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Sports tournaments, home–away assignments, and the break minimization problem

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Abstract

We consider the break minimization problem for fixing home–away assignments in round-robin sports tournaments. First, we show that, for an opponent schedule with n teams and $n-1$ rounds, there always exists a home–away assignment with at most $\frac{1}{4}n(n-2)$ breaks. Secondly, for infinitely many n, we construct opponent schedules for which at least $\frac{1}{6}n(n-1)$ breaks are necessary. Finally, we prove that break minimization for n teams and a partial opponent schedule with r rounds is an NP-hard problem for $r \geq 3$. This is in strong contrast to the case of $r = 2$ rounds, which can be scheduled (in polynomial time) without any breaks.

Keywords: Sports scheduling; Time tabling; Computational complexity

1. Introduction

Scheduling sports competitions is not an easy task. Over the last 30 years, the area of sports scheduling has generated a wealth of challenging combinatorial and algorithmical problems for the operational researcher and for the computer scientist. Concrete examples are the schedules of the Australian basketball league (De Werra, Jacot-Descombes and Masson [7]), the schedules of the Dutch football league (Schreuder [13]), and the schedules of the American baseball league (Russell and Leung [12]).

In this paper we will focus on scheduling round-robin sports tournaments with n teams that play $n-1$ rounds of matches against all other teams. Throughout the paper, we assume that n is an even integer. De Werra [3–6] introduced some fundamental mathematical models for round-robin tournaments that are based on edge-colorings of the complete graph. Designing a round-robin tournament is often done in two phases (we remark that there are other approaches that reverse the order of these two phases; see, for instance, Russell and Leung [12]):

- The first phase fixes the $\frac{1}{2}n$ matches for each of the $n-1$ rounds; the resulting schedule is called an *opponent schedule* S.
- The second phase decides, for every match in every round of the opponent schedule S, which team plays at home and which team plays away. The result is called a *home–away assignment* for the opponent schedule S.

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n	4	6	8	10	12	14	16	18	20	22	24	26
LB-EJR	2	4	8	12	18	26	32	44	54	64	74	90
LB-new	2	–	–	–	–	–	40	–	–	–	–	–
UB-new	2	4	12	16	30	36	56	64	90	100	132	144

Fig. 1. Some lower bounds (LB) and upper bounds (UB) on $b(n)$ for $n \leq 26$.

A home–away assignment induces for every team a so-called *home–away pattern* (HAP), that is, a sequence of $n-1$ pluses and minuses: the rth ($1 \leq r \leq n-1$) element in the HAP equals $+$, if the team plays at home in the rth round, and it equals $-$ if the team plays away in the rth round. For instance, the HAP "$++-++$" states that the corresponding team plays the third round away and the other rounds at home. A *break* occurs if two consecutive matches for a team are both played at home or both played away. In general, breaks are considered undesirable events. Hence, one of the main objectives in the second planning phase is to reach an assignment with a small number of breaks. By $B_{\min}(S)$ we denote the minimum total number of breaks over all possible home–away assignments for an opponent schedule S.

Trick [14] designed an algorithm that succeeds in computing $B_{\min}(S)$ for up to $n = 22$ teams. The combinatorics of the parameter $B_{\min}(S)$ is quite unclear, and Elf et al. [8] even conjecture that computing $B_{\min}(S)$ is NP-hard. It is easy to see that opponent schedules cannot have a home-away assignment with fewer than $n-2$ breaks (see also Lemma 1 in Section 2). For every even n, one can in fact find opponent schedules S for n teams with $B_{\min}(S) = n-2$. An elegant construction based on so-called canonical 1-factorizations is given by De Werra [4]. Quite recently, Miyashiro and Matsui [10] have shown that deciding whether a given opponent schedule can be implemented with exactly $n-2$ breaks can be done in polynomial time; this fact had already been conjectured in [8].

