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RELATING SYSTEMS PROPERTIES OF THE WAVE AND THE SCHRÖDINGER EQUATION

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Abstract. In this article we show that systems properties of the systems governed by the second order differential equation $\frac{d^2w}{dt^2} = -A_0 w$ and the first order differential equation $\frac{dz}{dt} = iA_0 z$ are related. This can be used to show that, for instance, exact observability of the N-dimensional wave equation implies the similar property for the N-dimensional Schrödinger equation.

1. **Introduction.** In Section 6.7 of [8], Tucsnak and Weiss study the relation between the second order state-output system

$$\frac{d^2w}{dt^2} = -A_0 w, \qquad y = C\frac{dw}{dt} \tag{1}$$

and the first order state-output system

$$\frac{dz}{dt} = iA_0 z, \qquad y = Cz. \tag{2}$$

As may be clear from this second system, the standard example is the wave equation (1) and the Schrödinger equation (2), with A_0 equal to the Laplacian. Among others, for these systems Tucsnak and Weiss showed that exact observability of (1) implies exact observability of (2). They do this under the assumption that A_0 is a positive definite operator from $D(A_0) \subset X_1$ to X_1, where X_1 is a Hilbert space. The approach they take is to use the classical change of variables for second order systems by defining the state and state space as

$$\begin{pmatrix} w \\ \frac{dw}{dt} \end{pmatrix} \in D(A_0^{\frac{1}{2}}) \oplus X_1.$$

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Here $D(A_0^{\frac{1}{2}})$ is regarded as a Hilbert space with the inner product

$$\langle w_1, w_2\rangle_{D(A_0^{\frac{1}{2}})} := \langle A_0^{\frac{1}{2}} w_1, A_0^{\frac{1}{2}} w_2\rangle, \qquad w_1, w_2 \in D(A_0^{\frac{1}{2}}),$$

where the latter is the inner product of X_1.

With this choice of state, the second order differential equation in (1) is written as

$$\frac{d}{dt}\begin{pmatrix} w \\ \frac{dw}{dt}\end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_0 & 0\end{pmatrix}\begin{pmatrix} w \\ \frac{dw}{dt}\end{pmatrix} \tag{3}$$

In the examples of the wave equation and the Schrödinger equation, the operator $-A_0$ is the product of the two formally adjoint operators: divergence $A_{12} = \text{div}$, $\text{div} f = \frac{\partial f_1}{\partial \zeta_1} + \frac{\partial f_2}{\partial \zeta_2} + \frac{\partial f_3}{\partial \zeta_3}$ and gradient $A_{21} = \nabla$, $\nabla x = \left(\frac{\partial x}{\partial \zeta_1}, \frac{\partial x}{\partial \zeta_2}, \frac{\partial x}{\partial \zeta_3}\right)^T$. This corresponds to the canonical case for physical systems models where the dynamical equations are actually systems of coupled conservation laws [9, 2, chap.4]. In this paper we shall consider this case, i.e., we assume that the operator $-A_0$ is the product of two operators A_{12} and A_{21} with A_{12} equal to minus the formal adjoint of A_{21} and vice versa. With the appropriate boundary conditions, the divergence and gradient operators satisfy this, see Example 1.

Considering the operator $-A_0$ as the product $-A_0 = A_{12}A_{21}$, the differential equation in (1) becomes

$$\frac{d^2 w}{dt^2} = A_{12}A_{21}w. \tag{4}$$

Thus $A_{21} : D(A_{21}) \subset X_1 \to X_2$, and $A_{12} : D(A_{12}) \subset X_2 \to X_1$, with X_1, X_2 Hilbert spaces. We define now the following change of coordinates

$$\begin{pmatrix} w_1 \\ w_2\end{pmatrix} = \begin{pmatrix} \frac{dw}{dt} \\ A_{21}w\end{pmatrix} \in X_1 \oplus X_2 \tag{5}$$

which combined with (4) leads to the following first order differential equation

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2\end{pmatrix} = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}. \tag{6}$$

Using this differential equation we may now comment more precisely how the decomposition of the operator relates to systems of conservation laws for the case of the wave equation and the Schrödinger equation. Equation (2), corresponding to the Schrödinger equation, may be written as the conservation law

$$\frac{dz}{dt} = -\text{div}\left(-i\,\nabla\left(\frac{\hbar}{2m}z\right)\right)$$

and equation (6) may be interpreted as a system of two coupled conservation laws

$$\frac{d}{dt}\begin{pmatrix} w_1 \\ w_2\end{pmatrix} = \begin{pmatrix} 0 & \text{div} \\ \nabla & 0\end{pmatrix}\begin{pmatrix} w_1 \\ w_2\end{pmatrix}, \tag{7}$$

in the generalised sense where both operators divergence, denoted by div and gradient, denoted by ∇ are considered as exterior derivatives of differential forms of degree 2 and 0, respectively [9].

