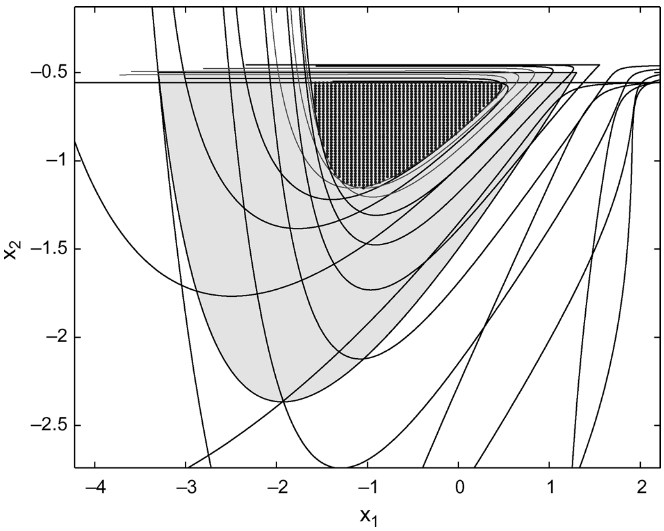

Fig. 3. Stability and H_∞ region.

In addition, while the arbitrarily chosen values for x_3 and γ resulted in a nonempty solution set in this example, there is no guarantee that this will be the case. Here, one can make use of H_∞ theory to investigate existence criteria.

VI. Concluding Remarks

This note gives a technique to compute the set of all first order stabilizing controllers which satisfy an H_∞ constraint for a given but arbitrary linear time-invariant plant. The result is based on determination of root invariant regions via D-decomposition and parameter mapping [19]. The method obviously extends to handle several H_∞ constraints. We believe that, due to the predominance of PID and first-order controllers, in practice and of H_∞ in theory, the result given here and in [16] represent a significant step in enabling the use of H_∞ optimal controllers in industrial control. As a final comment, we note that the set of controllers meeting the constraint seem to be nonfragile [18]. This raises the interesting question: Can we overcome the fragility problem if optimal control is used in conjunction with low order controllers?

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On H_2 Control of Systems With Multiple I/O Delays

Agoes A. Moelja, Gjerrit Meinsma, and Jacolien Kuipers

***Abstract*—The H_2 optimal control problem of systems with multiple I/O delays is solved. The problem is tackled by first transforming it to what is called the two-sided regulator problem. The latter is solved using orthogonal projection arguments and spectral factorization. The resulting controller is an interconnection of rational blocks, nonrational blocks having finite impulse response, and delay operators, all of which are implementable.**

***Index Terms*—Delay systems, H_2-control, linear quadratic Gaussian (LQG) control.**

I. Introduction

Time delays occur naturally in many physical systems. The presence of delays makes analysis and controller synthesis more challenging. Since the Smith predictor [22], there have been numerous attempts to control systems with delays optimally in some sense. In the area of H_∞ control, the books [7], [23] treat a general class of infinite dimensional H_∞ control problems that include systems with delays. Later, methods that are specifically tailored for systems with I/O delays were developed. The single delay case is treated in [12] and [13], while the

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solution of multiple I/O delays case may be found in [10] and [11]. Similar to the case in H_∞ control, in the area of H_2 [linear quadratic Gaussian (LQG)] control, general infinite dimensional theory may also be applied to systems with delays. Chapter 6 of [2] provides a detailed overview and references of the LQ theory for infinite dimensional systems. Along the infinite dimensional theory line, Delfour *et al.* [4]–[6] treated the LQ control of retarded differential equations. Extensions to general delay equations with delayed inputs and outputs may be found in [19], [20], and [3]. Earlier, Kleinman [9] came up with a solution of the LQG problem tailored for systems with a single I/O delay. Recently, it was shown in [13] that the solution for this problem may be obtained by converting the infinite dimensional problem to an equivalent finite dimensional optimization using loop shifting techniques originated from [1] and [27].

This note aims to solve the H_2-optimal control problem for systems with multiple I/O delays. For the case where the delays are present only either in the measurement output or in the control input, the problem is solved in [16], [18]. There, the one-sided problem is transformed to an LQR with input delays, which is solved in time domain using dynamic programming ideas. Another solution of the LQR problem with input delays may be found in [10], in which the problem is treated as the the limiting problem of an H_∞ problem. However, these results do not cover the general case where delays are present both in the measurement output and in the control input.

In this note, a frequency domain solution that covers the general case, where delays occur in both sides of the controller, is proposed. The approach is to first convert the standard problem of Fig. 1(a) to a one-block problem, a technique borrowed from [13]. The internal stability condition of the original problem is converted to a less demanding condition of the input-output transfer function being stable. This allows further transformation to what is called the two-sided regulator problem [Fig. 1(b)]. The latter is then solved iteratively using orthogonal projection arguments and spectral factorization. Each iteration in the solution reduces the number of distinct delays in the delay operators. This can be achieved by applying a special decomposition of the delay operators. The resulting optimal controller consists of rational blocks, finite impulse response (FIR) blocks, and delay operators, all of which are implementable. However, the method has its own drawback: It cannot be applied to cases with an unstable plant. Note that the idea of decomposing the delay operators was first used in [11] in the H_∞ context. This note applies the idea in the H_2 context. Furthermore, in relation to the time domain result in [16] and [18], this note offers an alternative frequency domain solution that is able to treat the case with delays on both sides of the controller. The method of this note has the potential for application. For example, the note [8] considers the problem of steel-sheet profile control at a hot strip mill, which has different delays in its measurement channels. The problem is formulated as an LQG problem, which is equivalent to the H_2 problem with one-sided delays. However, the method used in [8] requires that the noise and disturbance models are block-diagonal. Certain approximations have to be made to meet this requirement, resulting in a suboptimal controller. Using the method derived in this note, it is possible to compute the optimal H_2-controller for the same control problem without the approximations (see [16] for a time domain solution of the problem). The note is organized as follows. After the problem formulation and a preliminary result, the conversion of the standard problem to the two-sided regulator problem is elaborated, which is then solved in the subsequent sections.

Notations: The H_2-norm whenever finite is defined as $\|F(s)\|_2^2 = (1/2\pi)\int_{-\infty}^{\infty} \text{trace}[F^T(j\omega)F(j\omega)]d\omega$. A transfer matrix $F(s)$ is said to be *stable* if $F(s) \in H_\infty$, and *bistable* if both $F(s)$ and $F(s)^{-1}$ are stable. It is said to be *inner* if $F^\sim(s)F(s) = I$, where $F^\sim(s) := F^T(-s)$. It is proved in [24] that a transfer matrix is causal if it is proper. Here, a transfer matrix $F(s)$ is said to be *proper* if there exist $\rho \in \mathbb{R}$ such that $\sup_{\text{Re}(s)>\rho}\|F(s)\| < \infty$. In addition, $F(s)$ is said to be *strictly proper* if there is a $\rho \in \mathbb{R}$ such that $\lim_{s\to\infty,\text{Re}(s)>\rho}\|F(s)\| = 0$. A lower linear fractional transformation (LFT) of two transfer matrices $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22}\end{bmatrix}$ and U of appropriate dimension is defined as $F_\ell(M,U) = M_{11} + M_{12}U(I - M_{22}U)^{-1}M_{21}$. The notation $\{F\}_+$ and $F \in L_2$ denotes the stable part of F.

