p^{w_{cl}}_{cl} \circ q^{w_{cl}}_{cl}$ by

$$(p_{cl}\sharp_{cl}q_{cl})(x,\xi) = e^{\frac{i}{2}\sigma((D_x,D_\xi),(D_y,D_\eta))} p_{cl}(x,\xi) q_{cl}(y,\eta)|_{y=x,\eta=\xi}, \tag{7.7}$$

and for $p_{cl} \in S(m_1, g_{cl})$, $q_{cl} \in S(m_2, g_{cl})$, if $\{.,.\}$ denotes the Poisson bracket, then there is $r_{cl} \in S(m_1 m_2 \lambda^{-2}_{cl}, g_{cl})$ such that

$$p_{cl}\sharp_{cl}q_{cl} = p_{cl}q_{cl} + \frac{1}{2i}\{p_{cl}, q_{cl}\} + r_{cl}. \tag{7.8}$$

Recall eventually the Fefferman-Phong inequality that will be used in the next sections:

Proposition 7.1 *Let $p_{cl} \in S(m_{cl}, g_{cl})$. If $p_{cl} \geq 0$ then there is a real symbol $r_{cl} \in S(m_{cl}\lambda^{-2}_{cl}, g_{cl})$ such that $p^w_{cl} \geq r^w$. Hence if $m_{cl} = \lambda^2_{cl}$, then p^w_{cl} is bounded from below.*

Semiclassical Weyl-Hörmander Calculus

The original calculus already contains a parameter that plays the role of a Planck's constant, namely the inverse of the uncertainty parameter. In the semiclassical case this is made more explicit, but basically this is only a reduction to the original calculus by a change of variables.

For an admissible symbol p we first recall the definition of semiclassical Weyl quantization

$$p^w u = \frac{1}{(2\pi h)^n} \iint p\left(\frac{x+y}{2}, \xi\right) e^{i\langle x-y, \xi\rangle/h} u(y) dy d\xi, \quad u \in ß.$$

A straightforward computation shows that

$$p^w = p_{cl}^{w_{cl}} \qquad \text{where } p_{cl}(x,\xi) = p(x,h\xi). \tag{7.9}$$

Now observe that p belongs to a symbol class $S(m,g)$ for a Riemanian metric g and a positive function m if and only if $p_{cl} \in S(m_{cl}, g_{cl})$ where

$$m_{cl}(x,\xi) = m(x,h\xi), \quad g_{cl,(x,\xi)}(t,\tau) = g_{(x,h\xi)}(t,h\tau).$$

Using definition (7.2) and (7.5) for defining respectively g_{cl}, g, and λ_{cl}, λ, we also get

$$g^\sigma_{cl,(x,\xi)}(t,\tau) = h^{-2} g^\sigma_{(x,h\xi)}(t,h\tau), \quad \text{and} \quad \lambda_{cl}(x,\xi) = h^{-1}\lambda(x,h\xi). \tag{7.10}$$

As a consequence it is natural to introduce the following definitions in the semiclassical case:

Definition 7.2 *We say that g is an admissible (or semiclassically admissible) metric if*

$$\begin{array}{ll}
\forall X \in \mathbb{R}^{2n}, \;\; g_X \le h^{-2} g^\sigma_X \quad (\textit{i.e. } \lambda \ge h) & \textit{(Uncertainty Principle)}, \\
\exists C_0 > 0 \textit{ such that } g_X(X-Y) \le C_0^{-1} \Longrightarrow (g_X/g_Y)^{\pm 1} \le C_0 & \textit{(slowness)}, \\
\exists C_1, N_1 > 0 \textit{ such that } g_X/g_Y \le C_1 \left(1 + h^{-2} g^\sigma_X(X-Y)\right)^{N_1} & \textit{(temperance)},
\end{array} \tag{7.11}$$

for positive constants C_0, C_1, N_1.

A direct definition holds for semiclassical weights. Using this (note that all this is simply a change of variables) we can write

Lemma 7.3 *The metric g is an admissible metric of uncertainty parameter $\lambda(\ge h)$ if and only if g_{cl} is an admissible metric of uncertainty parameter $\lambda_{cl}(\ge 1)$, both uniformly in $0 < h \le 1$.*

We can therefore translate into the semiclassical point of view all the classical results. First observe that symbols of order 1 give bounded operators on $L^2(\mathbb{R}^n)$. Then the product formula is defined by

$$p^w \circ q^w = (p\sharp q)^w,$$

where

$$p\sharp q(x,\xi,h) = e^{\frac{ih}{2}\sigma((D_x,D_\xi),(D_y,D_\eta))} p(x,\xi,h) q(y,\eta,h)|_{y=x,\eta=\xi}.$$

The asymptotic expansion is then given for $p \in S(m_1,g)$, $q \in S(m_2,g)$ by

$$p\sharp q = pq + \frac{h}{2i}\{p,q\} + h^2 r, \tag{7.12}$$

where $r \in S(m_1 m_2 \lambda^{-2}, g)$. Recall eventually how to write the semiclassical Fefferman-Phong inequality that will be used in the text:

Proposition 7.4 *If $p \ge 0$ then there is a real symbol $r \in S(m\lambda^{-2}, g)$ such that $p^w \ge h^2 r^w$. Hence if $m = h^{-2}\lambda^2$, then p^w is bounded from below uniformly with respect to h.*

Remark 7.5 As an illustration, let us see what happens in the case of the constant metric $g = dx^2 + d\xi^2$. It is the one generally used in semiclassical work. We check immediately that it is admissible in the sense of definition 7.11, since $g_X = g_Y$ for all X, Y and that $g^\sigma/g = 1 \ge h$. Of course the translation procedure gives the Fefferman–Phong inequality: $p^w \ge -Ch^2$ if p is real non negative with all its derivatives bounded.

The microlocal metric Γ

We study now a particular metric used in the next sections.

Lemma 7.6 *the metric defined on* $\mathbb{R}^{2n}$ *by*

$$\Gamma = \frac{dx^2}{h^{2/3}} + \frac{d\xi^2}{\mu^2}, \qquad \textit{where } \mu^2 = p_1 + (h\lambda)^{2/3},$$

is (semiclassically) admissible.

Proof. Recall that we suppose that

$$\Gamma_0 = dx^2 + d\xi^2/\lambda^2,\ \lambda = \lambda(x,\xi) \geq 1,$$

is a *cl*-admissible metric. Let us prove the three points of (7.11). We first notice that

$$\Gamma^\sigma = \mu^2 dx^2 + h^{2/3} d\xi^2,$$

therefore the uncertainty parameter of Γ is $\mu h^{1/3}$ and we have for h small

$$\mu h^{1/3} \geq \lambda^{1/3} h^{2/3} \geq h^{2/3} \geq h,$$

therefore Γ satisfies the uncertainty principle.

Slowness of Γ. We take $X = (x,\xi)$ and $Y = (y,\eta)$ and we observe that if $\Gamma_X(X-Y) \leq C_0$ then

$$|x-y|^2 \leq C_0 h^{2/3}, \quad \text{and} \quad |\xi - \eta|^2 \leq C_0 \left(p_1(X) + (h\lambda)^{2/3}(X)\right). \tag{7.13}$$

Using a Taylor expansion and the fact that the the second derivative of p_1 is bounded, we can write that

$$\begin{aligned} p_1(Y) &\leq p_1(X) + |\nabla p_1||X-Y| + C|X-Y|^2 \\ &\leq p_1(X) + C'\sqrt{p_1}|X-Y| + C|X-Y|^2 \\ &\leq 2p_1(X) + C''|X-Y|^2, \end{aligned}$$

where for the second inequality we used inequality (2.11) for the non negative function p_1. Now use the fact that $\Gamma_X(X-Y) \leq C_0$. We get

$$\begin{aligned} p_1(Y) &\leq 2p_1(X) + C''|X-Y|^2 \\ &\leq 2p_1(X) + C''C_0\left(h^{2/3} + p_1(X) + (h\lambda)^{2/3}(X)\right) \\ &\leq C(p_1(X) + (h\lambda)^{2/3}(X)), \end{aligned} \tag{7.14}$$

since $\lambda \geq 1$. Formula (7.13) implies that

$$|x-y|^2 \leq C_0, \quad \text{and} \quad |\xi-\eta|^2 \leq C_0 \lambda^2(X),$$

and we get using the slowness of Γ_0 for C_0 sufficiently small that $(h\lambda)^{2/3}(Y) \le C'(h\lambda)^{2/3}(X)$. Using this and (7.14) yields

$$p_1(Y) + (h\lambda)^{2/3}(Y) \le C(p_1(X) + (h\lambda)^{2/3}(X)),$$

that is to say $\mu(X) \le C\mu(Y)$. This implies immediately that $\Gamma_Y \le C\Gamma_X$. Inverting the roles of X and Y proves the slowness of Γ.

Temperance of Γ. Again we denote $X = (x, \xi)$ and $Y = (y, \eta)$. Beginning from the first line of (7.14) we write

$$\begin{aligned} p_1(Y) &\le 2p_1(X) + C''|X-Y|^2 \\ &\le C\left(p_1(X) + (h\lambda)^{2/3}(X)\right)(1 + h^{-2/3}|X-Y|^2). \end{aligned} \tag{7.15}$$

Notice that

$$\begin{aligned} h^{-2}\Gamma_X^\sigma(X-Y) &= h^{-2}\left(\left(p_1(X) + (h\lambda)^{2/3}(X)\right)|x-y|^2 + h^{2/3}|\xi-\eta|^2\right) \\ &\ge h^{-4/3}|X-Y|^2 \\ &\ge h^{-2/3}|X-Y|^2, \end{aligned}$$

since $\lambda \ge 1$ and for $h \le 1$. Hence

$$p_1(Y) \le C\left(p_1(X) + (h\lambda)^{2/3}(X)\right)\left(1 + h^{-2}\Gamma_X^\sigma(X-Y)\right). \tag{7.16}$$

Since $\Gamma_0 = dx^2 + d\xi^2/\lambda^2$ is *cl*-temperate, there exists $C_0, N \ge 1$ such that

$$\Gamma_{0,X} \le C_0\Gamma_{0,Y}\left(1 + \Gamma_{0,X}^\sigma(X-Y)\right)^N.$$

Together with the fact that $\Gamma_0^\sigma = \lambda^2 dx^2 + d\xi^2$, this implies that

$$\begin{aligned} \lambda^2(Y) &\le C_0\lambda^2(X)\left(1 + \lambda^2(X)|x-y|^2 + |\xi-\eta|^2\right)^N \\ &\le C_0'\lambda^2(X)\left(1 + \lambda^{2/3}(X)|x-y|^2 + |\xi-\eta|^2\right)^{3N} \\ &\le C_0'\lambda^2(X)\left(1 + h^{-2}\left(((h\lambda)^{2/3}(X) + p_1(X))|x-y|^2 + h^{2/3}|\xi-\eta|^2\right)\right)^{3N}, \end{aligned} \tag{7.17}$$

since $h^{-4/3} \ge 1$ and $p_1 \ge 0$. Now we recognize in the parentheses a term of the form $h^{-2}\Gamma^\sigma$. Multiplying by h and raising to the power $1/3$ gives

$$(h\lambda)^{2/3}(Y) \le C(h\lambda)^{2/3}(X)\left(1 + h^{-2}\Gamma_X^\sigma(X-Y)\right)^N.$$

Together with (7.15) this gives

$$\mu^2(Y) \le C\mu^2(X)\left(1 + h^{-2}\Gamma_X^\sigma(X-Y)\right)^N,$$

which implies $\Gamma_X \le \Gamma_Y\left(1 + h^{-2}\Gamma_X^\sigma(X-Y)\right)^N$. Consequently Γ is (semiclassically) temperate. Eventually we have proven that Γ is a (semiclassically) admissible metric. □

8 Resolvent estimates away from the critical points when $|z| \gg h$

In this section we suppose that p satisfies hypotheses **(H2), (H3), (H4)** and we shall work away from a fixed neighborhood $\mathcal{B}$ of the critical points and for $|z| \gg h$. The main result of this section will be the estimate (8.20). At infinity in the phase space, we shall use the machinery of the Weyl calculus. Let us consider the following weight

$$\mu^2(x,\xi) = p_1(x,\xi) + (h\lambda(x,\xi))^{2/3}$$

We notice that $\mu \geq h^{1/3}$. We use the metric defined in lemma 7.6

$$\Gamma = \frac{dx^2}{h^{2/3}} + \frac{d\xi^2}{\mu^2}. \tag{8.1}$$

From the construction of the weight G in Proposition 2.1 (cf (2.1), (2.8)), we know that

$$g \stackrel{\text{def}}{=} G/h \in S\left(1, \Gamma\right) \text{ outside } \mathcal{B},$$

since $G = 0$ when $p_1 \geq 2(h\lambda)^{2/3}/M$. There is no restriction to extend g near the critical points and let it uniformly be in the class $S\left(1, \Gamma\right)$.

From Proposition 2.1, we have the following two estimates for our new g:

$$g \in S\left(1, \Gamma\right), \ \partial g \in S(\mu^{-1}, \Gamma). \tag{8.2}$$

We verify now that some other symbols are good symbols for the metric Γ. We first observe the evident fact that $S(m, \Gamma_0) \subset S(m', \Gamma)$ for all weights $m' \geq m$, since $\Gamma_0 \leq \Gamma$. From (1.4) we get

$$\partial p \in S(\lambda, dx^2 + d\xi^2/\lambda^2) \Rightarrow \partial p \in S(\mu^3 h^{-1}, \Gamma), \tag{8.3}$$

since $\mu^3 \geq h\lambda$. Of course in this new class, p, ∂p are no more symbols of order 2 and 1 respectively. Nevertheless the real part p_1 has a good behavior:

$$p_1 \in S(\mu^2, \Gamma). \tag{8.4}$$

Indeed, $0 \leq p_1 \leq \mu^2$ and since the second derivative of p_1 is bounded we use (2.11) to get $|\partial p_1| \leq C\sqrt{p_1} \leq C\mu$. Moreover, $\partial^2 p_1 \in S(1, dx^2 + d\xi^2/\lambda^2)$ gives $\partial^2 p_1 \in S(1, \Gamma)$. This implies (8.4).