1.1. Results of this paper

First, we will analyze the worst-case behavior of the problem parameter $B_{\min}$. How much damage can be done by short-sighted planning? How much can go wrong, if the first planning phase is done without taking the goals of the second planning phase into account? In order to approach these questions, we define $b(n)$ as the maximum value of $B_{\min}(S)$, where S runs over all possible opponent schedules with n teams. Elf et al. [8] detected a number of opponent schedules for $n \leq 26$ with many breaks. The resulting lower bounds on $b(n)$ are summarized in line LB-EJR of Fig. 1. In Section 2 we construct, for every $n = 4^k$, an opponent schedule S_n^* with $B_{\min}(S_n^*) \geq \frac{1}{6}n(n-1)$. The resulting new lower bounds on $b(n)$ for small values of n are summarized in line LB-new of Fig. 1. Also, in Section 3, we show that every opponent schedule S satisfies $B_{\min}(S) \leq \frac{1}{4}n(n-2)$ if n is of the form $4k$, and satisfies $B_{\min}(S) \leq \frac{1}{4}(n-2)^2$ if n is of the form $4k+2$. These upper bounds on $b(n)$ are summarized in line UB-new of Fig. 1.

The lower bound from Section 2 and the upper bound from Section 3 are quite close to each other. We conjecture that the lower bound is the true threshold, and that any opponent schedule S for n teams satisfies $B_{\min}(S) \leq \frac{1}{6}n(n-1)$.

In the second half of the paper, we then analyze break minimization for *partial* opponent schedules: a partial opponent schedule for n teams does not go over the full $n-1$ rounds, but only covers some smaller number $r < n-1$ of rounds; any pair of teams meets in at most one of these r rounds. Partial opponent schedules with $r = 2$ rounds behave quite nicely: they can always be scheduled without breaks, and a corresponding home–away assignment can be found in polynomial time; see Lemma 5 in Section 3. In strong contrast to this positive result, we will show in Section 4 that break minimization in partial opponent schedules with $r = 3$ rounds is an NP-hard problem. This hardness result carries over to all fixed numbers $r \geq 4$ of rounds.

2. Lower bounds

In this section we construct opponent schedules for which $B_{\min}$ is large. The combinatorics of opponent schedules is non-trivial. Even extending a partial opponent schedule with $n-k-1$ rounds to a full opponent schedule with $n-1$ rounds is not easy to do: to visualize this, we can construct a graph G on n vertices (representing the teams) and we connect two vertices, in case the corresponding teams have *not* played against each other so far in the partial opponent schedule. Then, scheduling the next round is equivalent to constructing a perfect matching in the k-regular graph G;

	$r+1$	$r+2$	$r+3$
a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

	$r+1$	$r+2$	$r+3$
a	+	+	−
b	−	+	−
c	+	−	+
d	−	−	+

Fig. 2. An opponent schedule for the teams a, b, c, d, and one possible home–away assignment with two breaks.

this is not always possible (see [11,2]). Also, scheduling all the remaining k rounds corresponds to constructing a proper k-edge coloring of G, which is an NP-hard problem for $k \geq 3$.

Lemma 1 (*Folklore*). *Each opponent schedule S has $B_{\min}(S) \geq n - 2$.*

Proof. First, observe that no two teams can have identical HAPs (otherwise, they could never play against each other). Consequently, there is at most one team with a breakless HAP that plays the first round at home $(+ - + - \cdots +)$, and there is at most one team with a breakless HAP that plays the first round away $(- + - + \cdots -)$. The HAPs of all remaining $n - 2$ teams contain at least one break. □

Our construction of opponent schedules with large $B_{\min}$ value starts with the observation that opponent schedules for a small number of teams have relatively many breaks: the total number of transitions equals $n(n - 2)$, while the number of breaks is at least $n - 2$. Therefore, a $\frac{1}{n}$-fraction of all transitions must be breaks! For $n = 4$, at least one quarter of all transitions are breaks; see Fig. 2 for an illustration. Now the main idea is to split the opponent schedule for n teams into many complete schedules for four teams.

Theorem 2. *For $n = 4^k$ teams with $k \geq 1$, there exists an opponent schedule S_n^* with $B_{\min}(S_n^*) \geq \frac{1}{6}n(n - 1)$.*

Proof. The construction is based on a block design [9]. A block will be a group of 4 teams, which will play a 4-team round-robin of 3 rounds. This implies that there are 2 breaks within a block. As there are in total $\frac{1}{2}n(n - 1)$ matches, and the 4-team round-robin contains 6 matches, we have to construct $\frac{1}{12}n(n-1)$ blocks; hence we will have $\frac{1}{6}n(n-1)$ breaks.