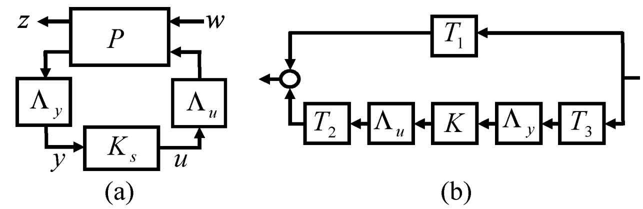

Fig. 1. Standard control system with (a) I/O delays and (b) two sided regulator problem.

II. Problem Formulation

We consider the control system in Fig. 1(a). Here the plant P is a rational transfer matrix having the following realization:

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22}\end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] \quad (1)$$

interconnected with a proper controller K_s and the multiple delay operators of the form

$$\Lambda_y = \text{diag}(e^{-sh_{y1}}, e^{-sh_{y2}}, \ldots, e^{-sh_{ym}}) \quad (2)$$

$$\Lambda_u = \text{diag}(e^{-sh_{u1}}, e^{-sh_{u2}}, \ldots, e^{-sh_{up}}) \quad (3)$$

where m and p are the dimension of y and u, respectively. We assume

A1) (C_2, A, B_2) is detectable and stabilizable;

A2) $R_1 = D_{12}^T D_{12} > 0$ and $R_2 = D_{21}D_{21}^T > 0$;

A3) $\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_{12}\end{bmatrix}$ and $\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_{21}\end{bmatrix}$ have full-column rank and full-row rank, respectively $\forall\omega \in \mathbb{R}$.

The problem is to find a stabilizing LTI causal controller K_s such that the H_2-norm of the transfer function from w to z is minimized.

III. Preliminary: Equivalent Finite Dimensional Stabilization

To transform the standard problem to the two-sided regulator problem, a result that allows the conversion of the internal stabilization problem of the infinite dimensional system of Fig. 1(a) to an equivalent finite dimensional problem is required. The following lemma is an extension of [14, Lemma 1] that treats the single delay case. The lemma is based on [15, Lemma 2]. Note that similar result is also independently developed in [21].

Lemma 1: Define the plant with the delay operators absorbed

$$\hat{P} = \begin{bmatrix} P_{11} & P_{12}\Lambda_u \\ \Lambda_y P_{21} & \Lambda_y P_{22}\Lambda_u \end{bmatrix} \quad (4)$$

where P, Λ_y, and Λ_u are given by (1), –(3). Also, define the plant $\tilde{P}$

$$\tilde{P} = \begin{bmatrix} P_{11} & \tilde{P}_{12} \\ \tilde{P}_{21} & \tilde{P}_{22}\end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & \tilde{B}_2 \\ \hline C_1 & 0 & D_{12} \\ \tilde{C}_2 & D_{21} & 0 \end{array}\right] \quad (5)$$

$$where \quad \tilde{C}_2 = \left[e^{-A^T h_{y1}}C_{2,1}^T \ \cdots \ e^{-A^T h_{ym}}C_{2,m}^T\right]^T \quad (6)$$

$$\tilde{B}_2 = \left[e^{-Ah_{u1}}B_{2,1} \ \cdots \ e^{-Ah_{up}}B_{2,p}\right] \quad (7)$$

with $B_{2,i}$ and $C_{2,i}$ are the ith column of B_2 and ith row of C_2, respectively. Then the control system $F_\ell(\hat{P}, K_s)$ is internally stable iff so is the control system $F_\ell(\tilde{P}, \tilde{K}_s)$, where there is a proper bijection between K_s and $\tilde{K}_s$ governed by the following equations:

$$\tilde{K}_s = (I + K_s\Phi_{22})^{-1}K_s \quad K_s = (I - \tilde{K}_s\Phi_{22})^{-1}\tilde{K}_s \tag{8}$$

$$\text{with} \quad \Phi_{22} = \tilde{P}_{22} - \Lambda_y P_{22}\Lambda_u. \tag{9}$$

Proof: See the Appendix. ■

IV. Conversion to Two-Sided Regulator Problem

The aim is to transform the standard problem of Fig. 1(a) to the two sided regulator problem of Fig. 1(b), where there is a proper bijection between K_s and K. In the first stage of the transformation, the standard problem is converted to a one-block problem, in which the rational plant has its (1,2) and (2,1) blocks invertible. Here, the internal stability requirement is transformed to a less demanding condition: the resulting one-block problem only requires the stability of the transfer function from the external input to the external output. This allows the second stage, which further transforms the problem to the two-sided regulator problem. An important feature of the latter problem is that with the help of Lemma 1, it may be proved that we may restrict ourself to stable controllers in the optimization. The subsequent lemma and theorem provide the details of the transformation. Note that the first stage (Lemma 2) was developed in [13] for solving the H_2 and H_∞ problems for systems with a single delay.

Lemma 2: Let X and Y be the stabilizing solutions of the following Riccati equations:

$$\left(A - B_2R_1^{-1}D_{21}^TC_1\right)^T X + X\left(A - B_2R_1^{-1}D_{21}^TC_1\right) - XB_2R_1^{-1}B_2^TX + C_1\left(I - D_{12}R_1^{-1}D_{12}^T\right)C_1 = 0 \tag{10}$$

$$\left(A - B_1D_{21}^TR_2^{-1}C_2\right)Y + Y\left(A - B_1D_{21}^TR_2^{-1}C_2\right)^T - YC_2^TR_2^{-1}C_2Y + B_1\left(I - D_{21}^TR_2^{-1}D_{21}\right)B_1^T = 0. \tag{11}$$

where $R_1 = D_{12}^TD_{12} > 0$ and $R_2 = D_{21}D_{21}^T > 0$. Define

$$F := -R_1^{-1}\left(B_2^TX + D_{12}^TC_1\right)$$
$$L := -\left(YC_2^T + B_1D_{21}^T\right)R_2^{-1}. \tag{12}$$

Consider the configuration of Fig. 1 where P, Λ_y, and Λ_u are given by (1), –(3). Then, the following hold.

1) K_s minimizes $\|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2$ if and only if K_s minimizes $\|F_\ell(G, \Lambda_uK_s\Lambda_y)\|_2$ with

$$G = \left[\begin{array}{c|cc} A & -LR_2^{\frac{1}{2}} & B_2 \\ \hline -R_1^{\frac{1}{2}}F & 0 & R_1^{\frac{1}{2}} \\ C_2 & R_2^{\frac{1}{2}} & 0 \end{array}\right]. \tag{13}$$

2) K_s internally stabilizes the control system of Fig. 1(a) if and only if $F_\ell(G, \Lambda_uK_s\Lambda_y) \in H_\infty$.