The norm of the second state space $X_1 \oplus X_2$ relates to the norm of the state space $D(A_0^{\frac{1}{2}}) \oplus X_1$ for (3) in the following way. The norm of the state $\begin{pmatrix} w \\ \frac{dw}{dt}\end{pmatrix} \in D(A_0^{\frac{1}{2}}) \oplus X_1$ equals

$$\left\|\begin{pmatrix} A_0^{\frac{1}{2}} w \\ \frac{dw}{dt}\end{pmatrix}\right\|^2 = \langle A_0^{\frac{1}{2}} w, A_0^{\frac{1}{2}} w\rangle + \left\|\frac{dw}{dt}\right\|^2 = \left\|\frac{dw}{dt}\right\|^2 + \langle w, A_0 w\rangle.$$

If $-A_{12} = A_{21}^*$, then since $A_0 = -A_{12}A_{21}$ the latter equals $\|A_{21}w\|^2$. Thus

$$\left\| \begin{pmatrix} A_0^{\frac{1}{2}} w \\ \frac{dw}{dt} \end{pmatrix} \right\|^2 = \left\| \begin{pmatrix} \frac{dw}{dt} \\ A_{21} w \end{pmatrix} \right\|^2 .$$

However the relation between the differential equations (3) and (6) is not straightforward, and how their systems properties relate, will hence be the topic of this paper. In the first instance we shall show that similar results as found in [8, Section 6.7] can be found when the system is written in the form of equation (6). In the second instance, we shall extend their result by not requiring anymore that $A_0 = -A_{12}A_{21}$ is positive definite.

2. **Generation of contraction semigroups.** In this section we investigate the properties of the operator in equation (6). We show that if it generates a contraction semigroup on the Hilbert space $X_1 \oplus X_2$, then its generates a unitary group.

We denote by A_{ext} the system operator in (6),

$$A_{\text{ext}} = \begin{pmatrix} 0 & A_{12} \\ A_{21} & 0 \end{pmatrix} \tag{8}$$

with domain $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$. This operator has some nice properties.

Lemma 2.1. *Consider* $A_{\text{ext}} = \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right)$ *with domain* $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$. *The following hold*

1. *The resolvent of* A_{ext} *is given by*

$$(sI - A_{\text{ext}})^{-1} = \begin{pmatrix} s(s^2 - A_{12}A_{21})^{-1} & A_{12}(s^2 I - A_{21}A_{12})^{-1} \\ A_{21}(s^2 - A_{12}A_{21})^{-1} & s(s^2 I - A_{21}A_{12})^{-1} \end{pmatrix}, \tag{9}$$

   where $A_{12}A_{21}$ *and* $A_{21}A_{12}$ *are defined on their natural domain, i.e.,* $D(A_{12}A_{21}) = \{x_1 \in D(A_{21}) \mid A_{21}x_1 \in D(A_{12})\}$.
2. *If* A_{ext} *generates a semigroup which is bounded by* $Me^{\omega t}$, $M \geq 1$, $\omega \geq 0$, *then this semigroup can be extended to a group which is bounded by* $Me^{\omega|t|}$.
3. *If* A_{ext} *generates a contraction semigroup, then this semigroup can be extended to a unitary group. Furthermore,* $A_{12} = -A_{21}^*$, $(A_{12}A_{21})^* = A_{12}A_{21}$ *and* $A_{12}A_{21}$ *is a non-positive operator, i.e.,* $\langle w_1, A_{12}A_{21}w_1\rangle \leq 0$ *for all* $w_1 \in D(A_{12}A_{21})$.

Proof. The resolvent identity (9) is obtained by direct computation.

From (9) we see that the resolvent at $-s$ equals that of s with a minus sign on the diagonals. Hence for $s \in \mathbb{C}$,

$$(sI + A_{\text{ext}})^{-1} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} (sI - A_{\text{ext}})^{-1} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$$

Since $\left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} -I & 0 \\ 0 & I \end{smallmatrix}\right) = \left(\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right)$, we see that for every $k \in \mathbb{N}$ there holds

$$(sI + A_{\text{ext}})^{-k} = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} (sI - A_{\text{ext}})^{-k} \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}. \tag{10}$$

Using this relation and the Hille-Yosida Theorem, [1, 3], the assertion 2. follows directly.