The blocks constitute a so-called (v, b, r, k, λ)-design, where the 5 parameters have the following meaning:

- There are $v = n$ teams.
- There are $b = \frac{1}{12}n(n - 1)$ blocks.
- Each teams appears in $r = \frac{1}{3}(n - 1)$ blocks. (In each block, a teams plays 3 matches.)
- Each block contains $k = 4$ teams.
- Each pair of teams occurs in exactly $\lambda = 1$ block. (Teams meet only once.)

Apart from these constraints, we need to place blocks in groups, consisting of $\frac{1}{4}n$ blocks, containing all teams exactly once.

There exists an elegant construction of the block-design above in the case $n = 4^k$, expressed in terms of lines in the k-dimensional space over the finite field with 4 elements. Let $\mathbb{F}$ denote this field with 4 elements. Every team in our construction corresponds to a point in $\mathbb{F}^k$; hence, there are $n = 4^k$ teams. A line corresponds to a block. These lines in $\mathbb{F}^k$ satisfy the following four useful properties:

- Every line contains exactly 4 points.
- For any two points in $\mathbb{F}^k$, there is a unique line that contains both points.
- The total number of lines is $\frac{1}{12}n(n - 1)$.
- The lines can be partitioned into $\frac{1}{3}(n - 1)$ families, such that each family contains $\frac{1}{4}n$ parallel[1] lines.

[1] A line ℓ' is parallel to ℓ if ℓ' can be obtained from ℓ by a translation.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T1	2	3	4	5	9	13	6	11	16	8	10	15	7	12	14
T2	1	4	3	6	10	14	5	12	15	7	9	16	8	11	13
T3	4	1	2	7	11	15	8	9	14	6	12	13	5	10	16
T4	3	2	1	8	12	16	7	10	13	5	11	14	6	9	15
T5	6	7	8	1	13	9	2	15	12	4	14	11	3	16	10
T6	5	8	7	2	14	10	1	16	11	3	13	12	4	15	9
T7	8	5	6	3	15	11	4	13	10	2	16	9	1	14	12
T8	7	6	5	4	16	12	3	14	9	1	15	10	2	13	11
T9	10	11	12	13	1	5	14	3	8	16	2	7	15	4	6
T10	9	12	11	14	2	6	13	4	7	15	1	8	16	3	5
T11	12	9	10	15	3	7	16	1	6	14	4	5	13	2	8
T12	11	10	9	16	4	8	15	2	5	13	3	6	14	1	7
T13	14	15	16	9	5	1	10	7	4	12	6	3	11	8	2
T14	13	16	15	10	6	2	9	8	3	11	5	4	12	7	1
T15	16	13	14	11	7	3	12	5	2	10	8	1	9	6	4
T16	15	14	13	12	8	4	11	6	1	9	7	2	10	5	3

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
T1	+	−	+	−	+	−	+	−	+	−	+	−	+	−	+
T2	−	+	+	−	+	−	+	−	−	+	−	−	+	−	+
T3	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
T4	+	−	−	+	−	+	−	+	+	−	+	+	−	+	−
T5	−	+	−	+	−	+	−	+	−	+	−	−	+	−	+
T6	+	−	−	+	−	+	−	+	+	−	+	−	+	−	+
T7	+	−	+	−	+	−	+	−	+	−	+	+	−	+	−
T8	−	+	+	−	+	−	+	−	−	+	−	+	−	+	−
T9	−	+	−	+	−	−	+	−	+	−	+	−	+	−	−
T10	+	−	−	+	−	−	+	−	−	+	−	−	+	−	−
T11	+	−	+	−	+	+	−	+	−	+	−	+	−	+	+
T12	−	+	+	−	+	+	−	+	+	−	+	+	−	+	+
T13	+	−	+	−	+	+	−	+	−	+	−	−	+	−	−
T14	−	+	+	−	+	+	−	+	+	−	+	−	+	−	−
T15	−	+	−	+	−	−	+	−	+	−	+	+	−	+	+
T16	+	−	−	+	−	−	+	−	−	+	−	+	−	+	+

Fig. 3. An opponent schedule S^*_{16} for 16 teams with at least 40 breaks, and a corresponding home–away assignment with exactly 40 breaks.