3) The squared optimal H_2-norm is

$$\begin{aligned}\min_{K_s} \|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2^2 &= \min_{K_s}\|F_\ell(P, K_s)\|_2^2 + \min_{K_s}\|F_\ell(G, \Lambda_uK_s\Lambda_y)\|_2^2 \\ &= \operatorname{tr}\left(B_1^TXB_1\right) + \operatorname{tr}(R_1FYF^T) \\ &\quad + \min_{K_s}\|F_\ell(G, \Lambda_uK_s\Lambda_y)\|_2^2.\end{aligned} \tag{14}$$

Proof: See [13]. ■

Theorem 3: Consider the problem of minimizing $\|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2$ over stabilizing, proper K_s, where P, Λ_u, and Λ_y are given by (1), –(3). Define K such that

$$K = K_s(I - \Lambda_yG_{22}\Lambda_uK_s)^{-1}$$
$$K_s = (I + K\Lambda_yG_{22}\Lambda_u)^{-1}K. \tag{15}$$

Then, the following statements hold.

1) There is a proper bijection between K and K_s.
2) K_s minimizes $\|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2$ if and only if K minimizes $\|G_{11} + G_{12}\Lambda_uK\Lambda_yG_{21}\|$ where G is given by (13).
3) $F_\ell(P, \Lambda_uK_s\Lambda_y)$ is internally stable if and only if $(G_{11} + G_{12}\Lambda_uK\Lambda_yG_{21}) \in H_\infty$.
4) The squared optimal H_2-norm is given by

$$\min_{K_s}\|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2^2 = \operatorname{tr}\left(B_1^TXB_1\right) + \operatorname{tr}(R_1FYF^T) + \min_K\|G_{11} + G_{12}\Lambda_uK\Lambda_yG_{21}\|_2^2. \tag{16}$$

5) If $F_\ell(P, \Lambda_uK_s\Lambda_y)$ is internally stable, then $K \in H_\infty$, i.e., K may be restricted to stable transfer functions in the minimization of $\|G_{11} + G_{12}\Lambda_uK\Lambda_yG_{21}\|$.

Proof: Statement 1) follows from the fact that G_{22} strictly proper. Suppose K is proper, then there exist a $\rho \in \mathbb{R}$ such that

$$\begin{aligned}&\lim_{s\to\infty,\mathrm{Re}(s)>\rho} K_s(s) \\ &= \lim_{s\to\infty,\mathrm{Re}(s)>\rho} (I + K(s)\Lambda_y(s)G_{22}(s)\Lambda_u(s))^{-1}K(s) \\ &= \lim_{s\to\infty,\mathrm{Re}(s)>\rho} K(s)\end{aligned}$$

implying that K_s is also proper. The converse may be proved similarly. To prove statement 2), recall that K_s minimizes $\|F_\ell(P, \Lambda_uK_s\Lambda_y)\|_2$ if and only if K_s minimizes $\|F_\ell(G, \Lambda_uK_s\Lambda_y)\|_2$ with G given by (13) [Lemma 2, statement 1)]. Furthermore, we have that

$$\begin{aligned}F_\ell(G, \Lambda_uK_s\Lambda_y) &= F_\ell\left(\begin{bmatrix} G_{11} & G_{12}\Lambda_u \\ \Lambda_yG_{21} & \Lambda_yG_{22}\Lambda_u \end{bmatrix}, K_s\right) \\ &= F_\ell\left(\begin{bmatrix} G_{11} & G_{12}\Lambda_u \\ \Lambda_yG_{21} & 0 \end{bmatrix}, K\right) \\ &= G_{11} + G_{12}\Lambda_uK\Lambda_yG_{21}.\end{aligned} \tag{17}$$

Statements 3) and 4) follow from statements 2) and 3) of Lemma 2. What is left is proving statement 5). First note that K may be written as $K = \tilde{K}_s(I - \tilde{G}_{22}\tilde{K}_s)^{-1}$, where $\tilde{G}_{22} = \tilde{C}_2(sI - A)^{-1}\tilde{B}_{P2}$, $\tilde{K}_s = (I + K_s\Phi_{22})^{-1}K_s$, and $\Phi_{22} = \tilde{G}_{22} - \Lambda_yG_{22}\Lambda_u$, with $\tilde{C}_2$ and $\tilde{B}_2$ given by (6) and (7). Next, recall that a controller internally stabilizes a rational plant P iff it stabilizes the plant's (2,2) part (see, e.g., [26, Lemma 11.2]). Using Lemma 1, it may be shown that it is also true even if delays are present in the I/O channels.[1] Now, suppose $F_\ell(P, \Lambda_uK_s\Lambda_y)$ is internally stable. Then, since $P_{22} = G_{22}$, K_s also stabilizes $\Lambda_yG_{22}\Lambda_u$. It follows from Lemma 1 that $\tilde{K}_s$ stabilizes $\tilde{G}_{22}$, implying that $F_\ell\left(\begin{bmatrix} 0 & I \\ I & \tilde{G}_{22}\end{bmatrix}, \tilde{K}_s\right)$ is internally stable. Hence, $F_\ell\left(\begin{bmatrix} 0 & I \\ I & \tilde{G}_{22}\end{bmatrix}, \tilde{K}_s\right) = K \in H_\infty$. ■

Remark 4: Statements 1)–4) of Theorem 3 are similar (with a slight modification) to results in [18, Lemmas 1 and 2]. Statement 5), however, is not in [17] and [28].

[1]To see this, consider the control system of Fig. 1(a). Let $\hat{P}$, $\tilde{P}$, and $\tilde{K}_s$ be respectively given by (4), (5), and (8). By Lemma 1 it follows that $F_\ell(\hat{P}, K_s)$ is internally stable iff $F_\ell(\tilde{P}, \tilde{K}_s)$ is internally stable. Since $\tilde{P}$ is rational, $F_\ell(\tilde{P}, \tilde{K}_s)$ is internally stable iff $\tilde{K}_s$ stabilizes $\tilde{P}_{22}$. Again, using Lemma 1, the desired result is obtained, i.e., $F_\ell(\hat{P}, K_s)$ is internally stable iff K_s stabilizes $\hat{P}_{22}$.

V. Two-Sided Regulator Problem

In this section and the subsequent sections, the two-sided regulator problem [Fig. 1(b)] is formulated and solved. The problem is finding a stable LTI controller K that minimizes

$$\|T_1 + T_2\Lambda_u K \Lambda_y T_3\|_2 \tag{18}$$

where T_1, T_2, and T_3 are LTI transfer functions that satisfy the following conditions.

A4) $T_1 \in H_2$ and $T_2, T_3 \in H_\infty$.

A5) T_2 and T_3 have, respectively, full-row rank and full-column rank on $j\mathbb{R} \cup \infty$,

while the delay operator Λ_u and Λ_y are given by (2) and (3). Without loss of generality, it is assumed that the delays in the delay operators are ordered according to their magnitude.