From the preceding section we know that Γ is a (semiclassically) admissible metric of uncertainty parameter $h^{1/3}\mu$. We have therefore the following symbolic expansion for the composition of $q_1 \in S(m_1, \Gamma)$ and $q_2 \in S(m_2, \Gamma)$:

$$q_1 \sharp q_2(x,\xi,h) = q_1 q_2(x,\xi,h) + \frac{h}{2i}\left\{q_1, q_2\right\}(x,\xi,h) + h^2 R_2(q_1,q_2)(x,\xi,h), \tag{8.5}$$

where

$$R_2(q_1,q_2) \in S(m_1 m_2 (h^{1/3}\mu)^{-2}). \tag{8.6}$$

This means that in the remainder of order two in the asymptotic expansion of the sharp product, we have a gain of $(h^{1/3}\mu)^{-1}$ to the square in addition to the gain of h^2 due to the semiclassical point of view. The Fefferman-Phong inequality reads for Γ:

Lemma 8.1 *Let m be an h-admissible weight and $q \in S(m,\Gamma)$. If $\mathrm{Re}\, q \geq 0$ then there is a real symbol $r \in S(mh^2(h^{1/3}\mu)^{-2})$ such that $\mathrm{Re}\,(q^w u, u) \geq (r^w u, u)$ for all $u \in ß$. In particular symbols in $S(h^{-2}(h^{1/3}\mu)^2,\Gamma)$ with non-negative real part correspond to operators with real part bounded from below by an h-independent constant in the operator sense.*

For the symbols we deal with, we noted in (8.2-8.3) that ∂p and ∂g have better symbolic estimates than the one given by the symbolic classes of p and g. This gives improvements to the symbolic calculus. Let us write explicitly the expansion of $q_1 \sharp q_2$ to the order d

$$\begin{aligned}(q_1\sharp q_2)(x,\xi,h) = &\sum_{j=0}^{d-1} \frac{h^j}{j!}\left(\frac{i}{2}\sigma\left(D_{x,\xi}, D_{y,\eta}\right)\right)^j q_1(x,\xi,h) q_2(y,\eta,h)|_{y=x,\eta=\xi} \\ &+ h^d R_d(q_1,q_2)(x,\xi,h),\end{aligned} \tag{8.7}$$

where

$$\begin{aligned}R_d(q_1,q_2)(x,\xi,h) = &\int_0^1 \frac{(1-\theta)^{d-1}}{(d-1)!} e^{\frac{i\theta h}{2}\sigma(D_{x,\xi},D_{y,\eta})} \\ &\left(\frac{i}{2}\sigma(D_{x,\xi},D_{y,\eta})\right)^d q_1(x,\xi,h) q_2(y,\eta,h)|_{y=x,\eta=\xi} d\theta.\end{aligned} \tag{8.8}$$

The order (as a symbol in a class $S(m,\Gamma)$), computed as in the classical case, is exactly the order of the symbol appearing on the second line

$$\left(\frac{i}{2}\sigma(D_{x,\xi},D_{y,\eta})\right)^d q_1(x,\xi,h) q_2(y,\eta,h)|_{y=x,\eta=\xi}.$$

Now return to the case of p and g with $d=2$. A straightforward computation using (8.2-8.3) gives that

$$\begin{aligned}&\left(\frac{i}{2}\sigma(D_{x,\xi},D_{y,\eta})\right)^2 g(x,\xi,h) p(y,\eta,h)|_{y=x,\eta=\xi} \\ &\qquad \in S(\mu^3 h^{-1} \times \mu^{-1} \times h^{-1/3}\mu^{-1},\Gamma) \subset S(h^{-4/3}\mu,\Gamma),\end{aligned}$$

hence

$$R_2(g,p) \in S(h^{-4/3}\mu,\Gamma),$$

so

$$g\sharp p = gp + \frac{h}{2i}\{g,p\} + r \quad \text{with} \quad r = h^2 R_2(g,p) \in S(h^{2/3}\mu,\Gamma). \tag{8.9}$$

(Note that this implies $r \in S(h^{1/3}\mu^2,\Gamma) \subset S(\mu^2,\Gamma)$ since $h^{1/3} \leq \mu$).

Let us now fix $\epsilon > 0$ and take $z \in \mathbb{C}$. We can write for $u \in ß$ using (8.9) that

$$\begin{aligned}\mathrm{Re}\ ((p^w - z)u, (1-\epsilon g)^w u) &= \mathrm{Re}\ \left(((1-\epsilon g)\sharp(p-z))^w u, u\right) \\ &= \left(((p_1 - \mathrm{Re}\, z)(1-\epsilon g) + \epsilon h\{p_2, g\}/2 - \epsilon \mathrm{Re}\, r)^w u, u\right),\end{aligned} \tag{8.10}$$

where $r \in S(\mu^2,\Gamma)$ was defined in (8.9). Let us study the first two terms in the asymptotic development of $\mathrm{Re}\,(p-z)\sharp(1-\epsilon g)$. For ϵ sufficiently small, we have from (2.3)

$$p_1 + \epsilon h\{p_2, g\}/2 \geq \epsilon_1\left((h\lambda)^{2/3} + p_1\right) = \epsilon_1 \mu^2, \tag{8.11}$$

when $|(x,\xi)| \geq \mathcal{O}(1)$ far from the critical points (recall that $G \stackrel{\text{def}}{=} hg$ in (2.3)). This means that $p_1 + \epsilon_0 h\{p_2, g\}/2$ is elliptic in $S(\mu^2, \Gamma)$ far from the critical points. Choose $\varphi \in \mathcal{C}_0^\infty$ equal to 1 in a neighborhood of the critical points, so that

$$p_1 + \epsilon h\{p_2, g\}/2 \geq \epsilon_1 \mu^2 - C\mu^2 \varphi(x,\xi). \tag{8.12}$$

Recall that $r \in S(h^{1/3}\mu^2, \Gamma)$. Using this and choosing ϵ sufficiently small yields

$$\begin{aligned} \mathrm{Re}\,(p-z)\sharp(1-\epsilon g) &= (p_1 - \mathrm{Re}\,z)(1-\epsilon g) + \epsilon h\{p_2, g\}/2 - \epsilon \mathrm{Re}\, r \\ &\geq c\mu^2 - 2\max(\mathrm{Re}\,(z), 0) - \mu^2 \varphi. \end{aligned} \tag{8.13}$$

Let us now introduce

$$Z \stackrel{\text{def}}{=} h^{2/3}|z|^{1/3}.$$

We follow the preceding computations, and get with $\epsilon_2 > 0$ that

$$\begin{aligned} &c\mu^2 - 2\max(\mathrm{Re}\,(z), 0) - C\mu^2\varphi \\ &\quad \geq (c/2)\mu^2 + \frac{c}{2}(\mu^2 - \epsilon_2 Z) + \Big(\frac{c\epsilon_2}{2} Z - 2\max(\mathrm{Re}\,(z), 0)\Big) - C\mu^2\varphi. \end{aligned} \tag{8.14}$$

We will bound from below each term of the right hand side. We assume that

$$\frac{c\epsilon_2}{2} Z \geq 4\mathrm{Re}\,(z). \tag{8.15}$$

It defines a region Σ in the complex plane, and if z is in this region the third term of (8.14) is bounded from below by cZ. To study the second term we observe that $\mu^2 \geq \epsilon_2 Z$ since $\lambda^2 \geq |z|\epsilon_2^3$. Now choose a cutoff function $\psi_1(t)$ supported in the ball of radius $2\epsilon_2^3$ and equal to one in the ball of radius ϵ_2^3. Then

$$\frac{c}{2}(\mu^2 - \epsilon_2 Z) \geq -c'' Z \psi_1^2(\lambda^2/|z|)$$

Summing up the preceding results we have obtained the following bound, where c, C denote fixed constants

$$(p_1 - \mathrm{Re}\,z)(1-\epsilon g) + \epsilon h\{p_2, g\}/2 + \epsilon \mathrm{Re}\, r \geq c(\mu^2 + Z) - CZ\psi_1^2(\lambda^2/|z|) - C\mu^2\varphi \tag{8.16}$$

Note that $\psi_1^2(\lambda^2/|z|) \in S(1, \Gamma_0)$. Now we want to go back to the operator side. We first notice that dividing the two sides of (8.16) by Z yields an inequality in $S(h^{-1}\mu^2, \Gamma)$ uniformly in z, which we recall can be arbitrarily large. Indeed the terms p_1, $h\{p_2, g\}$, r and μ^2 are in $S(\mu^2, \Gamma)$ and since $Z \gg h$ (from $|z| \gg h$) we get that these operators divided by Z are in $S(h^{-1}\mu^2, \Gamma)$. The others (divided by Z) are bounded by a constant since by hypothesis $\max\{\mathrm{Re}\,(z), 0\} \leq CZ$, and a fortriori are in $S(h^{-1}\mu^2, \Gamma)$.

Let us apply the inequality of Fefferman-Phong, Lemma 8.1, in this class to this operator. We get using (8.10-8.16) divided by Z and then multiplying by Z

$$\begin{aligned} ((\mu^2 + Z)^w u, u) \leq\; & C\mathrm{Re}\,((p^w - z)u, (1-\epsilon g)^w u) + CZ\left(\psi_1^2(\lambda^2/|z|)^w u, u\right) \\ & + C((\mu^2\varphi)^w u, u) + Zh^2 \mathrm{Re}\,(R^w u, u), \end{aligned} \tag{8.17}$$

where $h^2 R$ is of order $h^2(h^{-1}\mu^2)(\mu h^{1/3})^{-2} = h^{1/3}$ (recall that $\mu h^{1/3}$ is the uncertainty parameter of Γ). Choosing h small enough and using (8.19) below, gives

$$Zh^2 R^w \leq \frac{1}{4} Z \leq \frac{1}{2}(Z + \mu^2)^w,$$

and we therefore get for h small enough and an other constant C

$$((\mu^2+Z)^w u,u) \leq C\mathrm{Re}\,((p^w-z)u,(1-\epsilon g)^w u) + CZ\left(\psi_1^2(\lambda^2/|z|)^w u,u\right) + C((\mu^2\varphi)^w u,u). \quad (8.18)$$

We shall use the following

Lemma 8.2 *we have* $\left(\psi_1^2(\lambda^2/|z|)^w u,u\right) \leq \frac{C}{\max(1,|z|^2)}\|(p^w-z)u\|^2 + Ch\|u\|^2$.

Let us suppose for a while that this lemma is proven. We first write that for ϵ sufficiently small,

$$\mathrm{Re}\,((p^w-z)u,(1-\epsilon g)^w u) \leq C\|(p^w-z)u\|\|u\|.$$

Then we observe that $\mu^2 \geq 0$ and the Fefferman-Phong inequality in $S(\mu^2,\Gamma)$ yields $(\mu^2)^w \geq -Ch^{4/3}$. Since $Z \gg h$, we have for h sufficiently small

$$Z\|u\|^2 \leq 2((Z+\mu^2)^w u,u). \quad (8.19)$$

Then we use this result and the lemma which yields from (8.18) that

$$Z\|u\|^2 \leq C\|(p^w-z)u\|\|u\| + Z\frac{C}{\max(1,|z|^2)}\|(p^w-z)u\|^2 + CZh\|u\|^2 + C\|(\mu^2\varphi)^w u\|\|u\|.$$

Choosing h sufficiently small and noticing that $Z^2 \leq \max(1,|z|^2)$ yields the main result of this section

$$Z\|u\| \leq C\|(p^w-z)u\| + C\|(\mu^2\varphi)^w u\| \quad (8.20)$$

where we recall that $|z| \gg h$ and that $\mathrm{Re}\,(z) \leq CZ \stackrel{\mathrm{def}}{=} Ch^{2/3}|z|^{1/3}$.

It remains to prove Lemma 8.2.

Proof of Lemma 8.2. We first observe that for $|z| \ll \mathcal{O}(1)$, we have $\psi_1^2(\lambda^2/|z|) = 0$ since $\lambda \geq 1$ and the support of ψ_1 is bounded. Therefore we can suppose that $|z| \geq \mathcal{O}(1)$, since in the other case, the left member of the inequality in the lemma is zero. To prove the result we can go back to the original metric $dx^2 + d\xi^2/\lambda^2$. We first notice that since $p = \mathcal{O}(\lambda^2)$ we can choose the support of ψ_1 (i.e. ϵ_1 in (8.14)) such that

$$|p-z| \geq |z|/2 \text{ on the support of } \psi_1.$$

We notice also that uniformly with respect to z, we have

$$\frac{(p-z)}{|z|}\psi_1(\lambda^2/|z|) \in S(1, dx^2 + d\xi^2/\lambda^2).$$

We therefore have the following inequality in $S(1, dx^2+d\xi^2/\lambda^2)$:

$$\begin{aligned}\psi_1^2(\lambda^2/|z|) &\leq 4\frac{|p-z|^2}{|z|^2}\psi_1^2(\lambda^2/|z|) \\ &\leq 4\mathrm{Re}\,\frac{\overline{(p-z)}}{|z|}\psi_1(\lambda^2/|z|)\sharp\frac{(p-z)}{|z|}\psi_1(\lambda^2/|z|) + hR,\end{aligned} \quad (8.21)$$

where by the symbolic calculus, $hR \in S(h\lambda^{-1}, dx^2 + d\xi^2/\lambda^2) \subset S(h, dx^2 + d\xi^2/\lambda^2)$. Using the Gårding inequality for this inequality, we get

$$\left(\psi_1^2(\lambda^2/|z|)^w u, u\right) \leq \| \left(\frac{(p-z)}{|z|} \psi_1(\lambda^2/|z|) \right)^w u\|^2 + \mathcal{O}(h)\|u\|^2. \tag{8.22}$$

We next use the symbolic calculus and get from (8.7) to first order and using the notation from there

$$\psi_1(\lambda^2/|z|)\sharp \frac{p-z}{|z|} = \psi_1(\lambda^2/|z|) \left(\frac{p-z}{|z|} \right) + \frac{h}{|z|} R_1(\psi_1(\lambda^2/|z|), p). \tag{8.23}$$

Now observe that uniformly in $z \geq \mathcal{O}(1)$ we have

$$\partial \left(\psi_1(\lambda^2/|z|) \right) \in S(\lambda^{-1}, dx^2 + d\xi^2/\lambda^2), \quad \text{and} \quad \partial p \in S(\lambda, dx^2 + d\xi^2/\lambda^2).$$

The first fact follows from $\partial\lambda \in S(1, dx^2 + d\xi^2/\lambda^2)$ and the second from (1.4). Consequently we get a better estimate than the one that would be given by the classical symbolic calculus in the class associated with the metric $dx^2 + d\xi^2/\lambda^2$, namely

$$R_1(\psi_1(\lambda^2/|z|), p) \in S(1, dx^2 + d\xi^2/\lambda^2).$$

Since $|z| \geq \mathcal{O}(1)$, we get that

$$\frac{h}{|z|} R_1(\psi_1(\lambda^2/|z|), p) \in S(h, dx^2 + d\xi^2/\lambda^2).$$

Using this together with (8.22, 8.23) yields

$$\left(\psi_1^2(\lambda^2/|z|)^w u, u\right) \leq \| \left(\psi_1(\lambda^2/|z|)\right)^w \left(\frac{p-z}{|z|} \right)^w u\|^2 + \left(\mathcal{O}(h) + \mathcal{O}(h^2)\right) \|u\|^2. \tag{8.24}$$