The first two properties are straightforward to verify. For the third property, count the total number of lines. For any choice of two points, there is a unique line that contains both points. There are $\frac{1}{2}n(n-1)$ possibilities for choosing two points, and every line is specified by six of these possibilities. Hence, there are $\frac{1}{12}n(n-1)$ lines. For the fourth property, choose any line ℓ and any point P not on this line. There is a unique line through P that is parallel to ℓ. As in total there are n points and, as there are four points on each parallel line, we have $\frac{1}{4}n$ lines parallel to ℓ.

We structure the complete opponent schedule into $\frac{1}{3}(n-1)$ partial schedules. Every partial schedule consists of three consecutive rounds $3k-2$, $3k-1$, and $3k$, where $k = 1, \ldots, \frac{1}{3}(n-1)$. Every partial schedule corresponds to one family of $\frac{1}{4}n$ parallel lines in $\mathbb{F}^k$. For every parallel line $\{a, b, c, d\}$ in this family, we put the six matches between the four teams a, b, c, d into the three rounds of the partial schedule; see Fig. 2 for an illustration. This completes the construction of opponent schedule S^*_n.

Now let us consider an arbitrary home–away assignment for schedule S^*_n. Every line $\{a, b, c, d\}$ generates at least two breaks within the six matches played by the four teams a, b, c, d. Since altogether there are $\frac{1}{12}n(n-1)$ lines, this yields a total number of at least $\frac{1}{6}n(n-1)$ breaks for S^*_n. □

We now perform the above construction for 16 teams, that is, for $k = 2$ and $n = 16$. There are altogether 20 lines, and every line has four lines parallel to it (including the line itself). This yields five partial schedules, each containing three rounds. Fig. 3 gives one possible opponent schedule for this construction. Every corresponding home–away assignment has at least $\frac{1}{6}n(n-1) = 40$ breaks. Remarkably, the opponent schedule given in Fig. 3 can be scheduled with only $\frac{1}{2}n = 8$ breaks between the rounds $3k-1$ and $3k$ for $k = 1, 2, 3, 4, 5$; see Fig. 3.

Remark 3. The technique is based on constructing partial opponent schedules with $r = 3$ rounds. We can enforce that such partial schedules have at least $\frac{1}{2}n$ breaks. According to Rosa and Wallis [11], partial schedules with three rounds can always be extended to a complete opponent schedule, if $n \geq 8$. By using the partial schedules constructed above, we can construct, for all n of the form $4k$, an opponent schedule with at least $\frac{1}{2}n$ breaks in (say) the first three rounds. If n is of the form $4k + 2$, then we can apply the same construction for the first $n' = n - 6$ teams, and add three rounds for six teams with at least 2 breaks. This yields at least $\frac{1}{2}(n - 2)$ breaks for all $n \geq 6$.

3. Upper bounds

In this section, we describe a simple greedy approach for computing home–away assignments. The greedy approach works locally. It considers certain groups of consecutive rounds, and analyzes the local break structure within these groups. The following lemma gives a first crude estimate:

Lemma 4. *Each opponent schedule S for n teams satisfies*

$$B_{\min}(S) \leq \begin{cases} \frac{1}{2}n(n-2) & \text{if } n \text{ is of the form } 4k \\ \frac{1}{2}(n-2)^2 & \text{if } n \text{ is of the form } 4k+2. \end{cases}$$

Proof. Start with an arbitrary home–away assignment for S. Then perform the following step for $r = 1, 2, \ldots, n-2$:

> If the number of breaks between the rounds r and $r + 1$ is more than $\frac{1}{2}n$, then flip the home–away assignment for round $r + 1$.

Since the flipping translates every break into a non-break and every non-break into a break, the resulting number of breaks between rounds r and $r + 1$ is at most $\frac{1}{2}n$. In the case that $\frac{1}{2}n$ is odd (that is, n is of the form $4k + 2$), the number of breaks reduces to $\frac{1}{2}(n - 2)$, since the number of breaks between two consecutive rounds is always even. Multiplying by $n - 2$ yields the lemma. □

In the previous lemma, we used the fact that we can always make sure that there are not more than $\frac{1}{2}n$ breaks between any two consecutive rounds. The following lemma shows that we can do substantially better.