A6) Λ_u and Λ_y are of the form

$$\begin{aligned}\Lambda_\bullet &= \mathrm{diag}(e^{-sh_0}I_0, e^{-sh_1}I_1, \ldots, e^{-sh_N}I_N)\\ 0 &= h_0 < h_1 < \cdots < h_N\end{aligned} \tag{19}$$

where N is the number of nonzero distinct delays. Here, I_0 may be empty indicating that there are no channels that are not delayed. A special case of the problem (18) where $\Lambda_y T_3 = I$ is called one-sided regulator problem.

Remark 5: By Assumption A4), $T_1(s)$, $T_2(s)$, and $T_3(s)$ are assumed to be stable. This means that only stable plants can be handled by the method of this note, at least in its present form. In addition, Assumption A5) follows from assumptions A1)–A3). In fact, Assumption A5) is essentially equivalent to Assumptions A2) and A3).

The solution of the two-sided problem (18) is based on the solution of the one-sided regulator problem.

VI. Solution to the One-Sided Regulator Problem

This section provides the solution to a special case of the two sided problem of minimizing (18) where $\Lambda_y = I$, $T_3 = I$, i.e., $\min_{K\in H_2}\|T_1 + T_2\Lambda_\bullet K\|_2$. The problem is solved by devising a theorem (Theorem 8) that transforms the one-sided problem to another one-sided problem with one less distinct delays in the delay operator. Hence, by applying this theorem successively, we may reduce the delay operator to one with only two distinct delays. The simple structure of the delay operator in the reduced problem allows us to obtain the optimal controller (Corollary 9). The solution is based on spectral factorization theory, projection arguments, and a special decomposition of the delay operators. The subsequent lemmas provides the necessary ingredients for solving the one-sided regulator problem. The first lemma, which is based on the results in [12], presents the spectral factorization of a rational transfer matrix that is multiplied by a delay operator containing at most two distinct delays.

Lemma 6: Let F be a stable rational transfer function that has full column rank on $j\mathbb{R} \cup \infty$, and let Λ_s be a delay operator with at most two distinct delays of the form

$$\Lambda_s = \mathrm{diag}(e^{-sh_1}I_1, e^{-sh_2}I_2) \quad h_2 \geq h_1 \geq 0. \tag{20}$$

Define $\Pi := F^\sim F$ so that with the appropriate partitioning we have that

$$\Lambda_s^\sim F^\sim F\Lambda_s = \begin{bmatrix}\Pi_{11} & e^{-s(h_2-h_1)}\Pi_{12}\\ e^{s(h_2-h_1)}\Pi_{21} & \Pi_{22}\end{bmatrix}. \tag{21}$$

Define Φ_F and R such that

$$e^{-s(h_2-h_1)}\Pi_{11}^{-1}\Pi_{12} = R - \Phi_F \tag{22}$$

with R rational and Φ_F stable.[2] Also define

$$S := \begin{bmatrix}\Pi_{11} & \Pi_{11}R\\ R^\sim\Pi_{11} & \Pi_{22} - \Pi_{21}\Pi_{11}^{-1}\Pi_{12} + R^\sim\Pi_{11}R\end{bmatrix}. \tag{23}$$

Then, S is stable, rational, and proper. Define $\bar{F}_o$ such that $S = \bar{F}_o^\sim \bar{F}_o$, with $\bar{F}_o$ bistable. Then, we have that

$$\Lambda_s^\sim F^\sim F\Lambda_s = \begin{bmatrix}I_1 & 0\\ -\Phi_F^\sim & I_2\end{bmatrix}\bar{F}_o^\sim\bar{F}_o\begin{bmatrix}I_1 & -\Phi_F\\ 0 & I_2\end{bmatrix}. \tag{24}$$

Proof: See [12]. ■

The next lemma is a well-known projection result (see, e.g., [25]).

Lemma 7: Suppose $F_1, K \in H_2$, $F_2 \in H_\infty$, $F_2^\sim F_2 = I$, and $F_2^\sim F_1$ is antistable, then the following holds:

$$\|F_1 + F_2K\|_2^2 = \|F_1\|_2^2 + \|K\|_2^2. \tag{25}$$

It follows that $\|F_1 + F_2K\|_2$ is minimized over $K \in H_2$ for $K = 0$.

The last ingredient that is needed is a decomposition of the delay operator. Consider the delay operator (19) that contains N nonzero delays. The delay operator (19) may be factorized to two factors: A simple delay operator containing one nonzero delay and a multiple delay operator containing $N-1$ nonzero delays.

$$\Lambda_\bullet = \underbrace{\begin{bmatrix}I_0 & 0\\ 0 & e^{-sh_1}\mathrm{diag}(I_1,\ldots,I_N)\end{bmatrix}}_{=:\Lambda_s}\underbrace{\mathrm{diag}(I_0, I_1, \tilde{\Lambda}_r)}_{=:\Lambda_r} \tag{26}$$

$$\text{with } \tilde{\Lambda}_r = \mathrm{diag}\left(e^{-s(h_2-h_1)}I_2, \ldots, e^{-s(h_N-h_1)}I_N\right). \tag{27}$$

Hence, by extracting the simple delay operator Λ_s from (19), the remaining factor Λ_r contains one less nonzero delays. Now that all the ingredients are in place, we are ready to state the theorem that solves the one-sided regulator problem.

Theorem 8 (Delay Reduction Theorem): Consider the regulator problem

$$\min_{K\in H_2}\|T_1 + T_2\Lambda_\bullet K\|_2 \tag{28}$$

where T_1 and T_2 are rational transfer functions satisfying the assumptions

- $T_1 \in H_2$, $T_2 \in H_\infty$;
- T_2 has full-column rank on $j\mathbb{R} \cup \infty$;

and the delay operator $\Lambda_\bullet$ is of the form (19) containing N nonzero delays. Suppose $\Lambda_\bullet$ is factorized according to (26) and (27): $\Lambda_\bullet = \Lambda_s\Lambda_r = \Lambda_s\mathrm{diag}(I_0, I_1, \tilde{\Lambda}_r)$, where Λ_s and Λ_r contain one and $(N-1)$ nonzero delays, respectively. Then, there exists rational transfer functions $\tilde{T}_1, \tilde{T}_2$ and a nonrational transfer function Ψ having the properties[3]

- $\tilde{T}_1 \in H_2$, $\tilde{T}_2, \tilde{T}_2^{-1} \in H_\infty$;
- $\Psi, \Psi^{-1} \in H_\infty$;
- Ψ has finite impulse response;

such that

$$\begin{aligned}&\min_{K\in H_2}\|T_1 + T_2\Lambda_\bullet K\|_2^2\\ &\quad = \|T_1 - T_2\Lambda_s\tilde{T}_2^{-1}\tilde{T}_1\|_2^2 + \min_{\tilde{K}\in H_2}\|\tilde{T}_1 + \tilde{T}_2\Lambda_r\tilde{K}\|_2^2\end{aligned} \tag{29}$$

[2]Note that Φ_F may be constructed in the same way Φ_{22} is constructed in Lemma 1 and may be chosen to have finite impulse response with support on $[0, h_2 - h_1]$.