In particular, we see that if A_{ext} generates a contraction semigroup, then $-A_{\text{ext}}$ also generates a contraction semigroup. Combining this with Theorem 6.2.5 of [6] we have that A_{ext} generates a unitary group. The generator of a unitary group is skew-adjoint [3, Theorem II.3.24], and so $A_{12} = -A_{21}^*$. From this it follows that $A_{12}A_{21}$ is self-adjoint and non-positive. □

Since skew-adjoint operators generate unitary groups [6, Theorem 6.2.5], part 3 of Lemma 2.1 immediately gives the following result.

Theorem 2.2. *Let* $A_{\text{ext}} = \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right)$ *with domain* $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$ *generate a contraction semigroup, then*

$$A_S := iA_{12}A_{21} \tag{11}$$

with domain $D(A_S) = \{x_1 \in D(A_{21}) \mid A_{21}x_1 \in D(A_{12})\}$ *generates a unitary group on* X_1.

We apply the above result to the Schrödinger equation.

Example 1. The Schrödinger equation on Ω for a free particle is given by

$$\frac{\partial x}{\partial t}(\zeta,t) = i\frac{\hbar}{2m}\Delta x(\zeta,t), \quad \zeta \in \Omega,\ t \geq 0, \tag{12}$$

where $\hbar$ is the reduced Planck constant, m the mass of the particle, and Δ denotes the Laplacian, i.e., $\Delta x = \frac{\partial^2 x}{\partial \zeta_1^2} + \frac{\partial^2 x}{\partial \zeta_2^2} + \frac{\partial^2 x}{\partial \zeta_3^2}$. We write this Laplacian as

$$\Delta = \text{div} \cdot \nabla \tag{13}$$

with $\nabla x = \left(\frac{\partial x}{\partial \zeta_1}, \frac{\partial x}{\partial \zeta_2}, \frac{\partial x}{\partial \zeta_3} \right)^T$ and $\text{div} f = \frac{\partial f_1}{\partial \zeta_1} + \frac{\partial f_2}{\partial \zeta_2} + \frac{\partial f_3}{\partial \zeta_3}$. It is well-known that $-\nabla$ is the (formal) adjoint of the divergence div.

The A_{ext} associated to (13) is

$$A_{\text{ext}} = \begin{pmatrix} 0 & \sqrt{\frac{\hbar}{2m}}\text{div} \\ \sqrt{\frac{\hbar}{2m}}\nabla & 0 \end{pmatrix}.$$

As domain we choose

$$D(A_{\text{ext}}) = \left\{ \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \in L^2(\Omega)^2 \mid e_2 \in H_{\text{div}}(\Omega), e_1 \in H^1(\Omega) \text{ and } e_1 = 0 \text{ on } \partial\Omega \right\}.$$

For the precise definition of the Sobolev spaces H^1 and H_{div} we refer to [4]. The operator A_{ext} is written in the form of Theorem 2.2, and it generates a unitary group. The associated Schrödinger equation is given by

$$A_S = i\frac{\hbar}{2m}\Delta,$$

with domain

$$D(A_S) = \{e_1 \in H^1(\Omega) \mid \nabla e_1 \in H_{\text{div}}(\Omega) \text{ and } e_1 = 0 \text{ on } \partial\Omega\}.$$

This corresponds to a particle trapped in a potential well Ω, i.e, the associated potential is zero inside Ω and infinite outside Ω, see [5].

3. **Systems relations.** In the previous part we have seen that existence of solutions to the Schrödinger equation can be proved using the wave equation. In this section we study the relation between systems properties. Since the results for input operators are dual to the result for output operators, we shall concentrate on output operators.

We begin by recalling the *admissibility* for an output operator.

Definition 3.1. Let the operator A with domain $D(A)$ be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, and let C be a bounded linear operator from $D(A)$ to a second Hilbert space Y. C is a *finite-time admissible* output operator for A if there exists a $m > 0$ and $t_1 > 0$ such that

$$\int_0^{t_1} \|CT(t)x_0\|_Y^2 dt \leq m\|x_0\|^2 \quad \text{for all } x_0 \in D(A), \tag{14}$$

where $\| \|_Y$ denotes the norm on Y, whereas $\| \|$ on the right-hand side is the norm of X.

Hence admissibility implies that the output of the state-output equation $\dot{x} = Ax(t), y(t) = Cx(t)$ has a meaning as an L^2-function.

Since in this section we mainly work with generators of groups the following characterisation is useful. For the proof we refer to [10].