Since $\left(\psi_1(\lambda^2/|z|)\right)^w$ is bounded, we get the lemma. □

9 Resolvent estimates away from the critical points when z is small

We work again in this section with p satisfying **(H2), (H3), (H4)** and away from the critical points, but for small z. Here the spectral parameter will be denoted hz for $z = \mathcal{O}(1)$. We recall some notations of the preceding section, namely

$$\mu^2 = p_1 + (h\lambda)^{2/3}, \quad \Gamma = \frac{dx^2}{h^{2/3}} + \frac{d\xi^2}{\mu^2}.$$

As in the preceding section, we fix $\epsilon > 0$ and work with our operator p satisfying conditions (1.4-1.5). We can write for $u \in ß$

$$\begin{aligned} \text{Re } ((p^w - hz)u, (1-\epsilon g)^w u) &= \text{Re } \left(\left((1-\epsilon g)\sharp(p-hz)\right)^w u, u \right) \\ &= \left(\left((p_1 - \text{Re}\, hz)(1-\epsilon g) + \epsilon h \left\{p_2, g\right\}/2 + \epsilon \text{Re}\, r\right)^w u, u \right), \end{aligned} \tag{9.1}$$

following the same computations as in (8.10–8.16). We also get that

$$\begin{aligned}\mathrm{Re}\,(1-\epsilon g)\sharp(p-hz) &= (p_1-\mathrm{Re}\,hz)(1-\epsilon g)+\epsilon h\,\{p_2,g\}/2+\epsilon \mathrm{Re}\, r\\ &\geq c\mu^2-2\max(\mathrm{Re}\,(hz),0)-\mu^2\varphi,\end{aligned}\tag{9.2}$$

where $\varphi\in\mathcal{C}_0^\infty$ is equal to 1 in a neighborhood of the critical points, and where we recall that $r\in S(h^{2/3}\mu,\Gamma)$ was defined in (8.9). Of course outside this fixed neighborhood, and for h small enough, we have, using $\mu\geq h^{1/3}$,

$$\mu^2\gg 2\mathrm{Re}\,(hz),$$

therefore with a new function φ,

$$\mathrm{Re}\,(p-hz)\sharp(1-\epsilon g)\geq c\mu^2/2-\mu^2\varphi. \tag{9.3}$$

We can now use the Fefferman-Phong inequality (Lemma 8.1). Indeed, each term is in $S(\mu^2,\Gamma)$ and we get

$$((\mu^2)^w u,u)\leq C\mathrm{Re}\,((p^w-hz)u,(1-\epsilon g)^w u)+C((\mu^2\varphi)^w u,u)+\mathrm{Re}\,(R^w u,u), \tag{9.4}$$

where R is of order $h^2\times\mu^2\times(\mu h^{1/3})^{-2}=h^{4/3}$ from lemma 8.1 (recall that $\mu h^{1/3}$ is the uncertainty parameter of Γ). Choosing h small enough and noticing that

$$(\mu^2)^w\geq ch^{2/3},$$

gives

$$ch^{2/3}\|u\|^2\leq \big((\mu^2)^w u,u)\big)\leq C\mathrm{Re}\,((p^w-hz)u,(1-\epsilon g)^w u)+C((\mu^2\varphi)^w u,u). \tag{9.5}$$

We next write that for ϵ sufficiently small,

$$\mathrm{Re}\,((p^w-hz)u,(1-\epsilon g)^w u)\leq\|(p^w-hz)u\|\|u\|.$$

From this and (9.5) we get the main result of this section

$$ch^{2/3}\|u\|\leq C\|(p^w-hz)u\|+C\|(\mu^2\varphi)^w u\| \tag{9.6}$$

10 Proof of Theorem 1.2

In this section we shall glue together all the results of the Sections 4, 6, 8 and 9. We give the results here in the original variables and not on the FBI side.

In the following, we choose $u\in ß$ and we write $U=Tu$ where T is the FBI-Bargmann transform associated with the phase $i(x-y)^2/2$. We also denote by P the operator $(\chi_0 p)^w$ on the FBI side, where χ_0 is some $\mathcal{C}_0^\infty$ function equal to 1 in a very large compact set (including the critical points).

Proof of a). We suppose here that $h|z|\leq\mathcal{O}(h)$. Let us first recall the main result (9.6) of Section 9:

$$h^{2/3}\|u\|\leq C\|(p^w-hz)u\|+C\|(\mu^2\varphi)^w u\|, \tag{10.1}$$

where φ is a cutoff function equal to 1 near the critical points. We choose once and for all another cutoff function ψ equal to one in a larger neighborhood of the critical points, so that $\nabla\varphi\nabla\psi=0$. Then

$$h^{2/3}\|(1-\psi)^w u\|\leq C\|(p^w-hz)(1-\psi)^w u\|+C\|(\mu^2\varphi)^w(1-\psi)^w u\|.$$

Notice that $(\mu^2\varphi)^w(1-\psi)^w = \mathcal{O}(h^\infty)$ as a bounded operator in L^2 since the supports are disjoint. Moreover,

$$(p^w - hz)(1-\psi)^w = (1-\psi)^w(p^w - hz) + \frac{h}{2i}\{\psi, p\}^w + \mathcal{O}(h^2), \tag{10.2}$$

where $q \stackrel{\text{def}}{=} \frac{1}{2i}\{\psi, p\}$ is a symbol with $\operatorname{supp} q \subset \operatorname{supp} \nabla\psi$, so that the support of q is disjoint from the support of φ. Hence

$$h^{2/3}\|(1-\psi)^w u\| \le C\|(1-\psi)^w(p^w - hz)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\|.$$

The L^2-boundedness of $(1-\psi)^w$ and the fact that $h \le h^{2/3}$ give

$$h\|(1-\psi)^w u\| \le C\|(p^w - hz)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\|. \tag{10.3}$$

The main result of Section 6 on the FBI side states that

$$h\|U\|_{\Phi_0} \le \|(P - hz)U\|_{\Phi_0} + h^{5/6}\|(1-\chi)U\|_{\Phi_0}, \tag{10.4}$$

where χ is an arbitrary cutoff function equal to 1 in a neighborhood of the critical points. We can choose χ equal to 1 in a neighborhood of $\operatorname{supp}\psi$, where ψ is viewed as a function on the FBI-side (i.e. $\psi \circ \kappa^{-1}$ where κ is the canonical transform associated with the FBI transform T). With these notations we may write that $\varphi \prec \psi \prec \chi \prec \chi_0$ modulo a composition with κ. Coming back to the real side for the two first terms of this inequality, and using the metaplectic invariance gives

$$h\|u\| \le \|((\chi_0 p)^w - hz)u\| + h^{5/6}\|(1-\chi)U\|_{\Phi_0}, \tag{10.5}$$

and after replacing u by $\psi^w u$,

$$h\|\psi^w u\| \le \|((\chi_0 p)^w - hz)\psi^w u\| + h^{5/6}\|(1-\chi)T\psi^w u\|_{\Phi_0}. \tag{10.6}$$

Now we can treat the term $\|((\chi_0 p)^w - hz)\psi^w u\|$ as in (10.2) and get rid of the term χ_0 modulo a term of order h^∞ and we get with the same q

$$h\|\psi^w u\| \le \|(p^w - hz)u\| + h\|q^w u\| + \mathcal{O}(h^2)\|u\| + h^{5/6}\|(1-\chi)T\psi^w u\|_{\Phi_0}. \tag{10.7}$$

We shall use the following standard lemma for which we briefly review the proof at the end of this section.

Lemma 10.1 *We have* $\|(1-\chi)T\psi^w u\|_{\Phi_0} = \mathcal{O}(h^\infty)\|u\|$.

We can therefore write

$$h\|\psi^w u\| \le \|(p^w - hz)u\| + h\|q^w u\| + \mathcal{O}(h^2)\|u\|. \tag{10.8}$$

Let us now glue together the results (10.3), (10.8) to get

$$h\|(1-\psi)^w u\| + h\|\psi^w u\| \le C\|(p^w - hz)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\|. \tag{10.9}$$

For the term $Ch\|q^w u\|$ we simply apply (10.1) with u replaced by $q^w u$. This gives

$$h^{2/3}\|q^w u\| \le C\|(p^w - hz)q^w u\| + \|\varphi^w q^w u\|. \tag{10.10}$$

Since φ and q have disjoint support, we have $\varphi^w q^w = \mathcal{O}(h^\infty)$ as an operator in L^2. Besides we have

$$(p^w - hz)q^w = q^w(p^w - hz) + \mathcal{O}(h),$$

since q is with compact support. Therefore we get

$$h^{2/3}\|q^w u\| \leq C\|q^w(p^w - hz)u\| + Ch\|u\| \leq \|(p^w - hz)u\| + Ch\|u\|, \tag{10.11}$$

and eventually

$$h\|q^w u\| \leq \|(p^w - hz)u\| + Ch^{4/3}\|u\|. \tag{10.12}$$

Together with (10.9) this yields

$$h\|(1-\psi)^w u\| + h\|\psi^w u\| \leq C\|(p^w - hz)u\| + \mathcal{O}(h^{4/3})\|u\|, \tag{10.13}$$

and using the triangle inequality $\|u\| \leq \|(1-\psi)^w u\| + \|\psi^w u\|$,

$$h\|u\| \leq C\|(p^w - hz)u\| + \mathcal{O}(h^{4/3})\|u\|. \tag{10.14}$$

Taking h small enough completes the proof of part a) of the theorem. □

Proof of b). In this section we suppose that $|z| \gg h$. We also denote in the following

$$Z = |z|^{1/3}h^{2/3}.$$

We shall follow the proof of part a). We first recall the main result (8.20) of Section 8:

$$Z\|u\| \leq C\|(p^w - z)u\| + C\|(\mu^2\varphi)^w u\|, \tag{10.15}$$

where φ is a cutoff function equal to 1 near the critical points. As in the preceding section we choose once and for all another cutoff function ψ such that $\psi \succ \varphi$ and we write

$$Z\|(1-\psi)^w u\| \leq C\|(p^w - z)(1-\psi)^w u\| + C\|(\mu^2\varphi)^w(1-\psi)^w u\|.$$

As in (10.2), (10.3) we get

$$Z\|(1-\psi)^w u\| \leq C\|(p^w - z)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\|, \tag{10.16}$$

where we recall $q \overset{\text{def}}{=} \frac{1}{2i}\{\psi, p\}$ is a symbol with support in $\nabla\psi$.

We now recall the main result of Section 4 on the FBI side (see equation (4.15)):

$$Z\|U\|_{\Phi_0} \leq C\left(\|(P-z)U\|_{\Phi_0} + Z\|(1-\chi)U\|_{\Phi_0}\right), \tag{10.17}$$

where χ is an arbitrary cutoff function equal to 1 in a neighboorhood of the critical points. We choose $\chi \succ \psi$ where ψ is viewed as a function on the FBI side. With this notation we write as in the proof of a) that $\varphi \prec \psi \prec \chi \prec \chi_0$. Coming back to the real side for the two first terms of this inequality, and using the metaplectic invariance gives

$$Z\|u\| \leq C\left(\|((\chi_0 p)^w - z)u\| + Z\|(1-\chi)U\|_{\Phi_0}\right). \tag{10.18}$$

Taking $\psi^w u$ instead of u gives

$$Z\|\psi^w u\| \leq C\bigg(\|((\chi_0 p)^w - z)\psi^w u\| + Z\|(1-\chi)T\psi^w u\|_{\Phi_0}\bigg). \tag{10.19}$$

Now we can treat the term $\|(p^w - z)\psi^w u\|$ as in (10.2) and we get with the same q

$$Z\|\psi^w u\| \leq C\bigg(\|(p^w - z)u\| + h\|q^w u\| + \mathcal{O}(h^2)\|u\| + Z\|(1-\chi)T\psi^w u\|_{\Phi_0}\bigg). \tag{10.20}$$

Using Lemma 10.1 yields,

$$Z\|\psi^w u\| \leq C\|(p^w - z)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\| + Z\mathcal{O}(h^\infty)\|u\|. \tag{10.21}$$

Let us now combine (10.16), (10.21):

$$Z\|(1-\psi)^w u\| + Z\|\psi^w u\| \leq C\|(p^w - z)u\| + Ch\|q^w u\| + \mathcal{O}(h^2)\|u\| + Z\mathcal{O}(h^\infty)\|u\|. \tag{10.22}$$

Now we can use (10.11) and we get with new constants

$$Z\|(1-\psi)^w u\| + Z\|\psi^w u\| \leq C\|(p^w - z)u\| + \mathcal{O}(h^{4/3})\|u\| + \mathcal{O}(h^2)\|u\| + Z\mathcal{O}(h^\infty)\|u\|, \tag{10.23}$$

and the triangle inequality gives

$$Z\|u\| \leq C\|(p^w - z)u\|.$$

The proof of part b) of the theorem is complete. □

Proof of lemma 10.1. We have

$$\|(1-\chi)T\psi^w u\|^2_{\Phi_0} = (u, \psi^w T^*(1-\chi)^2 T\psi^w u) \leq \|u\|\,\|\psi^w T^*(1-\chi)^2 T\psi^w u\|,$$

where the adjoint T^* is w.r.t. the Φ_0 inner product. We will show that $T^*(1-\chi)^2 T$ is a pseudo-differential operator with Weyl symbol that is $\mathcal{O}(h^\infty)$ where $(1-\chi)^2 \circ \kappa$ and all its derivatives vanish. Since ψ and $(1-\chi)\circ\kappa$ have disjoint support, this shows the result.

To simplify the notation we do the computations for $T^*\chi T$. Recall that we use the transform (3.1), with $\varphi(t,y) = i(t-y)^2/2$. The constant C in (3.1) is given by $2^{-\frac{n}{2}}\pi^{-\frac{3n}{4}}$. The function Φ_0 equals $-(\operatorname{Im} t)^2/2$ and $(\chi\circ\kappa)(x,\xi) = \chi(x - i\xi)$. If we write $t = x - i\xi$, then the kernel $K(y,z)$ of $T^*\chi T$ is given by

$$K(y,z) = 2^{-n}(\pi h)^{-\frac{3n}{2}} \int e^{-i(x-y)\cdot\xi/h - (x-y)^2/(2h) + i(x-z)\cdot\xi/h - (x-z)^2/(2h)}\chi(x - i\xi)\,dx d\xi. \tag{10.24}$$

The phase function can be written as

$$i(y-z)\cdot\xi/h - (x - \frac{y+z}{2})^2/h - (y-z)^2/(4h).$$

For the last term in this expression we have from a Fourier transformation

$$e^{-(y-z)^2/(4h)} = (\pi h)^{-n/2}\int e^{i(y-z)\cdot(\eta-\xi)/h - (\eta-\xi)^2/h}\,d\eta.$$

Entering this in (10.24) we find that $K(y,z)$ equals

$$2^{-n}(\pi h)^{-2n}\int e^{i(y-z)\cdot\eta/h-(x-\frac{y+z}{2})^2/h-(\eta-\xi)^2/h}\chi(x-i\xi)\,dxd\xi d\eta.$$

This formally equals a Weyl pseudodifferential operator with symbol

$$\tilde{\chi}(y,\eta)=(\pi h)^{-n}\int e^{-(y-x)^2/h-(\eta-\xi)^2/h}\chi(x-i\xi)\,dxd\xi.$$

It is clear that $\tilde{\chi}$ has the correct symbol property, and that $\tilde{\chi}(y,\eta)=\mathcal{O}(h^\infty)$ for (y,η) such that $(\chi\circ\kappa)(y,\eta)$ and all its derivatives vanish. This completes the proof. □

11 Asymptotic expansion of the eigenvalues.

From apriori estimates to the resolvent.