Lemma 5 (*Cf.* [6]). *For any partial opponent schedule with n teams and only two rounds r and $r + 1$, there exists a home–away assignment A that has no breaks between these two rounds. Such a home–away assignment A can be computed in polynomial time.*

Proof. We construct a 2-regular graph G on $2n$ vertices in the following way. There are n vertices that correspond to the teams in round r, and n vertices that correspond to the teams in round $r + 1$. Two vertices are connected by an edge, if they correspond to the same team in rounds r and $r + 1$, or if they both correspond to teams in the same round that are opponents in this round. Since all points in G have degree 2, the connected components of G are cycles. Since each cycle has an equal number of teams in round r and in round $r + 1$, the cycle is an even cycle and hence 2-colorable. We color the vertices with two colors $+$ and $-$, and we consider this coloring as home–away assignment for the rounds r and $r + 1$. Indeed, each team plays once at home and once away (as the corresponding vertices are connected). And indeed, in every match in rounds r and $r+1$, one of the opponents plays at home and the other plays away (as the corresponding vertices are connected). □

By combining the techniques of Lemmas 4 and 5, we obtain the following theorem.

Theorem 6. *Each opponent schedule S for n teams satisfies*

$$B_{\min}(S) \leq \begin{cases} \frac{1}{4}n(n-2) & \text{if } n \text{ is of the form } 4k \\ \frac{1}{4}(n-2)^2 & \text{if } n \text{ is of the form } 4k+2. \end{cases}$$

Furthermore, a corresponding home–away assignment can be computed in polynomial time.

Proof. Consider an arbitrary opponent schedule S. We start with using Lemma 5 and obtain an initial home–away assignment A' without breaks between any odd-numbered round and the following even-numbered round, that is, without breaks between rounds r and $r+1$ for $r = 1, 3, 5, \dots, n-3$. Next, we improve the breaks between the even-numbered rounds and the following odd-numbered round in assignment A': for $r = 2, 4, 6, \dots, n-2$, we apply the technique from Lemma 4. If there are more than $\frac{1}{2}n$ breaks between rounds r and $r+1$, then we flip the home–away assignments in round $r+1$ and also those of round $r+2$. The number of breaks between rounds $r+1$ and $r+2$ remains at 0, whereas the number of breaks between rounds r and $r+1$ becomes $\leq \frac{1}{2}n$ in case n is of the form $4k$, and $\leq \frac{1}{2}(n-2)$ otherwise. Doing this $\frac{1}{2}(n-2)$ times, we end up with a home–away assignment A'' for S with at most the number of breaks stated above. □

A side-result of this section is that partial opponent schedules with $r = 3$ rounds always possess a home–away assignment with at most $\frac{1}{2}n$ breaks. According to Remark 3 this bound cannot be improved: for all $n \geq 4$ of the form $4k$, there exist partial opponent schedules with $r = 3$ rounds, for which every possible home–away assignment has at least $\frac{1}{2}n$ breaks.

4. The special case with a fixed number of rounds

The proof of Theorem 6 is based on the fact (Lemma 5) that finding an optimal solution for *two* consecutive rounds is easy. A natural extension of this approach would be to divide the rounds into groups of *three*, to find the optimal solution for every group, and then to combine and to flip these local solutions as we did in the proof of Theorem 6. However, in this section we will show that this extended approach most probably will not work out: we will show that the break minimization problem for three rounds belongs to the class of NP-hard problems. This implies that the case with three rounds is computationally intractable, and it also means that the combinatorics of this case is messy and difficult to grasp. An analogous statement holds for any fixed number $r \geq 4$ of rounds.

The NP-hardness proof for three rounds will be done by a polynomial time reduction from the following NP-hard version of the Max-Cut problem (see, for instance, Alimonti and Kann [1]):

PROBLEM: Cubic Max-Cut

INPUT: An undirected graph $G = (V, E)$ in which every vertex is incident to exactly three edges (this implies $|E| = \frac{3}{2}|V|$); a bound z.

QUESTION: Is there a partition of V into $V_1 \cup V_2$, such that at least z of the edges in E go between V_1 and V_2? (Edges between V_1 and V_2 are called *cut* edges, and the remaining edges are called *uncut*.)

For an arbitrary instance of Cubic Max-Cut, we will construct a corresponding instance of break minimization for an opponent schedule with three rounds. For every vertex $v \in V$ in the Max-Cut instance, we label the three incident edges with $A(v)$, $B(v)$, and $C(v)$, so that distinct edges get distinct labels. Then every edge $e = [u, v]$ receives two labels: one label $X(v)$ from vertex v and one label $Y(u)$ from vertex u, with $X, Y \in \{A, B, C\}$.