Lemma 3.2. *Let the operator A with domain $D(A)$ be the generator of the C_0-group $(T(t))_{t\in\mathbb{R}}$ on the Hilbert space X, and let C be a bounded linear operator from $D(A)$ to a second Hilbert space Y. Then C is finite-time admissible if and only if there exists a $M > 0$ and a $r \in \mathbb{R}$ such that for s with real part larger than r and all $x \in D(A)$ there holds*

$$\|Cx\| \leq M\|(sI - A)x\|.$$

The following lemma links admissibility of output operators for the two systems (6) and (11).

Lemma 3.3. *Let $A_{\text{ext}} = \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right)$ with domain $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$ be the generator of a contraction semigroup, and let C_{ext} be defined as*

$$C_{\text{ext}}x = C_1x_1, \quad \textit{where} \quad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(A_{\text{ext}}).$$

If C_{ext} is finite-time admissible for A_{ext}, then C_1 is finite-time admissible for A_S, see (11).

Proof. We begin by remarking that from our assumption on A_{ext} it follows that both operators generate a group, see Lemma 2.1 and Theorem 2.2. Hence we can apply Lemma 3.2 for both systems.

Since C_{ext} is finite-time admissible and A_{ext} generates a group, we know by Lemma 3.2 that for $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in D(A_{\text{ext}})$

$$\|C_{\text{ext}}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\| \leq M\left\|\begin{pmatrix} sI & -A_{12} \\ -A_{21} & sI \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\|, \quad \text{Re}(s) > r. \tag{15}$$

For $x_1 \in D(A_S)$ and $s \neq 0$, we define $x_2 = \frac{1}{s}A_{21}x_1$. Then $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in D(A_{\text{ext}})$ and using the definition of C_{ext}, we find

$$\|C_1x_1\| \leq M\|(sI - \frac{1}{s}A_{12}A_{21})x_1\| = \frac{M}{|s|}\|(is^2I - iA_{12}A_{21})x_1\|. \tag{16}$$

Next we show that this is uniformly bounded when the real part of is^2 is larger than or equal to 2. For that we write s as $s = a - ib$. Note that without loss of generality we may assume that $a = \text{Re}(s)$ is positive. The real part of $-is^2$ is given by $2ab$. Hence the real part of $-is^2$ is larger than or equal to two if and only if $ab \geq 1$. By our choice for the values of a, we see that a and b are now both positive. It is not hard to show that if $ab \geq 1$, then $|s|^2 = a^2 + b^2 \geq 2$. Applying this in equation (16), we have

$$\|C_1 x_1\| \leq \frac{M}{\sqrt{2}} \|(is^2 I - iA_{12}A_{21})x_1\|$$

for the real part of is^2 larger than or equal to two. So from Lemma 3.2 we conclude that C_1 is admissible for $A_S = iA_{12}A_{21}$. □

In fact we can also relate the exact observability of the extended system and the original system.

Definition 3.4. Let the operator A with domain $D(A)$ be the generator of the C_0-semigroup $(T(t))_{t\geq 0}$ on the Hilbert space X, let C be a bounded operator from $D(A)$ to a second Hilbert space Y, and let C be finite-time admissible. C is *exactly observable* for A if there exists a $m > 0$ and $t_1 > 0$ such that

$$\int_0^{t_1} \|CT(t)x_0\|_Y^2 dt \geq m\|x_0\|^2 \quad \text{for all } x_0 \in D(A), \tag{17}$$

where $\| \, \|_Y$ denotes the norm on Y and $\| \, \|$ the norm of X.

Since in this section we mainly work with generators of unitary groups the following characterisation is useful. For the proof we refer to [7].

Lemma 3.5. *Let the operator A with domain $D(A)$ be the generator of a unitary group on the Hilbert space X, and let C be a bounded linear operator from $D(A)$ to a second Hilbert space Y. Then C is exactly observable if and only if there exists a $m > 0$ such that for $\omega \in \mathbb{R}$ and $x \in D(A)$ there holds*

$$\|Cx\|^2 + \|(i\omega I - A)x\|^2 \geq m\|x\|^2.$$

Lemma 3.6. *Let $A_{\text{ext}} = \left(\begin{smallmatrix} 0 & A_{12} \\ A_{21} & 0 \end{smallmatrix}\right)$ with domain $D(A_{\text{ext}}) = D(A_{21}) \oplus D(A_{12})$ be the generator of a contraction semigroup. If $C_{\text{ext}} = (C_1 \; 0)$ is exactly observable for A_{ext}, then C_1 is exactly observable for $A_S = iA_{12}A_{21}$.*

Proof. By our assumption on A_{ext} it follows that both operators generate a unitary group, see Lemma 2.1 and Theorem 2.2. Hence we can use Lemma 3.5 for both systems.