In the previous sections we obtained apriori estimates for z in a subset of $\mathbb{C}$, given by

$$\|u\|\le C\|(P-z)u\|,\quad \forall u\in\mathcal{S}(\mathbb{R}^n). \tag{11.1}$$

We will show that such estimates imply the existence of the resolvent of P.

We will first establish this for one particular value of z. For this purpose we will use some functional analysis, and results given in section 5.2 of [8]. Following [8] we define $P:\mathcal{D}(P)\to L^2(\mathbb{R}^n)$ with domain $\mathcal{D}(P)=\mathcal{C}_0^\infty(\mathbb{R}^n)$. We let $\overline{P}$ be its closure (further on we will simply write P instead of $\overline{P}$ but for the moment we keep the distinction).

We show that $\mathcal{S}(\mathbb{R}^n)\subset\mathcal{D}(\overline{P})$. This follows if for $u\in\mathcal{S}(\mathbb{R}^n)$ there is a sequence $u_j\in\mathcal{C}_0^\infty(\mathbb{R}^n)$ with $u_j\to u$ in $L^2(\mathbb{R}^n)$ and $Pu_j\to Pu$ in $L^2(\mathbb{R}^n)$. Such a sequence is given by $u_j=\chi(\frac{x}{j})u(x)$, where $\chi\in C_0^\infty(\mathbb{R}^n,[0,1])$ is equal to 1 on a neighborhood of 0. We have $Pu_j=\chi(\frac{\cdot}{j})Pu+[P,\chi(\frac{\cdot}{j})]u\to Pu$, since the symbol of the commutator tends to zero in $S(\lambda,\Gamma_0)$. By the definition of $\overline{P}$ we have that in fact $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{D}(\overline{P})$.

Next we establish the existence of the resolvent for at least one value z_0 in the left complex half plane, when there is a real λ_0 such that $\overline{P}+\lambda_0$ is maximally accretive. For the Kramers-Fokker-Planck operator this property is established in proposition 5.5 of [8] with $\lambda_0=0$.

Proposition 11.1 *Assume that $\overline{P}+\lambda_0$ is maximally accretive. Then there is $\lambda_1>\lambda_0$ such that $(\overline{P}+\lambda_1)^{-1}$ exists and is a bounded operator on $L^2(\mathbb{R}^n)$.*

Proof. The accretivity of $\overline{P}+\lambda_0$ means that $((\overline{P}+\lambda_0)u,u)\ge 0$ for each $u\in\mathcal{D}(\overline{P})$. It follows that for each $\lambda>\lambda_0$ we have

$$\|(\overline{P}+\lambda)u\|\|u\|\ge((\overline{P}+\lambda_0)u,u)+(\lambda-\lambda_0)\|u\|^2\ge(\lambda-\lambda_0)\|u\|^2,\quad u\in\mathcal{D}(\overline{P}),$$

hence

$$\|u\|\le(\lambda-\lambda_0)^{-1}\|(\overline{P}+\lambda)u\|,\quad u\in\mathcal{D}(\overline{P}). \tag{11.2}$$

Hence $\overline{P}+\lambda$ is injective.

Suppose now that there is a sequence $u_j \in \mathcal{S}(\mathbb{R}^n)$ such that $(\overline{P}+\lambda)u_j \to v$ in $L^2(\mathbb{R}^n)$ for some $v \in L^2(\mathbb{R}^n)$. Denote $v_j = (\overline{P}+\lambda)u_j$. Then by the estimate (11.2) it follows that $\|u_j - u_k\| \to 0, j, k \to \infty$, hence u_j converges to an element u in $L^2(\mathbb{R}^n)$. Now $(u_j, v_j) \in \mathrm{graph}(\overline{P})$ and $u_j \to u, v_j \to v$ in $L^2(\mathbb{R}^n)$. Therefore, the range $\mathcal{R}(\overline{P})$ is closed. Theorem 5.4 of [8] and the fact that $\overline{P}$ is maximally accretive imply that for some $\lambda_1 > \lambda_0$, the range of $P+\lambda_1$ is also dense in $L^2(\mathbb{R}^n)$. It follows that $\overline{P}+\lambda_1$ is surjective, that the inverse $(\overline{P}+\lambda_1)^{-1}$ exists and that its norm is bounded by $\frac{1}{\lambda_1-\lambda_0}$. □

Remark 11.2 Alternatively we could use the following additional properties

$$\|u\| \leq C\|(P^* - \overline{z})u\|, \quad \forall u \in \mathcal{S}(\mathbb{R}^n), \tag{11.3}$$

$$u \in L^2(\mathbb{R}^n), (P-z)u \in \mathcal{S}(\mathbb{R}^n) \Rightarrow u \in \mathcal{S}(\mathbb{R}^n). \tag{11.4}$$

(Here we let $\mathcal{D}(P) = \{u \in L^2(\mathbb{R}^n)\,;\, Pu \in L^2(\mathbb{R}^n)\}$.) The first property is similar to the apriori estimate (11.1). The second property can for example be derived from hypoelliptic estimates in a chain of weighted Sobolev spaces as given for the Kramers-Fokker-Planck case in theorem 3.1d) of [9] (a result valid under somewhat different conditions than used here). In short the argument using (11.4) goes as follows. By a standard argument estimate (11.3) and the Hahn-Banach theorem imply the surjectivity of $P-z$. If $u \in \mathcal{D}(P)$ and $(P-z)u = 0$, then (11.4) implies that $u \in \mathcal{S}(\mathbb{R}^n)$ and (11.1) that $u = 0$. Hence $P : \mathcal{D}(P) \to L^2(\mathbb{R}^n)$ is injective, and $(P-z)^{-1} : L^2 \to L^2$ is bounded and $\|(P-z)^{-1}\| \leq C$. One can also show that under these assumptions $\mathcal{S}(\mathbb{R}^n)$ is dense in $\mathcal{D}(P)$ for the graph norm.

From now on we simply write P instead of $\overline{P}$. To obtain the resolvent, we consider an abstract situation. Let $P : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be a closed operator and assume (as we established above)

$$\mathcal{S}(\mathbb{R}^n) \text{ is dense in } \mathcal{D}(P). \tag{11.5}$$

Since $\mathcal{D}(P)$ has the norm $\|u\|_{\mathcal{D}(P)} = \|u\| + \|Pu\|$, this means that for every $u \in \mathcal{D}(P)$, there is a sequence $u_j \in \mathcal{S}$, $j = 1, 2, ...$, such that $u_j \to u$ and $Pu_j \to Pu$ in L^2.

Let $\Omega \subset \mathbb{C}$ be a connected open set. Let $z_0 \in \Omega$ an assume

$$(z_0 - P)^{-1} : L^2 \to \mathcal{D}(P) \text{ exists}, \tag{11.6}$$

$$\|u\| \leq C_K\|(P-z)u\|, \ \forall u \in \mathcal{S}, \ z \in K, \tag{11.7}$$

for every $K \subset\subset \Omega$.

Proposition 11.3 *Under these assumptions, $(z-P)^{-1}$ exists for every $z \in \Omega$.*

Proof. Using (11.5), we see that the apriori estimate in (11.7) extends to all $u \in \mathcal{D}(P)$. In particular, $z - P : \mathcal{D}(P) \to L^2$ is injective for $z \in \Omega$, so it remains to show that $z - P$ is surjective. If $(z_1 - P)^{-1}$ exists for some $z_1 \in K \subset\subset \Omega$, then (11.7) (extended to $\mathcal{D}(P)$) implies that $\|(z_1 - P)^{-1}\| \leq C_K$. Hence $\|(z - z_1)(z_1 - P)^{-1}\| < 1$ for $|z - z_1| < 1/C_K$, and we conclude that $z - P : \mathcal{D}(P) \to L^2$ has a right inverse of the form $(z_1 - P)^{-1}(1 + (z - z_1)(z_1 - P)^{-1})^{-1}$. If in addition, $z \in \Omega$, this right inverse is equal to the resolvent.

If $z \in \Omega$ is any given point, we take a smooth curve γ in Ω from z_0 to z, and cover γ by finitely many discs $D(z_j, r)$, $j = 0, 1, 2, ..., M$, such that $r < 1/C_\gamma$, $z_{j+1} \in D(z_j, r)$. Hence $(z - P)^{-1}$ exists. □

The same result is valid for Grushin problems. We keep the initial hypothesis about P. Let $R_- : \mathbb{C}^{N_0} \to L^2$, $R_+ : \mathcal{D}(P) \to \mathbb{C}^{N_0}$ be bounded and for simplicity independent of z. Put

$$\mathcal{P}(z) = \begin{pmatrix} P - z & R_- \\ R_+ & 0 \end{pmatrix} : \mathcal{D}(P) \times \mathbb{C}^{N_0} \to L^2 \times \mathbb{C}^{N_0}. \tag{11.8}$$

Assume still that (11.6) holds for some $z_0 \in \Omega$. Instead of (11.7), we assume

$$\|u\| + |u_-| \leq C_K(\|(P - z)u + R_-u_-\| + |R_+u|),\ z \in K,\ u \in \mathcal{S},\ u_- \in \mathbb{C}^{N_0}, \tag{11.9}$$

for every $K \subset\subset \Omega$. (Again, this extends to the case $u \in \mathcal{D}(P)$.)

Proposition 11.4 *Under the above assumptions, $\mathcal{P}(z)$ has a bounded inverse for every $z \in \Omega$.*

Proof. As before, we notice that (11.9) implies that

$$\|u\|_{\mathcal{D}(P)} + |u_-| \leq C_K(\|(P - z)u + R_-u_-\| + |R_+u|),\ u \in \mathcal{D}(P),\ u_- \in \mathbb{C}^{N_0},\ z \in K, \tag{11.10}$$

with a new constant C_K, so $\mathcal{P}(z)$ is injective for all $z \in \Omega$.

For $z = z_0$, $(P - z_0) : \mathcal{D}(P) \to L^2$ has a bounded inverse and is therefore a Fredholm operator of index 0. Hence,

$$Q := \begin{pmatrix} P - z_0 & 0 \\ 0 & O \end{pmatrix} : \mathcal{D}(P) \times \mathbb{C}^{N_0} \to L^2 \times \mathbb{C}^{N_0}$$

is Fredholm of index 0 and $\mathcal{P}(z_0)$ has the same property, being a finite rank perturbation of Q. Being injective by (11.9), it is bijective, and as in the preceding proof, we see that $\mathcal{P}(z)^{-1}$ exists for all $z \in \Omega$ with $|z - z_0| < 1/C_K$ if $z_0 \in K$. By the same procedure as above, we get the result. □

Grushin problem in the quadratic case

Let P_0 be a quadratic operator on $L^2(\mathbb{R}^n)$, so that P_0 has the Weyl symbol $\sum_{|\alpha+\beta|=2} a_{\alpha,\beta} x^\alpha \xi^\beta$ that we also denote by $P_0(x, \xi)$. (We can also add a constant to our symbol, but we shall avoid for simplicity to have linear terms in the symbol.) As in [15] we assume that P_0 is elliptic away from (0,0):

$$P_0(x, \xi) \neq 0,\ (x, \xi) \in \mathbb{R}^{2n} \setminus \{(0, 0)\}. \tag{11.11}$$

When $n > 1$ this implies that $P_0(\mathbb{R}^{2n})$ is a proper cone in $\mathbb{C}$ and when $n = 1$ we assume that so is the case. Then P_0 is a closed *operator* : $L^2 \to L^2$ with domain $\mathcal{D}(P_0) = \langle (x, D) \rangle^{-2}(L^2)$ and the assumption (11.5) is fulfilled. P_0 has discrete spectrum and the eigenvalues are computed in [15] as recalled in Section 5. They are contained in $P_0(\mathbb{R}^{2n})$.

Let $\lambda_0 \in \mathbb{C}$ be such an eigenvalue and let $E_{\lambda_0} \subset \mathcal{D}(P_0)$ be the corresponding space of generalized eigenvectors. Let $e_1, ..., e_{N_0}$ be a basis for E_{λ_0} and let $f_1, ..., f_{N_0} \in \mathcal{S}(\mathbb{R}^n)$ have the property that

$$\det((e_j|f_k)) \neq 0. \tag{11.12}$$

A possibly natural choice would be to let $f_1, ..., f_{N_0}$ be the dual basis in the space $E^*_{\overline{\lambda}_0}$ of generalized eigenvectors of P^*, associated to the eigenvalue $\overline{\lambda}_0$.

Put

$$R_- u_- = \sum u_-(j) e_j, \; R_+ u = ((u|f_j)) \in \mathbb{C}^{N_0},$$

for $u_- = (u_-(j)) \in \mathbb{C}^{N_0}$. For $\lambda \in \mathrm{neigh}\,(\lambda_0)$, the problem

$$(P_0 - \lambda)u + R_- u_- = v, \; R_+ u = v_+, \tag{11.13}$$

has a unique solution $(u, u_-) \in \mathcal{D}(P_0)$, for every $(v, v_+) \in L^2 \times \mathbb{C}^{N_0}$. In fact, let $\Pi : L^2 \to E_{\lambda_0}$ be the spectral projection and decompose $u = u' + u''$, $v = v' + v''$, with $u'' = \Pi u$, $u' = (1 - \Pi)u$ and similarly for v. Then the equation for u' is $(P_0 - \lambda)u' = v'$ and determines $u' \in \mathcal{D}(P)$ uniquely. u'' is completely determined by the condition $R_+ u'' = v_+ - R_+ u'$, thanks to the assumption (11.12). Finally u_- is determined by $R_- u_- = v' - (P_0 - \lambda)u'$.