[3]See the proof for the construction of $\tilde{T}_1$, $\tilde{T}_2$ and Ψ.

where there is a proper bijection between K and $\tilde{K}$

$$\tilde{K} = \Psi K. \tag{30}$$

Proof: Consider the problem (28). We begin by applying the factorization (26) to obtain

$$\|T_1 + T_2\Lambda_\bullet K\|_2 = \|T_1 + T_2\Lambda_s\Lambda_r K(s)\|_2. \tag{31}$$

Thus, by absorbing the second factor in (26) into the controller, the problem (28) is equivalent to the problem

$$\min_{K_r\in H_2} \|T_1 + T_2\Lambda_s K_r\|_2 \tag{32}$$

under the condition

$$K_r = \Lambda_r K,\ K \in H_2. \tag{33}$$

For the moment, let us forget about the condition (33) and concentrate in solving the problem (32). By construction Λ_s contains two distinct delays and is of the form (20), while by assumption T_2 is stable and has full-column rank on $j\mathbb{R}\cup\infty$. Hence, the spectral factorization of $T_2(s)\Lambda_s$ may be computed using Lemma 6

$$\Lambda_s^\sim T_2^\sim T_2\Lambda_s = T_{2,o}^\sim T_{2,o} \tag{34}$$

$$\text{where}\quad T_{2,o} = \bar{T}_{2,o}\begin{bmatrix} I_0 & -\Phi_T \\ 0 & \mathrm{diag}(I_1,\ldots,I_N)\end{bmatrix} \tag{35}$$

for a certain bistable rational transfer function $\bar{T}_{2,o}$ and a certain stable nonrational transfer function Φ_T that may be chosen to have finite impulse response. By defining the inner transfer function

$$T_{2,i} = T_2\Lambda_s T_{2,o}^{-1} \tag{36}$$

we may write the inner–outer factorization of $T_2\Lambda_s$

$$T_2\Lambda_s = T_{2,i}T_{2,o}. \tag{37}$$

Substituting (37) into (32), we obtain

$$\min_{K_r\in H_2}\|T_1 + T_2\Lambda_s K_r\|_2 = \min_{K_r\in H_2}\|T_1 + T_{2,i}T_{2,o}K_r\|_2. \tag{38}$$

Now, suppose we split K_r to two parts, a certain fixed part $K_{r,f}$ and a variable part $K_{r,v}$

$$K_r = K_{r,f} + K_{r,v} \tag{39}$$

then we may reformulate (38) as

$$\min_{K_{r,v}\in H_2}\left\|\underbrace{(T_1 + T_{2,i}T_{2,o}K_{r,f})}_{=:F_1} + \underbrace{T_{2,i}}_{=:F_2}T_{2,o}K_{r,v}\right\|_2 \tag{40}$$

provided that (39) is satisfied. Define F_1 and F_2 as in (40) and define the fixed part of K_r as

$$K_{r,f} := -T_{2,o}^{-1}\left\{T_{2,i}^\sim T_1\right\}_+ \tag{41}$$

then we have that $F_1 = T_1 - T_{2,i}\{T_{2,i}^\sim T_1\}_+$ and $F_2^\sim F_1 = (T_{2,i}^\sim T_1 - \{T_{2,i}^\sim T_1\}_+) \in H_2^\perp$. By Lemma 7, it follows that

$$\begin{aligned}
&\min_{K_{r,v}\in H_2}\|(T_1 + T_{2,i}T_{2,o}K_{r,f}) + T_{2,i}T_{2,o}K_{r,v}\|_2^2 \\
&= \min_{K_{r,v}\in H_2}\left(\|T_1 + T_{2,i}T_{2,o}K_{r,f}\|_2^2 + \|T_{2,o}K_{r,v}\|_2^2\right) \\
&= \|T_1 + T_{2,i}T_{2,o}K_{r,f}\|_2^2 + \min_{K_{r,v}\in H_2}\|T_{2,o}K_{r,v}\|_2^2.
\end{aligned} \tag{42}$$

For the special case of (28) where the delay operator $\Lambda_\bullet$ is already a simple delay operator of the form (20), we may choose $\Lambda_r = I$ so that the condition (33) becomes $K = K_r$ and the problem (28) is solved by taking $K_{r,v} = 0$ in the equivalent problem (42). This results in the optimal controller given by $K_{\mathrm{opt}} = K_{r,\mathrm{opt}} = K_{r,f}$, where $K_{r,f}$ is given by (41).[4]

Now, let us return to the general problem. In this case we cannot take $K_{r,v} = 0$, because it will result in a non-causal controller K.The solution is explained in what follows. From (42), we notice that the original problem (28) is equivalent to the problem

$$\min_{K_{r,v}\in H_2}\|T_{2,o}K_{r,v}\|_2 \tag{43}$$

under (33) and (39) with $\Lambda_r = \mathrm{diag}(I_0, I_1, \tilde{\Lambda}_r)$ and $K_{r,f} = -T_{2,o}^{-1}\{T_{2,i}^\sim T_1\}_+$. Substituting (39) into (43), we obtain

$$\begin{aligned}
\|T_{2,o}K_{r,v}\|_2 &= \|-T_{2,o}K_{r,f} + T_{2,o}K_r\|_2 \\
&= \left\|\left\{T_{2,i}^\sim T_1\right\}_+ + T_{2,o}K_r\right\|_2.
\end{aligned} \tag{44}$$

We proceed with substituting the first condition (33) and the expression (35) for $T_{2,o}$ into (44)

$$\begin{aligned}
&\left\|\left\{T_{2,i}^\sim T_1\right\}_+ + T_{2,o}K_r\right\|_2 \\
&= \left\|\left\{T_{2,i}^\sim T_1\right\}_+ + \bar{T}_{2,o}\begin{bmatrix} I_0 & -\Phi_T \\ 0 & \mathrm{diag}(I_1, I_r)\end{bmatrix}\right. \\
&\quad \left.\times\begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & \tilde{\Lambda}_r\end{bmatrix} K\mathrm{diag}(I_0, I_1, \tilde{\Lambda}_r)K\right\|_2
\end{aligned}$$

where $I_r = \mathrm{diag}(I_2,\ldots,I_N)$, and by appropriately partitioning Φ_T to $[\Phi_{T1}\ \Phi_{T2}]$

$$\begin{aligned}
&= \left\|\left\{T_{2,i}^\sim T_1\right\}_+ + \bar{T}_{2,o}\begin{bmatrix} I_0 & 0 & 0 \\ 0 & I_1 & 0 \\ 0 & 0 & \tilde{\Lambda}_r\end{bmatrix}\right. \\
&\quad \left.\times\begin{bmatrix} I_0 & -\Phi_{T1} & -\Phi_{T2}\tilde{\Lambda}_r \\ 0 & I_1 & 0 \\ 0 & 0 & I_r\end{bmatrix} K\right\|_2.
\end{aligned} \tag{45}$$