Since C_{ext} is exactly observable for A_{ext}, there exists an $m > 0$ such that for all $\omega \in \mathbb{R}$,

$$\|C_{\text{ext}} x\|^2 + \|(i\omega I - A_{\text{ext}})x\|^2 \geq m\|x\|^2, \qquad x \in D(A_{\text{ext}}). \tag{18}$$

We must show that there exists a $m_1 > 0$ such that for all $q \in \mathbb{R}$

$$\|C_1 x_1\|^2 + \|(iqI - iA_{12}A_{21})x_1\|^2 \geq m_1\|x_1\|^2, \qquad x_1 \in D(A_S). \tag{19}$$

Substituting $\omega = 0$ and $\left(\begin{smallmatrix} x_1 \\ \frac{1}{\sqrt{m}} iA_{21}x_1 \end{smallmatrix}\right)$ with $x_1 \in D(A_S)$ in equation (18) gives

$$\|C_1 x_1\|^2 + \|\frac{1}{\sqrt{m}} A_S x_1\|^2 + \|A_{21}x_1\|^2 \geq m\left(\|x_1\|^2 + \|\frac{1}{\sqrt{m}} A_{21}x_1\|^2\right).$$

Hence

$$\|C_1 x_1\|^2 + \frac{1}{m}\|A_S x_1\|^2 \geq m\|x_1\|^2.$$

Since we may without loss of generality assume that $m < 1$, equation (19) holds for $q = 0$ with $m_1 = m^2$. Furthermore, since the left hand-side of (19) is continuous in q, we have that there exists an interval $(-q_0, q_0)$ with $q_0 \in (0, 1]$ such that (19) holds with $m_1 = m^2/2$.

So it remains to show that (19) holds for $q \geq q_0$ and for $q \leq -q_0$.

Since $A_{12}A_{21}$ is non-positive, $qI - A_{12}A_{21}$ is boundedly invertible for $q \geq q_0$. Thus (19) holds for some (positive) m_1 when $q \geq q_0$.

For $x_1 \in D(A_S)$, we have that $A_{21}x_1 \in D(A_{21})$. Hence $x := \begin{pmatrix} x_1 \\ \frac{-i}{\omega}A_{21}x_1 \end{pmatrix}$ is in the domain of A_{ext}, and if we substitute this in equation (18), we find

$$\|C_1x_1\|^2 + \| \left(i\omega + i\frac{1}{\omega}A_{12}A_{21}\right) x_1\|^2 \geq m \left[\|x_1\|^2 + \|\frac{i}{\omega}A_{21}x_1\|^2\right].$$

Since $q_0 \leq 1$, we find for $\omega \geq \sqrt{q_0}$

$$\begin{aligned}
\|C_1x_1\|^2 &+ \|(i(-\omega^2)I - iA_{12}A_{21})x_1\|^2 \\
&= \|C_1x_1\|^2 + \omega^2\|(i\omega I + i\frac{1}{\omega}A_{12}A_{21})x_1\|^2 \\
&\geq \|C_1x_1\|^2 + q_0\|(i\omega + i\frac{1}{\omega}A_{12}A_{21})x_1\|^2 \\
&\geq q_0 \left[\|C_1x_1\|^2 + \|(i\omega + i\frac{1}{\omega}A_{12}A_{21})x_1\|^2\right] \\
&\geq q_0 m \left[\|x_1\|^2 + \|\frac{i}{\omega}A_{21}x_1\|^2\right] \geq q_0 m\|x_1\|^2.
\end{aligned} \tag{20}$$

So we have for $q \leq -q_0$ that

$$\|C_1x_1\|^2 + \|(iqI - iA_{12}A_{21})x_1\|^2 \geq q_0 m\|x_1\|^2. \tag{21}$$

Summarising, we see that (19) holds some $m_1 > 0$ and all $q \in \mathbb{R}$. Applying the result of Miller once more, we conclude that C_1 is exactly observable for A_S. □

For applications of the above results we refer to chapter 6 and 7 of [8].

4. **Conclusion.** In this paper we have highlighted the links existing between the properties of systems associated with operators that can be decomposed into a product of skew-symmetric operators and the properties of some associated extended operators. In the Schrödinger equation case it is an alternative to the classical change of variables that allows to relate the system solutions to the ones of the wave equation. The interest of the proposed approach is that the variables and the spaces (and consequently the norms) that are used for the decomposition of the system operator are the physical ones, avoiding some restrictive assumptions on the positive definitiveness of the operator to prove the existence of these links. More precisely we have deduced from the existence of a contraction semigroup for the wave equation the existence of a contraction semigroup for the Schrödinger equation. Furthermore, we showed that admissibility and exact observability of the one induces the same properties for the other.

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