If we introduce the solution operator

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}, \text{ by } \begin{pmatrix} u \\ u_- \end{pmatrix} = \mathcal{E} \begin{pmatrix} v \\ v_+ \end{pmatrix},$$

then we also know that

$$\begin{pmatrix} \Lambda^{2-k} & \\ 0 & 1 \end{pmatrix} \mathcal{E} \begin{pmatrix} \Lambda^k & 0 \\ 0 & 1 \end{pmatrix} \text{ is bounded} \tag{11.14}$$

for every $k \in \mathbb{R}$, when $\Lambda = \langle (x, D) \rangle$. We also notice that if $M(\lambda)$ denotes the matrix of $(\lambda - P_0)_{|E_{\lambda_0}}$ with respect to the basis $e_1, ..., e_{N_0}$, then

$$E_{-+}(\lambda) = M(\lambda)((e_k|f_j))^{-1}. \tag{11.15}$$

We now choose P_0 as in Proposition 5.1, acting on $H_{\Phi^0_\epsilon}$, where Φ^0_ϵ is a quadratic form with $\Lambda_{\Phi^0_\epsilon} = \kappa_T(\Lambda_{\epsilon G^0})$ and G^0 is a real quadratic form chosen as in 5.3. Here $\epsilon > 0$ is small and fixed and the earlier assumptions are fulfilled with the real phase space replaced by $\Lambda_{\Phi^0_\epsilon}$. As in Section 5 we now work with the h-quantization. Then, if λ_0 is an eigenvalue of the $(h = 1)$ quantization, we get the well-posed Grushin problem

$$(P_0 - hz)u + R_- u_- = v, \; R_+ u = v_+, \tag{11.16}$$

for z in some fixed neighborhood of λ_0. Here, we take

$$R_+ u(j) = (u|f_{j,h})_{L^2_{\Phi^0_\epsilon}}, \; R_- u_- \sum u_-(j) e_{j,h}, \tag{11.17}$$

with $f_{j,h}(x) = h^{-\frac{n}{2}} f_j(\frac{x}{\sqrt{h}})$, and similarly for $e_{j,h}$, so that

$$R_+ = \mathcal{O}(1) : L^2_{\Phi^\epsilon_0} \to \mathbb{C}^{N_0}, \; R_- = \mathcal{O}(1) : \mathbb{C}^{N_0} \to H_{\Phi^\epsilon_0},$$

uniformly, when $h \to 0$. More precisely, we have (cf. Proposition 5.1):

Proposition 11.5 *For every $(v, v_+) \in H_{\Phi^\epsilon_0} \times \mathbb{C}^{N_0}$, the problem (11.16) has a unique solution in the same space and the solution satisfies: $d^2 u \in L^2_{\Phi^\epsilon_0}$. Moreover, for every fixed $k \in \mathbb{R}$, we have the apriori estimate*

$$h\|(1 + \frac{d^2}{h})^{1-k} u\| + |u_-| \le C(\|(1 + \frac{d^2}{h})^{-k} v\| + h|v_+|). \tag{11.18}$$

Proof. When $h = 1$, we simply translate the earlier result for (11.13) into a result for (11.16) and get the estimate

$$\|(1 + d^2)^{1-k}u\| + |u_-| \le C(\|(1 + d^2)^{-k}v\| + |v_+|). \tag{11.19}$$

Now consider (11.16) for other values of h, and indicate the h-dependence by means of super/subscripts. Let $Uf(x) = h^{\frac{n}{2}} f(\sqrt{h}x)$, so that U is unitary: $H_{\Phi_0^\epsilon,h} \to H_{\Phi_0^\epsilon,1}$. We further have

$$UP_0^h = hP_0^1U,\ UR_-^h = R_-^1,\ R_+^h = R_+^1U,$$

and the problem (11.16) (with general h) can be transformed into

$$h(P_0^1 - z)Uu + R_-^1u_- = Uv,\ R_+^1Uu = v_+, \tag{11.20}$$

that we write as

$$(P_0^1 - z)hUu + R_-^1u_- = Uv,\ R_+^1hUu = hv_+. \tag{11.21}$$

Applying (11.19) to this system, we get

$$h\|(1 + d^2)^{1-k}Uu\| + |u_-| \le C(\|(1 + d^2)^{-k}Uv\| + h|v_+|).$$

Here

$$d(x)Uu(x) = U(d(\frac{x}{\sqrt{h}})u(x)) = U(\frac{d(x)}{\sqrt{h}}u(x)),$$

and using the unitarity of U, we get (11.18). $\Box$

We can rewrite (11.18) equivalently as

$$\|(h + d^2)^{1-k}u\| + h^{-k}|u_-| \le C(\|(h + d^2)^{-k}v\| + h^{1-k}|v_+|). \tag{11.22}$$

In the following, it will be convenient to replace the f_j in the definition of R_+^1 by $\chi_k f_j$, where $\chi_R(x) = \chi(\frac{x}{R})$ for some sufficiently large $R > 0$. Correspondingly, f_j^h is replaced by $\chi_R(\frac{x}{\sqrt{h}})f_j^h$. This will be only a small modification of R_+^h and does neither affect the well-posedness of (11.16) nor the estimates (11.18), (11.22).

Mimicking Proposition 5.2, we have

Proposition 11.6 *Let $\chi_0 \in C_0^\infty(\mathbb{C}^n)$ be fixed and $= 1$ near $x = 0$, and fix $k \in \mathbb{R}$. Then for z in a neighborhood of λ_0, independent of k, we have the following estimate for the problem (11.16) in $H_{\Phi_0^\epsilon}$ (for $\epsilon > 0$ small and fixed and for h sufficiently small):*

$$\|(h + d^2)^{1-k}\chi_0u\| + h^{-k}|u_-| \le C(\|(h + d^2)^{-k}\chi_0v\| + h^{1-k}|v_+| + h^{\frac{1}{2}}\|1_Ku\|), \tag{11.23}$$

where K is any fixed neighborhood of supp χ_0.

Proof. Let Π denote the orthogonal projection onto the holomorphic functions as in Section 5. Applying $\Pi\chi_0$ to the first equation in (11.16), we get after some simple calculations, (using also that $\Pi R_- = R_-$, $u = \Pi u$):

$$\begin{aligned} (P_0 - hz)\Pi\chi_0u + R_-u_- &= \Pi\chi_0v + [P_0, \Pi\chi_0]u + \Pi(1 - \chi_0)R_-u_-, \\ R_+\Pi\chi_0u &= v_+ - R_+(1 - \chi_0)u - R_+(1 - \Pi)\chi_0u. \end{aligned} \tag{11.24}$$

Here (5.29) tells us that

$$\|(h+d^2)^{-k}[P_0,\Pi\chi_0]u\| \le \mathcal{O}(h)\|1_K u\|.$$

Since the e_j decay exponentially and $(h+d^2)^{-k}\Pi(h+d^2)^k$ is uniformly bounded in our weighted L^2 space, it is also clear that

$$\|(h+d^2)^{-k}\Pi(1-\chi_0)R_-u_-\| \le \mathcal{O}(h^\infty)|u_-|,$$

and since χf_j has compact support, we have $R_+(1-\chi_0)=0$, when $h>0$ is small enough. We also have $|R_+(1-\Pi)\chi_0 u| \le \mathcal{O}(h^\infty)\|1_K u\|$. Applying this and (11.22) to the problem (11.24), we get

$$\begin{aligned}\|(h+d^2)^{1-k}\Pi\chi_0 u\| + h^{-k}|u_-| \le \quad & \|(h+d^2)^{-k}\Pi\chi_0 v\| + \mathcal{O}(h)\|1_K u\| + \mathcal{O}(h^\infty)|u_-| \\ & +h^{1-k}|v_+| + \mathcal{O}(h^\infty)\|1_K u\|.\end{aligned} \tag{11.25}$$

According to (5.24), we have

$$\|(h+d^2)^{1-k}(1-\Pi)\chi_0 u\| \le \mathcal{O}(h^{\frac{1}{2}})\|1_K u\|,$$

and using this and

$$\|(h+d^2)^{-k}\Pi\chi_0 v\| \le C\|(h+d^2)^{-k}\chi_0 v\|$$

in (11.25), we get (11.23). □

Remark 11.7 Return to the case $h=1$ and choose P_0 as after the equation (11.15). Since P_0 is elliptic on $\Lambda_{\Phi_\epsilon^0}$ for $0<\epsilon\le\epsilon_0$ with ϵ_0 small enough, an easy deformation argument shows that the spectrum of P_0 on $H_{\Phi_\epsilon^0}$ is independent of ϵ, and similarly for the generalized eigenvectors. The aim of this remark is to show that there exists a $\delta>0$ such that the generalized eigenvectors e_j satisfy

$$e_j \in H_{\Phi_0^0-\delta|x|^2}. \tag{11.26}$$

Rather than using deformation arguments as elsewhere in this paper, we shall employ the alternative method of Fourier integral operators with complex phase, and more precisely we shall study the evolution equation associated to P_0.

Let us first recall some elementary facts from complex symplectic geometry (as in [17] and further references given there): On $\mathbb{C}_x^n\times\mathbb{C}_\xi^n$, we have the complex symplectic $(2,0)$-form $\sigma=\sum_1^n d\xi_j\wedge dx_j$ and the real symplectic forms $\mathrm{Re}\,\sigma$, $-\mathrm{Im}\,\sigma$. If t is a vector field on $\mathbb{C}^{2n}$ of type (1,0), we let $\widehat{t}=t+\overline{t}$ be the associated real vector field. Then if f is a holomorphic function, we let H_f denote the Hamilton field (of type (1,0)) with respect to σ and if g is a real-valued C^1-function, we let $H_g^{\mathrm{Re}\,\sigma}$, $H_g^{-\mathrm{Im}\,\sigma}$ denote the Hamilton field of g with respect to $\mathrm{Re}\,\sigma$ and $-\mathrm{Im}\,\sigma$ respectively. Then we have the relations,

$$\widehat{H}_f = H^{\mathrm{Re}\,\sigma}_{\mathrm{Re}\,f} = -H^{-\mathrm{Im}\,\sigma}_{\mathrm{Im}\,f}, \quad \widehat{H}_{if} = -H^{\mathrm{Re}\,\sigma}_{\mathrm{Im}\,f} = -H^{-\mathrm{Im}\,\sigma}_{\mathrm{Re}\,f}.$$

Using Fourier integral operators with quadratic phase in the complex domain, we see that if $0\le t\le t_0$, and $u_0\in H_{\Phi_0}$, with $\Phi_0=\Phi_0^0$, then we can solve the heat equation

$$\frac{\partial}{\partial t}u(t,x)+P_0u(t,x)=0,\ u(0,x)=u_0(x),$$

and the solution operator e^{-tP_0} is bounded $H_{\Phi_0} \to H_{\Phi_t}$, where

$$\Lambda_{\Phi_t} = \exp{(t\widehat{H}_{\frac{1}{i}P_0})}(\Lambda_{\Phi_0}) = \exp{(tH^{-\operatorname{Im}\sigma}_{\operatorname{Re}P_0})}(\Lambda_{\Phi_0}). \tag{11.27}$$

We further have the eikonal equation for $\Phi(t,x) = \Phi_t(x)$:

$$\frac{\partial\Phi}{\partial t} + \operatorname{Re}P_0(x, \frac{2}{i}\frac{\partial\Phi}{\partial x}) = 0, \tag{11.28}$$

corresponding to the manifold

$$\tau = \frac{\partial\Phi}{\partial t},\ \xi = \frac{2}{i}\frac{\partial\Phi}{\partial x},$$

in $\mathbb{R}^2_{t,\tau} \times \mathbb{C}^{2n}_{x,\xi}$, which is Lagrangian for the symplectic form $d\tau\wedge dt - \operatorname{Im}\sigma$. (To get (11.28) at least formally, differentiate $\|u(t,.)\|^2_{H_{\Phi(t,.)}}$ with respect to t). Since $\operatorname{Re}P_0$ is constant along the flow of $H^{-\operatorname{Im}\sigma}_{\operatorname{Re}P_0}$, we know that $\operatorname{Re}P_0|_{\Lambda_{\Phi_t}} \geq 0$, so (11.28) shows that $\frac{\partial\Phi}{\partial t} \leq 0$ and hence that

$$t \mapsto \Phi_t(x) \text{ is decreasing.} \tag{11.29}$$

Let $L_0 \subset \Lambda_{\Phi_0}$ be the subspace, defined by $\operatorname{Re}P_0 = 0$ and notice that $\operatorname{Re}P_0 \sim \operatorname{dist}(\cdot, L_0)^2$ on Λ_{Φ_0}. In general, if f is a smooth function on an IR-manifold Λ, and $\widetilde{f}$ denotes an almost holomorphic extension of f to a neighborhood of Λ, then at the points where df is real, the Hamilton field $H^{\sigma_\Lambda}_f$ of f with respect to the real symplectic form $\sigma_\Lambda = \sigma_{|\Lambda}$ is equal to $\widehat{H}_{\widetilde{f}}$. Applying this to $f = \frac{1}{i}P_0$, we get at the points of L_0:

$$\widehat{H}_{\frac{1}{i}P_0} = H^{\sigma_{\Lambda_{\Phi_0}}}_{\operatorname{Im}P_0}.$$

Let $L_t = \exp{(t\widehat{H}_{\frac{1}{i}P_0})}(L_0)$ (cf (11.27)). Since $H^{\sigma_{\Lambda_{\Phi_0}}}_{\operatorname{Im}P_0}$ is transversal to L_0 away from 0, we deduce that $t \mapsto \Pi_x(L_t)$ moves transversally to $\Pi_x(L_0)$ away from 0 and since by (11.28),

$$\frac{\partial\Phi}{\partial t} \sim -\operatorname{dist}\,(x, \Pi_x(L_t))^2,$$

we conclude that for small t

$$\Phi_t(x) \leq \Phi_0(x) - \frac{t^3}{C}|x|^2. \tag{11.30}$$

If e_j is an eigenvector of P_0: $P_0e_j = \lambda_j e_j$, $\lambda_j \in \mathbb{C}$, we first recall that $e_j \in H_{\Phi_0+\epsilon|x|^2}$ for every $\epsilon > 0$, and conclude that $e^{-tP_0}e_j \in H_{\Phi_t+\epsilon|x|^2}$ for every $\epsilon > 0$. On the other hand, $e^{-tP_0}e_j = e^{-t\lambda_j}e_j$, so $e_j = e^{t\lambda_j}e^{-tP}e_j \in H_{\Phi_t+\epsilon|x|^2}$ for every $\epsilon > 0$. Taking $t > 0$ small but fixed, we then obtain (11.26) from (11.30). If λ_j is a multiple eigenvalue, we also have to take into account the possible Jordan blocks in the action of P_0 on the corresponding generalized eigenspace, but this only requires minor modifications in the argument and we get (11.26) in general.

Estimate for the semi-global problem

We now consider the situation in Section 6. P is now an h-pseudodifferential operator acting in $H_\Phi = H_{\Phi_\epsilon}$, and we define $R_+ = R^h_+$, $R_- = R^h_-$ as in the preceding subsection. As in Section 6, P_0 is now the quadratic approximation of P at $(0,0)$ and we shall use the fact that $\Phi_\epsilon = \Phi^0_\epsilon$ for

$|x| \leq \sqrt{Ah}$ for $A \gg 1$. Recall the estimate (11.23) for solutions to (11.16) (for P_0 and with norms in $H_{\Phi^0_\epsilon}$)).