Hence, we obtain the equivalent problem (29) by setting

$$\tilde{T}_1 = \left\{T_{2,i}^\sim T_1\right\}_+, \tilde{T}_2 = \bar{T}_{2,o}, \Psi = \begin{bmatrix} I_0 & -\Phi_{T1} & -\Phi_{T2}\tilde{\Lambda}_r \\ 0 & I_1 & 0 \\ 0 & 0 & I_r\end{bmatrix} \tag{46}$$

in (42) and (45). Notice that $\tilde{T}_1$ is stable, $\tilde{T}_2$ is bistable, and rational, and Ψ is bistable and has a realization with finite impulse response. What remains is to show that $\tilde{T}_1$ is rational. First note that according to Lemma 6, Φ_T is given by $\Phi_T = R - e^{-sh_1}\Pi_{11}^{-1}\Pi_{12}$, for certain

[4]This corollary is formally stated later in Corollary 9.

rational transfer matrices R, Π_{12}, and Π_{11}. The fact that $\tilde{T}_1$ is rational may be observed from the following expressions:

$$\begin{aligned}\tilde{T}_1 &= \{T_{2i}^{\sim} T_1\}_+ = \left\{(T_2 \Lambda_s T_{2o}^{-1})^{\sim} T_1\right\}_+ \\ &= \left\{\left(T_2 \begin{bmatrix} I_0 & 0 \\ 0 & e^{-sh_1} I_{1\sim N}\end{bmatrix}\begin{bmatrix} I_0 & \Phi_T \\ 0 & I_{1\sim N}\end{bmatrix}\bar{T}_{2o}^{-1}\right)^{\sim} T_1\right\}_+ \\ &= \left\{\left(T_2 \begin{bmatrix} I_0 & R - e^{-sh_1}\Pi_{11}^{-1}\Pi_{12} \\ 0 & e^{-sh_1} I_{1\sim N}\end{bmatrix}\bar{T}_{2o}^{-1}\right)^{\sim} T_1\right\}_+ \\ &= \left\{\bar{T}_{2o}^{-\sim}\begin{bmatrix} I_0 & 0 \\ R^{\sim} - e^{sh_1}\Pi_{12}^{\sim}\Pi_{11}^{-\sim} & e^{sh_1} I_{1\sim N}\end{bmatrix} T_2^{\sim} T_1\right\}_+ .\end{aligned}$$

where $I_{1\sim N} = \operatorname{diag}(I_1, \ldots, I_N)$. The last expression shows that the elements of $T_{2i}^{\sim} T_1$ are either a rational transfer function or a rational transfer function multiplied by e^{sh_1}. Both have a rational stable part. ■

The proof of Theorem 8 results in the solution of a special case of the one-sided regulator problem, where the delay operator only have at most two distinct delays.

Corollary 9: Consider a special case of the regulator problem (28) problem in which the delay operator $\Lambda_\bullet = \Lambda_s$ is a simple delay operator of the form (20) containing at most two distinct delays. Let $T_{2,o}$ be defined such that $\Lambda_s^{\sim} T_2^{\sim} T_2 \Lambda_s = T_{2,o}^{\sim} T_{2,o}$ with $T_{2,o}$ bistable. Define $T_{2,i} = T_2 \Lambda_s T_{2,o}^{-1}$, then the H_2 optimal controller that minimizes $\|T_1 + T_2 \Lambda_s K\|_2$ is given by

$$K_{\text{opt}} = -T_{2,o}^{-1}\left\{T_{2,i}^{\sim} T_1\right\}_+ . \tag{47}$$

VII. Solution to the Two-Sided Regulator Problem

Now, let us return to solving the two-sided regulator problem of minimizing $\|T_1 + T_2 \Lambda_u K \Lambda_y T_3\|_2$ over $K \in H_2$ where conditions A4)–A6) of Section V are satisfied. Suppose that the delay operators Λ_u and Λ_y have N and M nonzero distinct delays, respectively. The solution consists of two stages, the first of which deals with $T_2\Lambda_u$ and the second deals with $\Lambda_y T_3$. The following provides a sketch of the solution.

We begin by absorbing $\Lambda_y T_3$ into the controller, transforming the problem to $\min_{K_u \in H_2} \|T_1 + T_2 \Lambda_u K_u\|_2$ under the condition $K_u = K \Lambda_y T_3$, $K \in H_2$. This is a one-sided problem, to which we may apply Theorem 8. By applying this theorem N times, the problem will be reduced to $\min_{K_u \in H_2} \|\tilde{T}_1 + \tilde{T}_2 \Psi K_u\|_2$ for certain rational transfer functions $\tilde{T}_1 \in H_2, \tilde{T}_2 \in H_\infty$, and a certain nonrational transfer function $\Psi \in H_\infty$. Moreover, $\tilde{T}_2$ and Ψ are in fact bistable. By substituting the condition $K_u = K\Lambda_y T_3$, $K \in H_2$ back into the regulator problem and absorb the bistable factor $(\tilde{T}_2\Psi)$ into the controller, we obtain the problem $\min_{K_y \in H_2} \|\tilde{T}_1 + K_y \Lambda_y T_3\|_2 = \min_{K_y^T \in H_2} \|\tilde{T}_1^T + T_3^T \Lambda_y K_y^T\|_2$ where $K_y = \tilde{T}_2 \Psi K$. The latter problem is a one-sided regulator problem and may be solved by applying Theorem 8 $(M-1)$ times followed by application of Corollary 9. The previous discussion is formally stated in the following theorem.

Theorem 10: Consider the problem of minimizing $\|T_1 + T_2 \Lambda_u K \Lambda_y T_3\|_2 K \in H_2$ where conditions A4)–A6) of Section V are satisfied. Suppose that Λ_u and Λ_y have N and M nonzero distinct delays, respectively. Then, the optimal controller may be obtained using the following algorithm.

1) Absorb $\Lambda_y T_3$ into the controller, transforming the problem to

$$\min_{K_u \in H_2} \|T_1 + T_2 \Lambda_u K_u\|_2 \tag{48}$$

under the condition

$$K_u = K \Lambda_y T_3 \quad K \in H_2. \tag{49}$$

2) Apply Theorem 8 N times, so that the problem (48) reduces to the regulator problem

$$\min_{K_u \in H_2} \|\tilde{T}_1 + \tilde{T}_2 \Psi K_u\|_2 \tag{50}$$

for certain rational transfer functions $\tilde{T}_1 \in H_2, \tilde{T}_2 \in H_\infty$, and a certain non-rational transfer function $\Psi \in H_\infty$ with $\tilde{T}_2^{-1}$, $\Psi^{-1} \in H_\infty$.