We construct a break minimization instance that has six teams for every vertex $v \in V$: the three teams $A_1(v)$, $B_1(v)$, $C_1(v)$ are the so-called 1-teams corresponding to v, and the three teams $A_2(v)$, $B_2(v)$, $C_2(v)$ are the so-called 2-teams corresponding to v. Altogether, this yields $n = 6|V|$ teams. The matches in the partial opponent schedule S with rounds 1, 2, and 3 are defined as follows.

- In the first round, there are three matches for every vertex $v \in V$: $A_1(v)$ versus $C_2(v)$, and $A_2(v)$ versus $B_1(v)$, and $B_2(v)$ versus $C_1(v)$.
- In the second round, there are three matches for every vertex $v \in V$: $A_1(v)$ versus $A_2(v)$, and $B_1(v)$ versus $B_2(v)$, and $C_1(v)$ versus $C_2(v)$.
- In the third round, there are two matches for every edge $e \in E$: if edge e has been labeled $X(v)$ and $Y(u)$ with $v, u \in V$ and $X, Y \in \{A, B, C\}$, then there are the two matches $X_1(v)$ versus $Y_1(u)$ and $X_2(v)$ versus $Y_2(u)$.

The first- and second-round matches for the six teams corresponding to vertex v are also summarized in the following table:

	$A_1(v)$	$A_2(v)$	$B_1(v)$	$B_2(v)$	$C_1(v)$	$C_2(v)$
Round 1	$C_2(v)$	$B_1(v)$	$A_2(v)$	$C_1(v)$	$B_2(v)$	$A_1(v)$
Round 2	$A_2(v)$	$A_1(v)$	$B_2(v)$	$B_1(v)$	$C_2(v)$	$C_1(v)$

Applying the technique of Lemma 5, we see that there are only two possibilities for scheduling the matches in this table without introducing breaks between the first round and the second round:

- One possibility is that all 1-teams play the first round at home, and the second round away. Symmetrically, the 2-teams play the first round away, and the second round at home. This home–away assignment is called the 1-*assignment* for the six teams.
- The other possibility is that all 1-teams play the first round away, and the second round at home. Symmetrically, the 2-teams play the first round at home, and the second round away. This home–away assignment is called the 2-*assignment* for the six teams.

All other assignments create at least two breaks for the six teams between the first and the second round.

Lemma 7. *If the Max-Cut instance has answer YES, then the constructed opponent schedule S has a home–away assignment with at most* $2(|E| - z)$ *breaks.*

Proof. Consider a partition $V_1 \cup V_2$ of the vertex set V that cuts at least z edges. For every vertex $v \in V_1$, we use the 1-assignment to fix the locations of the matches in the first two rounds, and for every vertex $v \in V_2$ we use the 2-assignment for the matches in the first two rounds. This fixes all the matches in the first two rounds without breaks between the first and second round.

Now consider the third-round matches "$X_1(v)$ versus $Y_1(u)$" and "$X_2(v)$ versus $Y_2(u)$" that correspond to an edge $e \in E$ that has labels $X(v)$ and $Y(u)$ with $v, u \in V$ and $X, Y \in \{A, B, C\}$.

(Case 1): First consider the case where the vertices v and u are on different sides of the partition $V_1 \cup V_2$. By symmetry, we may assume that $v \in V_1$ and $u \in V_2$ holds. Then the teams for v use the 1-assignment, whereas the teams for u use the 2-assignment. Consequently, in the second round, $X_1(v)$ and $Y_2(u)$ play away, whereas $X_2(v)$ and $Y_1(u)$ play at home. In the third round, we make $X_1(v)$ and $Y_2(u)$ play at home and $X_2(v)$ and $Y_1(u)$ play away. This fixes both matches "$X_1(v)$ versus $Y_1(u)$" and "$X_2(v)$ versus $Y_2(u)$" without break between the second and third rounds.

(Case 2): Next consider the case where both vertices v and u are on the same side of the partition $V_1 \cup V_2$. We assume that $v, u \in V_1$. Then in the second round, $X_1(v)$ and $Y_1(u)$ play away, whereas $X_2(v)$ and $Y_2(u)$ play at home. Independently of how we fix the locations of the third-round matches "$X_1(v)$ versus $Y_1(u)$" and "$X_2(v)$ versus $Y_2(u)$", we will always create exactly two breaks between the second and third rounds.