Again, we want to replace the fixed cutoff χ_0 in (11.23) by $\chi_0(\frac{x}{\sqrt{Ah}})$ and consider the change of variables $x = \sqrt{Ah}\widetilde{x}$, $hD_x = \sqrt{Ah}\widetilde{h}D_{\widetilde{x}}$, $\widetilde{h} = 1/A$.

$$P_0(x, hD_x; h) = \frac{h}{\widetilde{h}} P_0(\widetilde{x}, \widetilde{h}D_{\widetilde{x}}; h) =: \frac{h}{\widetilde{h}} \widetilde{P}_0,$$

and with $d = d(x)$, $\widetilde{d} = d(\widetilde{x})$:

$$h + d^2 = \frac{h}{\widetilde{h}}(\widetilde{h} + \widetilde{d}^2),\ e^{-2\Phi^0(x)/h} = e^{-2\Phi^0(\widetilde{x})/\widetilde{h}},$$

and we relate our unknown functions by the unitary relation

$$u(x) = (Ah)^{-\frac{n}{2}}\widetilde{u}(\widetilde{x}),\ h^{\frac{n}{2}}u(x) = \widetilde{h}^{\frac{n}{2}}\widetilde{u}(\widetilde{x}). \tag{11.31}$$

With these substitutions, the problem (11.16) becomes

$$\frac{h}{\widetilde{h}}(\widetilde{P}_0 - \widetilde{h}z)\widetilde{u} + \widetilde{R}_- u_- = \widetilde{v},\ \widetilde{R}_+\widetilde{u} = v_+, \tag{11.32}$$

and we can apply (11.23) to this new problem. A straightforward calculation gives

$$\begin{aligned} \|(h+d^2)^{1-k}\chi_0(\frac{x}{\sqrt{Ah}})u\| + h^{-k}|u_-| \leq\ & C\Big(\|(h+d^2)^{-k}\chi_0(\frac{x}{\sqrt{Ah}})v\| \\ & + h^{1-k}|v_+| + \frac{1}{\sqrt{A}}\|(h+d^2)^{1-k}1_K(\frac{x}{\sqrt{Ah}})u\|\Big). \end{aligned} \tag{11.33}$$

This estimate will be applied with $k = 1/2$.

We now return to the full operator P (on the FBI-side) and the norms and scalar products will now be with respect to $e^{-2\Phi/h}L(dx)$, $\Phi = \Phi_\epsilon$, with $\epsilon > 0$ small and fixed. Recall however that $\Phi = \Phi^\epsilon_0$ in $|x| \leq \sqrt{Ah}$. We consider the semi-global Grushin problem

$$(P - hz)u + R_- u_- = v,\ R_+ u = v_+, \tag{11.34}$$

in some fixed bounded open set containing the (projections of the) critical points. For simplicity, we assume that the critical set is reduced to a single point, corresponding to $x = 0$. Let $\chi \in C_0^\infty$ be equal to 1 near 0.

Notice that by Remark 11.7, $e^{-\Phi/h}R_- u_-$ is exponentially small away from any fixed neighborhood of $x = 0$. Apply (6.14) with $(P - hz)u = v - R_- u_-$:

$$\|\Lambda u\|^2 \leq C'\mathrm{Re}\,(\chi v|u) + C\|\Lambda^{-1}R_- u_-\|\|\Lambda u\| + C(\chi_0^2(\frac{x}{\sqrt{Ah}})\Lambda u|\Lambda u) + C\|(1-\chi)\Lambda u\|\|\Lambda u\|. \tag{11.35}$$

Here Λ was defined in (6.13). Using Remark 11.7 it is easy to check that

$$\|\Lambda^{-1}R_- u_-\| \leq \frac{C}{\sqrt{h}}|u_-|, \tag{11.36}$$

and (11.35) becomes

$$\|\Lambda u\|^2 \le C\Big(\|\Lambda^{-1}v\|\|\Lambda u\| + \frac{1}{\sqrt{h}}|u_-|\|\Lambda u\| + \|\Lambda\chi_0(\frac{x}{\sqrt{Ah}})u\|^2 + \|(1-\chi)\Lambda u\|\|\Lambda u\|\Big). \quad (11.37)$$

Apply "$2ab \le \alpha a^2 + \alpha^{-1}b^2$" with suitable α's to the 1st, 2nd and the 4th terms of the right hand side and bootstrap away the $\|\Lambda u\|^2$ terms. After removing the squares, we get

$$\|\Lambda u\| \le C\Big(\|\Lambda^{-1}v\| + \frac{1}{\sqrt{h}}|u_-| + \|\Lambda\chi_0(\frac{x}{\sqrt{Ah}})u\| + \|(1-\chi)\Lambda u\|\Big). \quad (11.38)$$

Apply (11.33) (for the Grushin problem for P_0) with $k = 1/2$:

$$\begin{aligned}
&\|\Lambda\chi_0(\frac{x}{\sqrt{Ah}})u\| + h^{-1/2}|u_-| \\
&\le C\Big(\|\Lambda^{-1}\chi_0(\frac{x}{\sqrt{Ah}})v\| + \|\Lambda^{-1}\chi_0(\frac{x}{\sqrt{Ah}})(P-P_0)u\| + h^{\frac{1}{2}}|v_+| + \frac{1}{\sqrt{A}}\|\Lambda 1_K(\frac{x}{\sqrt{Ah}})u\|\Big) \\
&\le C\Big(\|\Lambda^{-1}\chi_0(\frac{x}{\sqrt{Ah}})v\| + C(A)h^{\frac{1}{2}}\|\Lambda u\| + h^{\frac{1}{2}}|v_+| + \frac{1}{\sqrt{A}}\|\Lambda 1_K(\frac{x}{\sqrt{Ah}})u\|\Big),
\end{aligned}$$

where we used (6.16), to get the last estimate. Use this estimate in (11.38) after adding $h^{-1/2}|u_-|$ to both sides:

$$\|\Lambda u\| + h^{-\frac{1}{2}}|u_-| \le C\Big(\|\Lambda^{-1}v\| + C(A)h^{\frac{1}{2}}\|\Lambda u\| + h^{\frac{1}{2}}|v_+| + \frac{1}{\sqrt{A}}\|\Lambda 1_K(\frac{x}{\sqrt{Ah}})u\| + \|(1-\chi)\Lambda u\|\Big),$$

and choosing first A large enough and then $h > 0$ small enough, we get the basic apriori estimate for the problem (11.34):

$$\|\Lambda u\| + h^{-\frac{1}{2}}|u_-| \le C\big(\|\Lambda^{-1}v\| + h^{\frac{1}{2}}|v_+| + \|(1-\chi)\Lambda u\|\big). \quad (11.39)$$

The global Grushin problem.

Now let P be as in Theorem 1.2. Applying the inverse FBI-transform we have the obvious analogue of the Grushin problem and for that problem, we still have (11.39) provided that we define Λ to be a suitable h-pseudodifferential operator whose symbol is equivalent to $(h + \min(d^2, (Ahd)^{2/3}))^{1/2}$, and interpret χ as a pseudodifferential cutoff. From (11.39), we get the weaker estimate

$$h\|\psi^w u\| + |u_-| \le C(\|v\| + h|v_+| + h^{5/6}\|(1-\chi)\psi^w u\|), \quad (11.40)$$

analogous to (10.7). This leads to

$$h\|\psi^w u\| + |u_-| \le C(\|v\| + h|v_+| + h\|q^w u\| + \mathcal{O}(h^2)\|u\|), \quad (11.41)$$

which is analogous to (10.8). On the other hand, we have (10.3), and as in Section 10, we finally get the global apriori estimate (analogous to (10.14)):

$$h\|u\| + |u_-| \le C(\|v\| + h|v_+|). \quad (11.42)$$

We are therefore exactly in the situation of the beginning of this section and from Proposition 11.4 we get that the Grushin problem is well-posed.

Asymptotics for E_{-+} and for the eigenvalues.

For simplicity, we continue to assume that $\mathcal{C}$ is reduced to a single point, $(0,0)$. We may assume that the global Grushin problem for the original operator P, considered in the preceding subsection, is of the form

$$(P-hz)u+R_-u_-=v,\ \ R_+u=v_+, \tag{11.43}$$

where z varies in a fixed neighborhood of an eigenvalue $\lambda_0\in\mathbb{C}$, of the quadratic approximation P_0 (with $h=1$) of P at $(0,0)$, and where

$$R_-u_-=\sum_{j=1}^{N_0}u_-(j)e_j^h(x), R_+u(j)=(u|f_j^h(x)). \tag{11.44}$$

Here

$$e_j^h(x)=h^{-\frac{n}{4}}e_j(\frac{x}{\sqrt{h}}), f_j^h(x)=h^{-\frac{n}{4}}f_j(\frac{x}{\sqrt{h}}), \tag{11.45}$$

and $e_1,...,e_{N_0}$ form a basis for the generalized eigenspace E_{λ_0} of P_0, associated to λ_0. It is well known that we may take e_j of the form

$$e_j(x)=p_j(x)e^{i\Phi_0(x)}, \tag{11.46}$$

where p_j is a polynomial and $\Phi_0(x)$ is a complex quadratic form such that $\Lambda_{\Phi_0}=\{(x,\Phi_0'(x))\}$ is the stable outgoing manifold Λ^0 for the $\frac{1}{i}H_{P_0}$-flow and (by Remark 11.7) we know that

$$\operatorname{Im}\Phi_0 \text{ is positive definite.} \tag{11.47}$$

We may assume that the f_j have an analogous form:

$$f_j(x)=q_j(x)e^{i\Psi_0(x)}, \tag{11.48}$$

with q_j polynomial and Ψ_0 a quadratic form with $\operatorname{Im}\Psi_0$ positive definite.

Let $\Lambda_\pm$ be the stable outgoing $(+)$ and incoming $(-)$ manifolds through (0,0) for the $\frac{1}{i}H_p$-flow, where p is the principal symbol of P. Then $\Lambda_\pm$ are complex Lagrangian manifolds defined to infinite order at $(0,0)$ and $\Lambda_+^0=T_{(0,0)}\Lambda_+$. Let κ be a complex canonical transformation: $\operatorname{neigh}((0,0);\mathbb{C}^{2n})\to\operatorname{neigh}((0,0);\mathbb{C}^{2n})$, mapping $\{\xi=0\}$ to Λ_+ and $\{x=0\}$ to Λ_-. Let U be a formal elliptic Fourier integral operator of order 0 quantizing κ, and consider

$$U^{-1}PU:=\widetilde{P},$$

whose symbol is well-defined mod $\mathcal{O}((x,\xi)^\infty+h^\infty)$. The principal symbol $\widetilde{p}$ of $\widetilde{P}$ then vanishes on $\{x=0\}$ and on $\{\xi=0\}$ and therefore takes the form

$$\widetilde{p}=\sum_{|\alpha|=|\beta|=1}a_{\alpha,\beta}(x,\xi)x^\alpha\xi^\beta. \tag{11.49}$$

Using for simplicity the classical quantization of symbols, we get

$$\widetilde{P}=\sum_{|\alpha|=|\beta|=1}a_{\alpha,\beta}(x,hD)x^\alpha(hD)^\beta+ha(x,hD;h), \tag{11.50}$$

where a is a classical symbol of order 0. (We are now working with formal Taylor series at $(x,\xi)=(0,0)$.)

Put

$$\mathcal{P}^m_{\mathrm{hom}}=\{\sum_{|\alpha|=m} b_\alpha\big(\frac{x}{\sqrt{h}}\big)^\alpha\}. \tag{11.51}$$

Here b_α will in general be functions of h. When they are not, we say that $\sum_{|\alpha|=m} b_\alpha\big(\frac{x}{\sqrt{h}}\big)^\alpha$ is homogeneous of order 0 in h (or even independent of h, with $x/\sqrt{h}$ viewed as independent variables). Then in the obvious way,

$$(\frac{x}{\sqrt{h}})^\gamma(\sqrt{h}D)^\delta:\mathcal{P}^m_{\mathrm{hom}}\to\mathcal{P}^{m+\gamma-\delta}_{\mathrm{hom}}$$

is homogeneous of degree 0 in h.

Write

$$\frac{1}{h}\widetilde{P}=\sum_{|\alpha|=|\beta|=1} a_{\alpha,\beta}(x,hD)(\frac{x}{\sqrt{h}})^\alpha(\sqrt{h}D)^\beta+a(x,hD;h). \tag{11.52}$$

Write $a\sim\sum_0^\infty h^j a_j$ and Taylor expand $a(x,hD;h)$ at $(0,0)$:

$$a(x,hD;h)=\sum_{j=0}^\infty\sum_{\gamma,\delta} h^{j+\frac{|\delta|}{2}+\frac{|\gamma|}{2}}\frac{a^{(\gamma)}_{j(\delta)}(0,0)}{\gamma!\delta!}(\frac{x}{\sqrt{h}})^\delta(\sqrt{h}D)^\gamma. \tag{11.53}$$

If $|\delta|-|\gamma|=k\in\mathbf{Z}$, then $|\gamma|+|\delta|=|k|+2\min(|\delta|,|\gamma|)$, so the general term in the last sum can be written

$$h^{j+\frac{|k|}{2}+\min(|\gamma|,|\delta|)}\frac{a^{(\gamma)}_{j(\delta)}(0,0)}{\gamma!\delta!}(\frac{x}{\sqrt{h}})^\delta(\sqrt{h}D)^\gamma.$$

In conclusion the block matrix of

$$a(x,hD;h):\bigoplus_0^\infty\mathcal{P}^m_{\mathrm{hom}}\to\bigoplus_0^\infty\mathcal{P}^m_{\mathrm{hom}},$$

is $(h^{\frac{|j-k|}{2}}A_{j,k})$, where $A_{j,k}=\sum_{\nu=0}^\infty A^\nu_{j,k}h^\nu$, and $A^\nu_{j,k}:\mathcal{P}^k_{\mathrm{hom}}\to\mathcal{P}^j_{\mathrm{hom}}$ is homogeneous of degree 0. (We then say that $A_{j,k}$ is a classical symbol of order 0.)

The same discussion applies to $a_{\alpha,\beta}(x,hD)$ and hence also to $h^{-1}\widetilde{P}$, whose matrix is

$$(h^{\frac{|j-k|}{2}}P_{j,k}),\ P_{j,k}=\sum_{\nu=0}^\infty P^\nu_{j,k}h^\nu, \tag{11.54}$$

where $P^\nu_{j,k}:\mathcal{P}^k_{\mathrm{hom}}\to\mathcal{P}^j_{\mathrm{hom}}$ is homogeneous of degree 0. The leading part of $h^{-1}\widetilde{P}$ is given by

$$\frac{1}{h}\widetilde{P}_0:=\sum_{|\alpha|=|\beta|=1} a_{\alpha,\beta}(0,0)(\frac{x}{\sqrt{h}})^\alpha(\sqrt{h}D)^\beta+a_0(0,0),$$

in the following sense: $h^{-1}\widetilde{P}_0$ has a block diagonal matrix in $\bigoplus_0^\infty\mathcal{P}^m_{\mathrm{hom}}$, and $P^0_{j,j}$ is equal to the restriction of $h^{-1}\widetilde{P}_0$ to $\mathcal{P}^j_{\mathrm{hom}}$.