3) Substitute the condition (49) back into the regulator problem (50) and absorb the bistable factor $(\tilde{T}_2\Psi)$ into the controller, which results in the problem

$$\min_{K_y \in H_2} \|\tilde{T}_1 + K_y \Lambda_y T_3\|_2 = \min_{K_y^T \in H_2} \left\|\tilde{T}_1^T + T_3^T \Lambda_y K_y^T\right\|_2 \tag{51}$$

where there is a stable bijection between K and K_y governed by the equation

$$K_y = \tilde{T}_2 \Psi K. \tag{52}$$

4) Apply Theorem 8 $(M-1)$ times to the right-hand side of (51) to obtain a regulator problem with simple delay operator containing at most two distinct delays. Solve the latter problem using Corollary 9.

VIII. Numerical Example

Consider the two sided regulator setup [Fig. 1(b)], where $T_1 = \operatorname{diag}(\beta/s+1, \gamma/s+1)$, $T_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1\end{bmatrix}$, $T_3 = \begin{bmatrix} 1 & 0 \\ 1 & 1\end{bmatrix}$, $\Lambda_u(s) = \operatorname{diag}(1, e^{-sh_u})$, and $\Lambda_y(s) = \operatorname{diag}(1, e^{-sh_y})$. We begin applying Theorem 10 by absorbing $\Lambda_y T_3$ into K to obtain (48). Next, we apply Theorem 8 to (48). First we need to compute the bistable spectral factor $T_{2,o}$ such that $\Lambda_u^{\sim} T_2^{\sim} T_2 \Lambda_u = T_{2,o}^{\sim} T_{2,o}$. We proceed by using the formulas of Lemma 6. First, we compute $\Pi = T_2^{\sim} T_2 = \begin{bmatrix} 1 & 1 \\ 1 & 2\end{bmatrix}$. We choose $\Phi_u = -e^{-sh_u}\Pi_{11}^{-1}\Pi_{12} = -e^{-sh_u}$ and $R = 0$. Observe that Φ_u and R are stable and rational, respectively, and satisfy (22). Using (23), we compute $S = I = \operatorname{diag}(1,1)$. Since the bistable $\bar{T}_{2,o} = I = \operatorname{diag}(1,1)$ trivially satisfy $S = \bar{T}_{2,o}^{\sim}\bar{T}_{2,o}$, it follows from (24) that a bistable spectral factor $T_{2,o}$ is given by

$$T_{2,o} = \bar{T}_{2,o}\begin{bmatrix} 1 & -\Phi_u \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & e^{-sh_u} \\ 0 & 1\end{bmatrix}.$$

This results in the inner–outer factorization $T_2\Lambda_u = T_{2,o}T_{2,i}$ with $T_{2,i} = T_2 \Lambda_u T_{2,o}^{-1} = \operatorname{diag}(1, e^{-sh_u})$ Now, we proceed by computing $\tilde{T}_1(s), \tilde{T}_2(s)$, and $\Psi(s)$ using (46)

$$\begin{aligned}\tilde{T}_1 &= \left\{T_{2,i}^{\sim} T_1\right\}_+ = \left\{\begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{sh_u}}{s+1}\end{bmatrix}\right\}_+ = \begin{bmatrix} \frac{\beta}{s+1} & 0 \\ 0 & \frac{\gamma e^{-h_u}}{s+1}\end{bmatrix} \\ \tilde{T}_2 &= \bar{T}_{2,o} = \begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}, \Psi(s) = \begin{bmatrix} 1 & -\Phi_u \\ 0 & 1\end{bmatrix} = \begin{bmatrix} 1 & e^{-sh_u} \\ 0 & 1\end{bmatrix}.\end{aligned}$$

Hence, (48) reduces to (50). Substituting (49) back into (50) and taking its transpose, we obtain the problem $\min_{K_y^T \in H_2} \|\tilde{T}_1^T + T_3^T \Lambda_y K_y^T\|_2$ with $K_y = \tilde{T}_2 \Psi K$. To solve this problem, first we need to compute the bistable spectral factor $T_{3,o}$ satisfying $\Lambda_y^{\sim}(s)(T_3^T)^{\sim} T_3^T \Lambda_y =$

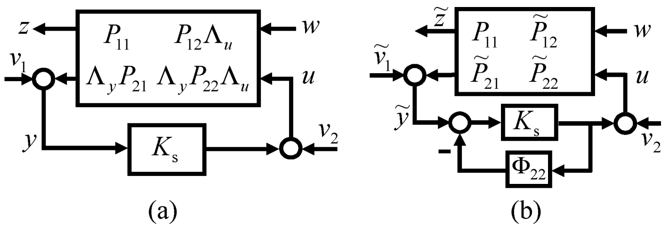

Fig. 2. Proof of Lemma 1.

$T_{3,o}^{\sim}T_{3,o}$. Since $T_3^T = T_2$ and the delay operator Λ_y is structurally identical to Λ_u, it follows that the inner factor $T_{3,i}$ satisfying $T_3^T\Lambda_y = T_{3,o}T_{3,i}$ is given by $T_{3,i} = \operatorname{diag}(1, e^{-sh_y})$. We have that $\{T_{3,i}^{\sim}\tilde{T}_1\}_+ = \{\operatorname{diag}(\beta/s+1, \gamma e^{sh_y}e^{-h_u}/s+1)\}_+ = \operatorname{diag}(\beta/s+1, \gamma e^{sh_y}e^{-h_u}/s+1)$ and the optimal controller $K_{y,\mathrm{opt}}^T$ of the single-sided problem (51) may obtained using (47) of Corollary 9: $K_{y,\mathrm{opt}}^T = -T_{3,o}^{-1}\{T_{3,i}^{\sim}(s)\tilde{T}_1\}_+$. The optimal controller of the original two sided regulator problem may be recovered using $K_{\mathrm{opt}} = \Psi^{-1}\tilde{T}_2^{-1}K_{y,\mathrm{opt}}$, resulting in

$$K_{\mathrm{opt}} = \begin{bmatrix} -\frac{\beta}{s+1} - \frac{\gamma e^{-s(h_u+h_y)}e^{-(h_u+h_y)}}{s+1} & \frac{\gamma e^{-sh_u}e^{-(h_u+h_y)}}{s+1} \\ \frac{\gamma e^{-sh_y}e^{-(h_u+h_y)}}{s+1} & -\frac{\gamma e^{-(h_u+h_y)}}{s+1} \end{bmatrix}.$$

IX. Concluding Remarks

This note proposes a solution of the standard H_2-optimal control problem of systems with multiple I/O delays using frequency domain methods. The approach is to convert the standard problem to the two sided regulator problem, which is solved using frequency domain techniques. However, in its present form, the method cannot handle unstable plants. Further research is needed to find a remedy.

Appendix

Proof of Lemma 1

First, define the following transfer matrices:

$$\Phi_{12} = \tilde{P}_{12} - P_{12}\Lambda_u, \Phi_{21} = \tilde{P}_{21} - \Lambda_y P_{21}. \quad (54)$$

It may be verified that Φ_{12}, Φ_{21}, and Φ_{22} are stable, and in particular Φ_{22} is strictly proper. It may be shown that the setup in Fig. 2(a), through block diagram manipulations similar to the technique used in [14] and [27] and the use of the (54) and (9), may be transformed to the setup in Fig. 2(b) with

$$\tilde{v}_1 = v_1 - \Phi_{22}v_2 - \Phi_{21}w, \tilde{z} = z + \Phi_{12}u, \tilde{y} = y + \Phi_{22}(u - v_2). \quad (55)$$

The (55) suggest that the output signals (y, u, z) of Fig. 2(a) may be expressed in terms of the output signals $(\tilde{y}, u, \tilde{z})$ and the input signals $(\tilde{v}_1, v_2, w)$ of Fig. 2(d)

$$\begin{bmatrix} y \\ u \\ z \end{bmatrix} = \begin{bmatrix} I & -\Phi_{22} & 0 \\ 0 & I & 0 \\ 0 & -\Phi_{12} & I \end{bmatrix}\begin{bmatrix} \tilde{y} \\ u \\ \tilde{z} \end{bmatrix} + \begin{bmatrix} 0 & \Phi_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}. \quad (56)$$

Notice that all transfer matrices in (56) are stable. Moreover, (55) also suggests a bistable mapping between the input signals of the two stabilization setups

$$\begin{bmatrix} v_1 \\ v_2 \\ w \end{bmatrix} = \begin{bmatrix} I & \Phi_{22} & \Phi_{21} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}. \quad (57)$$

Next, let us denote T as the transfer matrix from (v_1, v_2, w) to (y, u, z) in Fig. 2(a), i.e.

$$\mathrm{col}[y, u, z] = T\ \mathrm{col}[v_1, v_2, w]. \quad (58)$$

Suppose K_s stabilizes $\hat{P}$, then T is stable. Substituting (56) and (57) into (58), we obtain

$$\begin{bmatrix} \tilde{y} \\ u \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} I & \Phi_{22} & 0 \\ 0 & I & 0 \\ 0 & \Phi_{12} & I \end{bmatrix} \times \left(T\begin{bmatrix} I & \Phi_{22} & \Phi_{21} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} - \begin{bmatrix} 0 & \Phi_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\begin{bmatrix} \tilde{v}_1 \\ v_2 \\ w \end{bmatrix}$$

showing the transfer function from $(\tilde{v}_1, v_2, w)$ to $(\tilde{y}, u, \tilde{z})$ in Fig. 2(d) is stable. It follows that $\tilde{K}_s$ stabilizes $\tilde{P}$. The converse may be proved in a similar manner. The prove of the proper bijection between K_s and $\tilde{K}_s$ may be obtained using similar arguments as in the prove of statement 1) of Theorem 3.

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Robust Kalman Filter for Descriptor Systems

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***Abstract*—This note is concerned with the problem of state estimation for descriptor systems subject to uncertainties. A Kalman type recursive algorithm is derived. Numerical examples are included to demonstrate the performance of the proposed robust filter.**

***Index Terms*—Descriptor systems, discrete-time filters, robust estimation, robust filtering, robustness.**

I. Introduction

Analysis and design of descriptor systems (also known as singular systems or implicit systems) have received great attention in the literature. This is because systems in descriptor formulation frequently arises naturally in the process of modeling of economical systems [9], image modeling [6], and robotics [10]. Besides, the descriptor formulation contains the usual state–space system as a special case and can describe some dynamical systems for which state–space description does not exist [18].

For discrete-time descriptor systems, there has been an intensive study on the Kalman filtering problem (see, e.g., [1]–[4], [8], [11], [12], [17], and [19]). Different formulations have been proposed in order to deal with this problem. In [1], the state estimation problem was solved by transforming a descriptor system in an extended nondescriptor system. In a direct descriptor context, one can consider the least square method [2], [7], [8]) the maximum likelihood criterion [11], [12], the minimum-variance estimation [5], and ARMA innovation model [4], [19]. This variety reflects the well known fact that, for usual state space systems under Gaussian assumption, the Gauss–Markov estimate is identical to the minimum-variance estimate, which, in turns, is identical to the maximum-likelihood and identical to the deterministic weighted least-square estimate with an appropriate quadratic functional.

The estimation algorithms for descriptor systems considered so far in the literature assume that the model of the plant is known exactly. However, models in engineering systems are only approximate. For usual state space systems, generalizations of the classical Kalman filter to encompass systems with norm bounded uncertainties have been the focus of [13]–[16], and the references therein. For the case when uncertain noise covariances are considered on descriptor system filtering, a guaranteed estimation performance filter is deduced in [17]. To the best of authors knowledge, robust descriptor filters to deal with uncertainties in the system matrices E_{i+1}, F_i, and H_i (see the model (1), Section II) have not been considered in the literature yet. To solve this problem, this note extends to descriptor systems a robust procedure for usual state space systems developed by [13]. This robust framework was chosen since the resulting state–space robust filter has presented some advantages if compared with other usual robust filters including the guaranteed cost filter (see the comparisons among the filters in [13]). This note develops robust Kalman type recursion for the most general case, when the descriptor system is rectangular. It is shown that the proposed filter collapses to the nominal descriptor Kalman filter when the system is not subject to uncertainties. When it is reduced to usual state–space systems, it provides alternative robust recursions to that presented by [13] (see more details in Remark IV.2).

This note is organized as follows. In Section II, it is stated the problem of robust estimation as a problem of optimum estimation for systems subject to uncertainties. In Section III, it is revisited the descriptor Kalman filter for systems without uncertainties. In Section IV, it is proposed a robust Kalman filter for uncertain descriptor systems. Necessary and sufficient conditions for convergence of the robust Riccati equation and for the stability of the steady-state robust filter are given for the system with constant parameters. In Section V, simulation results are presented to demonstrate the performance of the descriptor robust filter developed in this note. The notation is standard: $\Re$ is the set of real numbers, $\Re^n$ is the set of n-dimensional vectors whose elements are in $\Re$, $\Re^{m\times n}$ is the set of $m \times n$ real matrices, A^T and $A^\dagger$ are the transpose and the pseudoinverse of the matrix A, respectively, $P > 0 (P \geq 0)$ denotes a positive–definite (semidefinite) matrix, $\|x\|$ is the Eucledian norm of x, $\|x\|_P$ is the weighted norm of x defined by $\|x\|_P = (x^T P x)^{1/2}$.

II. Problem Statement

Consider the uncertain discrete-time linear stochastic descriptor system

$$\begin{aligned}(E_{i+1} + \delta E_{i+1})x_{i+1} &= (F_i + \delta F_i)x_i + w_i, \qquad i = 0, 1, \ldots\\ z_i &= (H_i + \delta H_i)x_i + v_i \end{aligned} \tag{1}$$

where $x_i \in \Re^n$ is the descriptor variable; $z_i \in \Re^p$ is the measured output; $w_i \in \Re^m$, and $v_i \in \Re^p$ are the process and measurement noises; $E_{i+1} \in \Re^{m\times n}$, $F_i \in \Re^{m\times n}$, and $H_i \in \Re^{p\times n}$ are the known

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