Now we shall exploit that the exponent $|j-k|/2$ in (11.54) is an integer precisely when j and k have the same parity. We therefore introduce

$$\mathcal{F}_e = \bigoplus_0^\infty \mathcal{P}_{\mathrm{hom}}^{2k},\ \mathcal{F}_o = \bigoplus_0^\infty \mathcal{P}_{\mathrm{hom}}^{2k+1}. \tag{11.55}$$

Then $h^{-1}\widetilde{P} : \mathcal{F}_e \oplus \mathcal{F}_o \to \mathcal{F}_e \oplus \mathcal{F}_o$ has the block diagonal matrix

$$\begin{pmatrix} P_{e,e}, & P_{e,o} \\ P_{o,e} & P_{o,o} \end{pmatrix}, \tag{11.56}$$

where $P_{e,e}, P_{o,o}, h^{-1/2}P_{e,o}, h^{-1/2}P_{o,e}$ are classical symbols of order 0.

The Grushin problem for $\widetilde{P}$ that we obtain from (11.43) is

$$(\widetilde{P} - hz)u + \widetilde{R}_- u_- = v,\ \widetilde{R}_+ u = v_+, \tag{11.57}$$

with

$$\widetilde{R}_- = U^{-1}R_-,\ \widetilde{R}_+ = R_+U. \tag{11.58}$$

We want to decompose $\widetilde{R}_\pm$ into even and odd degrees.

Return to P, P_0 and notice that $[P_0, \iota] = 0$, where ι is the involution $\iota(u)(x) = u(-x)$. Consequently, E_{λ_0} is invariant under ι and splits into $E_{\lambda_0}^e \oplus E_{\lambda_0}^o$, with $\iota = 1$ on $E_{\lambda_0}^e$ and $\iota = -1$ on $E_{\lambda_0}^o$. Let the corresponding dimensions be N_e, N_o, so that $N_0 = N_e + N_o$. We may assume that e_j is even for $1 \le j \le N_e$ and odd for $N_e + 1 \le j \le N_0$, and we may choose f_j with the same properties. Then $p_j(x), q_j(x)$ are even when $1 \le j \le N_e$ and odd otherwise.

Now, write

$$e_j^h(x) = h^{-\frac{n}{4} - \frac{m_j}{2}} a_j(x;h) e^{i\frac{\Phi_0(x)}{h}}, \tag{11.59}$$

where

$$a_j \sim \sum_{\nu=0}^\infty a_j^\nu(x) h^\nu,\ \text{and } a_j(x) = \mathcal{O}(|x|^{(m_j - 2\nu)_+}), \tag{11.60}$$

and actually, $a_j(x;h) = h^{m_j/2} p_j(x/\sqrt{h})$, with $m_j = d^o p_j$. m_j is even when $1 \le j \le N_e$, and odd otherwise. Assume to fix the ideas that $j \le N_e$. Then

$$U^{-1}(e_j^h) = h^{-\frac{n}{4} - \frac{m_j}{2}} \widetilde{a}_j(x;h) e^{i\frac{F(x)}{h}}, \tag{11.61}$$

where $\widetilde{a}_j$ satisfies (11.60). Moreover,

$$F(x) = \mathcal{O}(x^3), \tag{11.62}$$

since Λ_{Φ_0} is tangent to Λ_Φ so that Λ_F is tangent to $\{\xi = 0\}$.

Taylor expanding $\widetilde{a}_j$ and $e^{iF/h} = \sum_0^\infty (iF(x))^k/(k!h^k)$, we see that

$$h^{\frac{n}{4}} U^{-1}(e_j^h) \in \bigoplus_0^\infty \mathcal{P}_{\mathrm{hom}}^m, \tag{11.63}$$

and when m is even, the component in $\mathcal{P}^m_{\mathrm{hom}}$ is a classical symbol of order 0 (and the order tends to $-\infty$ like $-m/2$, when $m \to \infty$), while the component in $\mathcal{P}^m_{\mathrm{hom}}$ is of order $h^{1/2}$, when m is odd). The case $j \geq N_e + 1$ is treated similarly, and we conclude that

$$\widetilde{R}_- = \begin{pmatrix} \widetilde{R}^{ee}_- & \widetilde{R}^{eo}_- \\ \widetilde{R}^{oe}_- & \widetilde{R}^{oo}_- \end{pmatrix} : \mathbb{C}^{N_e} \oplus \mathbb{C}^{N_o} \to \mathcal{F}_e \oplus \mathcal{F}_o, \tag{11.64}$$

where $h^{n/4}\widetilde{R}^{ee}_-$, $h^{n/4}\widetilde{R}^{oo}_-$, $h^{n/4-1/2}\widetilde{R}^{eo}_-$, $h^{n/4-1/2}\widetilde{R}^{oe}_-$ are classical symbols of order 0.

Next, we do the same work with $\widetilde{R}_+$ and start from

$$\widetilde{R}_+ u(j) = (u|{}^t\!U(f_{j,h})). \tag{11.65}$$

Possibly after a slight perturbation of Ψ, we may assume that

$${}^t\!U(f_{j,h}) = h^{-\frac{n}{4}} h^{-\frac{\widetilde{m}_j}{2}} \widetilde{b}_j(x;h) e^{\frac{i}{h}G(x)}, \tag{11.66}$$

where $\widetilde{m}_j$, $\widetilde{b}_j$ have the same properties as $m_j, \widetilde{a}_j$ above, and $\widetilde{m}_j$ is even for $1 \leq j \leq N_e$ and odd otherwise. Moreover $\det G''(0) \neq 0$, and the scalar product in (11.65) should be computed as a formal stationary phase integral. In doing so, we apply the complex Morse lemma (to ∞ order at $x = 0$) to reduce G to a quadratic form. If α is a formal diffeomorphism with $\alpha(0) = 0$, and $Au = \alpha^* u = u \circ \alpha$, then $A : \bigoplus_0^\infty \mathcal{P}^m_{\mathrm{hom}} \to \bigoplus_0^\infty \mathcal{P}^m_{\mathrm{hom}}$ has the same block matrix structure as $h^{-1}\widetilde{P}$ in (11.54). From these facts, we get

$$\widetilde{R}_+ = \begin{pmatrix} \widetilde{R}^{ee}_+ & \widetilde{R}^{eo}_+ \\ \widetilde{R}^{oe}_+ & \widetilde{R}^{oo}_+ \end{pmatrix} : \mathcal{F}_e \oplus \mathcal{F}_o \to \mathbb{C}^{N_e} \oplus \mathbb{C}^{N_o}, \tag{11.67}$$

where $h^{-n/4}\widetilde{R}^{ee}_+$, $h^{-n/4}\widetilde{R}^{oo}_+$, $h^{-n/4-1/2}\widetilde{R}^{eo}_+$, $h^{-n/4-1/2}\widetilde{R}^{oe}_+$ are classical symbols of order 0.

Consider the rescaled problem which is equivalent to (11.57):

$$(\frac{1}{h}\widetilde{P} - z)u + h^{\frac{n}{4}}\widetilde{R}_- u_- = v,\ \ h^{-\frac{n}{4}}\widetilde{R}_+ u = v_+, \tag{11.68}$$

or in matrix form

$$\widetilde{\mathcal{P}}(z)\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}.$$

Let

$$\mathcal{E} = \begin{pmatrix} \widetilde{E} & \widetilde{E}_+ \\ \widetilde{E}_- & \widetilde{E}_{-+} \end{pmatrix} : \bigoplus_0^\infty \mathcal{P}^m_{\mathrm{hom}} \oplus \mathbb{C}^{N_0} \to \bigoplus_0^\infty \mathcal{P}^m_{\mathrm{hom}} \oplus \mathbb{C}^{N_0},$$

be the inverse. Decomposing

$$\widetilde{\mathcal{P}}(z) = \begin{pmatrix} \widetilde{\mathcal{P}}^{ee} & \widetilde{\mathcal{P}}^{eo} \\ \widetilde{\mathcal{P}}^{oe} & \widetilde{\mathcal{P}}^{oo} \end{pmatrix} : (\mathcal{F}_e \oplus \mathbb{C}^{N_e}) \oplus (\mathcal{F}_o \oplus \mathbb{C}^{N_o}) \to (\mathcal{F}_e \oplus \mathbb{C}^{N_e}) \oplus (\mathcal{F}_o \oplus \mathbb{C}^{N_o}),$$

where $\widetilde{\mathcal{P}}^{ee}$, $\widetilde{\mathcal{P}}^{oo}$, $h^{-\frac{1}{2}}\widetilde{\mathcal{P}}$, $h^{-\frac{1}{2}}\widetilde{\mathcal{P}}$ are classical symbols of order 0, we get the same decomposition for $\mathcal{E}(z)$. In particular,

$$\widetilde{E}_{-+}(z) = \begin{pmatrix} \widetilde{E}^{ee}_{-+} & \widetilde{E}^{eo}_{-+} \\ \widetilde{E}^{oe}_{-+} & \widetilde{E}^{oo}_{-+} \end{pmatrix} : \mathbb{C}^{N_e} \oplus \mathbb{C}^{N_o} \to \mathbb{C}^{N_e} \oplus \mathbb{C}^{N_o} \tag{11.69}$$

has the same structure. The determinant of this matrix is a classical symbol of order 0. In fact,

$$\begin{pmatrix} h^{\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} \widetilde{E}_{-+}(z) \begin{pmatrix} h^{-\frac{1}{2}} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \widetilde{E}^{ee}_{-+} & h^{\frac{1}{2}} \widetilde{E}^{eo}_{-+} \\ h^{-\frac{1}{2}} \widetilde{E}^{oe}_{-+} & \widetilde{E}^{oo}_{-+} \end{pmatrix}$$

has the same determinant and is a classical symbol of order 0 of the form

$$\det \widetilde{E}_{-+}(\lambda) \sim \det E^0_{-+}(\lambda) + h f_1(\lambda) + h^2 f_2(\lambda) + \ldots, \tag{11.70}$$

where $E^0_{-+}(\lambda)$ is the matrix given in (11.15) (there denoted without the superscript 0). In particular $E^0_{-+}(\lambda) = (\lambda - \lambda_0)^{N_0} f(\lambda)$ with $f(\lambda) \neq 0$, in the space of holomorphic functions in a neighborhood of λ_0. This could also be deduced from the well known formula

$$N_0 = \operatorname{tr} \frac{1}{2\pi i} \int_{\gamma_0} (\lambda - P_0)^{-1} d\lambda = \frac{1}{2\pi i} \int \frac{\frac{d}{d\lambda} \det E^0_{-+}}{\det E^0_{-+}(\lambda)} d\lambda,$$

where γ_0 is a closed contour around λ_0. (We have of course the similar formula for P, E_{-+}, permitting to identify the zeros of E_{-+} and the eigenvalues of P, counted with their multiplicities.)

Using Puiseux series for the partial sums in (11.70) we conclude that the eigenvalues of $h^{-1}\widetilde{P}$ close to λ_0 have complete asymptotic expansions in powers of h^{1/N_0}:

$$\lambda(h) = \lambda_0 + c_1 h^{1/N_0} + c_2 h^{2/N_0} + \ldots.$$

Finally, it is clear from the construction, that if

$$\mathcal{E} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}$$

is the inverse of the global problem for P, then modulo $\mathcal{O}(h^\infty)$:

$$E_{-+}(z;h) = h\widetilde{E}_{-+}(z;h). \tag{11.71}$$

and hence the true eigenvalues of P have the same asymptotic expansions as above. This completes the proof of Theorem 1.3.

12 The evolution problem.

Let P be a closed densely defined unbounded operator acting on a complex Hilbert space $\mathcal{H}$. Assume that the spectrum of P is contained in

$$\operatorname{Re} z \geq \frac{1}{C} \langle \operatorname{Im} z \rangle^\delta - C, \tag{12.1}$$

for some constants $C, \delta > 0$. Assume also that

$$\|(z - P)^{-1}\| \leq \frac{C}{\langle z \rangle^\delta}, \text{ for } \operatorname{Re} z \leq \frac{1}{2C} \langle \operatorname{Im} z \rangle^\delta - 2C. \tag{12.2}$$

For $t > 0$ we put

$$E(t) = \frac{1}{2\pi i} \int_\gamma e^{-tz} (z - P)^{-1} dz, \tag{12.3}$$

where γ is a contour to the left of the spectrum which outside a compact set coincides with the curve

$$\operatorname{Re} z = \frac{1}{3C} \langle \operatorname{Im} z \rangle^{\delta}, \tag{12.4}$$

and oriented in the direction of decreasing $\operatorname{Im} z$. Clearly the integral converges and defines a bounded operator which depends smoothly on t. We have

$$(\partial_t + P)E(t) = 0, \quad PE(t) = E(t)P. \tag{12.5}$$

When $u \in \mathcal{D}(P)$ (the domain of P) we also have

$$\lim_{t \to 0} E(t)u = u. \tag{12.6}$$

In fact, let z_0 be to the left of γ and write

$$u = (z_0 - P)^{-1}v, \ v \in \mathcal{H}. \tag{12.7}$$

The resolvent identity gives

$$(z - P)^{-1}(z_0 - P)^{-1} = \frac{1}{(z - z_0)}((z_0 - P)^{-1} - (z - P^{-1}),$$

so for $t > 0$, we have

$$E(t)u = \frac{1}{2\pi i} \int_{\gamma} e^{-tz} \frac{1}{(z - z_0)} (z_0 - P)^{-1} v dz - \frac{1}{2\pi i} \int_{\gamma} e^{-tz} \frac{1}{(z - z_0)} (z - P)^{-1} v dz. \tag{12.8}$$

Here the first integral vanishes since we can push the contour to the right and exploit the decay of the exponential. The second integral allows a limit when $t \to 0$, so we get

$$\lim_{t \to 0} E(t)u = -\frac{1}{2\pi i} \int_{\gamma} \frac{1}{(z - z_0)} (z - P)^{-1} v dz. \tag{12.9}$$

Here the integrand is of norm $\mathcal{O}(\langle z \rangle^{-1-\delta})$ in view of (12.2) and we can push the contour to the left (around z_0) and apply the residue theorem to get

$$\lim_{t \to 0} E(t)u = (z_0 - P)^{-1}v = u,$$

and (12.6) follows.

In the following we assume that P satisfies the assumptions **(H1)**–**(H5)** so that Theorem 1.2 gives a localization of the spectrum to a union of a conic neighborhood of the open positive axis and a infinite cusp away from the origin. We introduce 2 contours γ and $\widetilde{\gamma}$. Both contours are given by

$$\operatorname{Re} z = \frac{1}{C_0} h^{\frac{2}{3}} |\operatorname{Im} z|^{\frac{1}{3}} \tag{12.10}$$

in the region $\operatorname{Re} z > bh$. Here C_0 and b are positive constants such that b is different from the real parts of the eigenvalues of the quadratic approximations of P with $h = 1$. In the region $\operatorname{Re} z \le bh$, γ is given by $\operatorname{Re} z = bh$ while $\widetilde{\gamma}$ joins $bh + iC_0^3 b^3 h$ to $bh - iC_0^3 b^3 h$ further to the left so that $\widetilde{\gamma}$ is entirely to the left of the spectrum of P while γ will have a fixed finite number of eigenvalues,

$\lambda_0, ..., \lambda_{N-1}$ to its left. Let $\gamma_{\rm int}$ denote the vertical part of γ in the region $\mathrm{Re}\, z = bh$ and let $\gamma_{\rm ext}$ denote the part of γ in the region $\mathrm{Re}\, z \geq bh$.

On the exterior piece we have

$$\|(z-P)^{-1}\| \leq \frac{\mathcal{O}(1)}{h^{2/3}|\mathrm{Im}\, z|^{1/3}}, \tag{12.11}$$

and on the interior piece we have

$$\|(z-P)^{-1}\| \leq \frac{\mathcal{O}(1)}{h}. \tag{12.12}$$

This holds since we have chosen b so that the distance from $\gamma_{\rm int}$ to the spectrum of P is $\geq h/C$. Further to the left in the region $\mathrm{Re}\, z \leq bh$, we also have $\|(z-P)^{-1}\| = \mathcal{O}(h)$ when $\mathrm{dist}\,(z, \{\lambda_0, .., \lambda_{N-1}\}) \geq h/C$.

Assume for simplicity that the eigenvalues of the different quadratic approximations are simple and distinct. Then

$$e^{-tP/h} = \frac{1}{2\pi i}\int_{\widetilde{\gamma}} e^{-tz/h}(z-P)^{-1}dz = \sum_{0}^{N-1} e^{-t\lambda_j/h}\Pi_{\lambda_j} + \frac{1}{2\pi i}\int_{\gamma} e^{-tz/h}(z-P)^{-1}dz. \tag{12.13}$$

Here Π_{λ_j} is the (rank one) spectral projection associated to λ_j, since the distance from λ_j to the other eigenvalues is $\geq h/C$, Π_{λ_j} is uniformly bounded in norm when $h \to 0$.

Remark 12.1 If we drop the assumption on the eigenvalues of the quadratic approximations, then for instance two eigenvalues λ_1 and λ_2 can be very close together but separated from the others by h/C, and (since we are dealing with a non-selfadjoint operator) we can not state that Π_1 and Π_2 are uniformly bounded when $h \to 0$. But the sum $\Pi_1 + \Pi_2$ will have this property and so will the term $e^{-\lambda_1 t}\Pi_1 + e^{-\lambda_2 t}\Pi_2$ in (12.13). This kind of situation will appear when there is a symmetry, and to have a more complete understanding in that case would include problems about the tunnel effect.

We estimate the last integral in (12.13) using the decomposition $\gamma = \gamma_{\rm int} \cup \gamma_{\rm ext}$:

$$\frac{1}{2\pi i}\int_{\gamma_{\rm int}} e^{-tz/h}(z-P)^{-1}dz = \mathcal{O}(h)e^{-\frac{t}{h}bh}\frac{1}{h} = \mathcal{O}(1)e^{-bt},$$

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_{\rm ext}} e^{-tz/h}(z-P)^{-1}dz &= \mathcal{O}(1)\int_{C_0^3 b^3 h}^{\infty} e^{-\frac{t}{C_0 h}h^{\frac{2}{3}}y^{\frac{1}{3}}}\frac{1}{h^{\frac{2}{3}}y^{\frac{1}{3}}}dy \\ &= \frac{\mathcal{O}(1)}{t^2}\int_{tb}^{\infty} e^{-x}x dx \\ &= \mathcal{O}(1)(\frac{1}{t}+\frac{1}{t^2})e^{-tb}.\end{aligned}$$

Here and below, we let the prefactors $\mathcal{O}(1)$ depend on b, C_0 Combining this with (12.13), we get

$$e^{-tP/h} = \sum_{0}^{N-1} e^{-t\lambda_j/h}\Pi_{\lambda_j} + \mathcal{O}(1)(1+\frac{1}{t}+\frac{1}{t^2})e^{-tb}. \tag{12.14}$$

It is quite possible that the last estimate improves for small t when we let $e^{-tP/h}$ act on elements in the domain of P. Since P is accretive with $\mathrm{Re}\, P \geq -Ch$ we can get rid of the terms $1/t$ and $1/t^2$. The proof of Theorem 1.4 is complete.

13 Application to the Kramers-Fokker-Planck operator

In this section we prove Theorem 1.1 and compute the eigenvalues of the Kramers-Fokker-Planck operator with quadratic potential V :

$$P = v \cdot h\partial_x - V'(x) \cdot h\partial_v + \frac{\gamma}{2}(-(h\partial_v)^2 + v^2 - hn). \tag{13.1}$$

We will recall the classical procedure to obtain this operator by conjugation from

$$P_{\mathrm{FP}} = v \cdot h\partial_x - V'(x) \cdot h\partial_v - \frac{1}{2}\gamma h\partial_v \cdot (h\partial_v + 2v). \tag{13.2}$$

In this article V is a $\mathcal{C}^\infty$ potential with bounded derivatives of second and higher order, and x, $v \in \mathbb{R}^n$. We suppose that V has a finite number of critical points.

We observe (see for example [9]) that the first two terms form the Hamilton field X_0 of the Hamiltonian q where

$$q(x,v) = \frac{1}{2}v^2 + V(x), \qquad X_0 = v \cdot h\partial_x - V'(x) \cdot h\partial_v,$$

when v is considered as the dual variables of x. The Maxwellian is defined by

$$M = e^{-\frac{2}{h}(\frac{v^2}{2} + V(x))},$$

and we get the formally conjugated operator $P = M^{1/2} P_{\mathrm{FP}} M^{-1/2}$ in (13.1).

Metrics and hypotheses

In this section we check that the Kramers-Fokker-Planck operator satisfies the hypotheses of the main theorem under the simple assumptions that V is a Morse function with bounded derivatives of order 2 and higher, such that $|V'(x)| \geq 1/C$ when $|x| \geq C$. Denote by (ξ, η) the variable dual to (x, v). Then

$$P = p^w - \frac{\gamma h n}{2}, \quad \text{with } \; p = \frac{\gamma}{2}(v^2 + \eta^2) + iv \cdot \xi - iV'(x) \cdot \eta.$$

(Note that for this operator $p^w = p(x, hD_x)$). We introduce the natural weight associated to P

$$\lambda^2(x, \xi, v, \eta) = 1 + (V'(x))^2 + \xi^2 + v^2 + \eta^2,$$

and the metric

$$\Gamma_0 = dx^2 + dv^2 + \frac{d\xi^2 + d\eta^2}{\lambda^2}.$$

We note that λ is $\mathcal{C}^\infty$ and that

$$\lambda \in S(\lambda, \Gamma_0), \quad \lambda' \in S(1, \Gamma_0),$$

since V is with second derivative bounded. Let us now check that p satisfies the symbolic estimates (1.4). Denoting $p = p_1 + ip_2$ and $\rho = (x, v, \xi, \eta)$ we get

$$
\begin{aligned}
p_1(\rho) &= \frac{\gamma}{2}(v^2 + \eta^2), \\
p_2(\rho) &= v \cdot \xi - V'(x) \cdot \eta, \\
\partial p_1(\rho) &= \gamma(0, v, 0, \eta), \\
\partial p_2(\rho) &= \gamma(-V''(x) \cdot \eta, \xi, v, -V'(x)), \\
\partial^2 p_1(\rho) &= \gamma \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & Id & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Id \end{pmatrix}, \\
H_{p_2}(\rho) &= v \cdot \partial_x - V'(x) \cdot \partial_v + \eta \cdot V'' \cdot \partial_\xi - \xi \cdot \partial_\eta, \\
H_{p_2} p_1(\rho) &= -\gamma \xi \cdot \eta - \gamma V'(x) \cdot v, \\
\partial H_{p_2} p_1(\rho) &= (-\gamma V''(x) \cdot v, -\gamma V'(x), -\gamma\eta, -\gamma\xi), \\
H_{p_2}^2 p_1(\rho) &= -\gamma v \cdot V''(x) \cdot v + \gamma(V'(x))^2 - \gamma\eta \cdot V''(x) \cdot \eta + \gamma\xi^2.
\end{aligned}
\tag{13.3}
$$

We get directly, using that the derivatives of V of order 2 and higher are bounded, that

$$
p_1 \geq 0, \quad p \in S(\lambda^2, \Gamma_0), \quad \partial p \in S(\lambda, \Gamma_0), \quad \partial^2 p_1 \in S(1, \Gamma_0), \quad \partial H_{p_2} p_1 \in S(\lambda, \Gamma_0). \tag{13.4}
$$

Besides, let us denote by $\{\rho_j\}$ the critical points of p. We notice that they are of the form $(x_j, 0, 0, 0)$ where $\{x_j\}$ are the critical points of V. By δ^2 we denote a $\mathcal{C}^\infty$ function equivalent to the distance to the set $\{\rho_j\}$. Then for ϵ_0 sufficiently small we have

$$
\begin{aligned}
p_1 + \epsilon_0 H_{p_2}^2 p_1 &= \gamma \left(\epsilon_0 (V'(x))^2 + \epsilon_0 \xi^2 + v \cdot (Id/2 - \epsilon_0 V''(x)) \cdot v + \eta \cdot (Id/2 - \epsilon_0 V''(x)) \cdot \eta \right) \\
&\sim \begin{cases} \delta^2 & \text{in a fixed compact set including the } \rho_j\text{s}, \\ \lambda^2 & \text{away from a neighborhood of the } \rho_j\text{s}. \end{cases}
\end{aligned}
\tag{13.5}
$$

The last thing to check is that the metric Γ_0 is (classically in the sense of (7.3)) admissible. A simple adaptation of Proposition 5.11 in [8] shows that Γ_0 is *cl*-admissible. Note that it is therefore semiclassically admissible (in the sense of (7.11)) since in that case weaker assumptions are needed.

As a consequence we can apply to the Kramers-Fokker-Planck operator $P = p^w - \gamma hn/2$ the main Theorem 1.2. In order to be complete we compute now the eigenvalues of the quadratic approximation P_0 of P near the critical points.

Eigenvalue computation

Here we will compute explicitly the λ_j that occur in the formula for the spectrum given in Proposition 5.1. In addition we compute the constant term that also contributes to the eigenvalues. Thus we obtain the spectrum up to $o(h)$. We assume that p has a single critical point at $x = 0$ and that V is quadratic. After a simultaneous orthogonal change of coordinates in x and in v, we may assume that

$$
V(x) = \tfrac{1}{2} \sum_{j=1}^{n} d_j x_j^2. \tag{13.6}
$$

(The assumption that V is a Morse function implies that all the d_j are different from 0.)

With this choice of V, the operator P_{FP} equals

$$P_{\mathrm{FP}} = -\tfrac{1}{2}\gamma h n + (x, v, D_x, D_v) W (x, v, D_x, D_v)^T. \tag{13.7}$$

where the matrix W is given by

$$W = \begin{pmatrix} 0 & 0 & 0 & -\frac{i}{2} h V''_{xx} \\ 0 & 0 & \frac{i}{2} h I & -\frac{i}{2}\gamma h I \\ 0 & \frac{i}{2} h I & 0 & 0 \\ -\frac{i}{2} h V''_{xx} & -\frac{i}{2}\gamma h I & 0 & \frac{1}{2} h^2 \gamma I \end{pmatrix}.$$

As we explained, the operators P and P_0 of Proposition 5.1 are obtained by conjugation, which corresponds to a complex symplectic coordinate transformation of the symbol. Since the eigenvalues of the linearization of the Hamilton flow are invariant under such a transformation, we can use the unconjugated operator P_{FP} to compute them.

The matrix W is of the form $W = \frac{1}{2}\begin{pmatrix} 0 & ihA^T \\ ihA & h^2 B \end{pmatrix}$, where the $2n \times 2n$ matrix $A = \begin{pmatrix} 0 & I \\ -V''_{xx} & -\gamma I \end{pmatrix}$ is the linearization of the vector field component of P_{FP} for $h = 1$. The matrix corresponding to the linearization of the Hamilton field is given by

$$W' = \begin{pmatrix} ihA & h^2 B \\ 0 & -ihA^T \end{pmatrix}. \tag{13.8}$$

Because of (13.6), the eigenvalues of A are obtained simply by diagonalizing the 2×2 matrices $\begin{pmatrix} 0 & 1 \\ -d_j & -\gamma \end{pmatrix}$. We find that the eigenvalues of A are given by

$$\nu_{j,1} = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - 4d_j}, \qquad \nu_{j,2} = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 4d_j},$$

with $j = 1, \dots, n$. Let $s_{j,k}$ denote the sign of the real part of $\nu_{j,k}$

$$s_{j,1} = \operatorname{sgn}(\operatorname{Re}(\nu_{j,1})) = -1, \quad s_{j,2} = \operatorname{sgn}(\operatorname{Re}(\nu_{j,2})) = -\operatorname{sgn}(d_j).$$

It follows that the eigenvalues of W' are given by

$$ih\nu_{j,1}, \quad ih\nu_{j,2}, \quad -ih\nu_{j,1}, \quad -ih\nu_{j,2},$$

and that the ones with positive imaginary part are given by

$$\begin{aligned} ihs_{j,1}\nu_{j,1} &= \frac{i}{2}\gamma h + \frac{i}{2} h\sqrt{\gamma^2 - 4d_j}, \\ ihs_{j,2}\nu_{j,2} &= -\operatorname{sgn}(d_j)\left(-\frac{i}{2}\gamma h + \frac{i}{2} h\sqrt{\gamma^2 - 4d_j}\right). \end{aligned}$$

The constant term in (13.7) satisfies $-\frac{1}{2}\gamma h n = \frac{1}{2}\operatorname{tr}(A) = \frac{1}{2}\sum_{j=1}^n (\nu_{j,1} + \nu_{j,2})$. Thus the spectrum of the quadratic operator is given by

$$\left\{ h \sum_{j=1}^{n} \left((\tfrac{1}{2} + \tfrac{1}{2}s_{j,1} + k_{j,1}) s_{j,1}\nu_{j,1} + (\tfrac{1}{2} + \tfrac{1}{2}s_{j,2} + k_{j,2}) s_{j,2}\nu_{j,2} \right) \; ; \; k_{j,1}, k_{j,2} \in \mathbb{N} \right\}.$$

Remark 13.1 In the case of quadratic Kramers-Fokker-Planck, the lowest eigenvalue of the spectrum is 0 if and only if all the $s_{j,1}, s_{j,2}$ are equal to -1, i.e. if $x = 0$ is a minimum of V.

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