To summarize: the matches for any cut edge e between V_1 and V_2 can be assigned without break, and the matches for any uncut edge can be assigned with two breaks. Since there are at most $|E| - z$ uncut edges, we end up with at most $2(|E| - z)$ breaks. □

Lemma 8. *If the constructed opponent schedule S has a home–away assignment with at most* $2(|E| - z)$ *breaks, then the Max-Cut instance has answer YES.*

Proof. Consider a home–away assignment A' with at most $2(|E| - z)$ breaks for opponent schedule S. We will now slightly modify assignment A' and enforce a uniform combinatorial structure for its first- and second-round matches. For some fixed vertex v, we consider the locations of the teams $A_1(v)$, $B_1(v)$, $C_1(v)$ in the second round of assignment A'.

(Case 1): If all three teams play at home, we simply change their first-round locations to the 2-assignment. Symmetrically, if all three teams play away, then we change their first-round locations to the 1-assignment. In either case, we do not create additional breaks.

(Case 2): If two teams, say $A_1(v)$ and $B_1(v)$, play at home whereas $C_1(v)$ plays away, then the six teams for vertex v incur at least 2 breaks between rounds one and two. We change the first- and second-round locations of these six teams to the 2-assignment; in the second round we only move the location of the match $C_1(v)$ versus $C_2(v)$. This decreases the number of breaks between the first and second rounds by at least 2. On the other hand, we create at most 2 new breaks for $C_1(v)$ and $C_2(v)$ between the second and third rounds. All in all, the number of breaks does not go up. The case where two teams play away whereas one team plays at home can be handled symmetrically, changing to 1-assignments now.

We repeat this process for every vertex $v \in V$. Eventually, we end up with a home–away assignment A'', in which, for every vertex v, the six corresponding teams play their first- and second-round matches either according to their

1-assignment or according to their 2-assignment. Since we never increase the number of breaks, the resulting assignment A'' has at most $2(|E| - z)$ breaks.

From assignment A'', we define the following partition $V_1 \cup V_2$ of the vertex set V: vertex v is put into part V_1, if the six teams for vertex v use the 1-assignment in the first and second rounds of A''; otherwise, vertex v is put into part V_2. Consider an edge $e \in E$ that has labels $X(v)$ and $Y(u)$ with $v, u \in V$ and $X, Y \in \{A, B, C\}$. As in the proof of the previous lemma, the edges between vertices within V_1 or within V_2 (the uncut edges) create exactly 2 breaks. Since assignment A'' has at most $2(|E| - z)$ breaks, there are at least z cut edges. Hence the constructed partition solves the Max-Cut instance. □

Lemmas 7 and 8 together imply the correctness of our reduction. This yields the following theorem.

Theorem 9. *Break minimization in partial opponent schedules with n teams and three rounds is NP-hard.* □

Next, we want to extend the statement in Theorem 9 to the cases with a fixed number $r \geq 4$ of rounds. Consider some break minimization instance for an opponent schedule S with n teams $T_1, T_2, \ldots, T_n$ and r rounds. We create the following new opponent schedule S' with $2n$ teams $T_1, T_2, \ldots, T_n$ and $T'_1, T'_2, \ldots, T'_n$, and with $r + 1$ rounds:

- If, in the kth round ($1 \leq k \leq r$) of the original schedule S, team T_i plays against team T_j, then in the kth round of schedule S' we make team T_i play against T_j and we make team T'_i play against T'_j.
- In round $r + 1$ of schedule S', team T_i plays against team T'_i for $i = 1, \ldots, n$.

We claim that schedule S has a home–away assignment with at most b breaks, if and only if schedule S' has a home–away assignment with at most $2b$ breaks.

First, assume that schedule S has a home–away assignment with b breaks. We construct the following home–away assignment for S'. If, in one of the first r rounds of S, team T_i plays at home (respectively, away), then in the corresponding round of S', team T_i also plays at home (respectively, away) whereas team T'_i plays away (respectively, at home). Then the matches in the last round $r + 1$ can be fixed easily without any breaks between rounds r and $r + 1$.

Next, assume that schedule S' has a home–away assignment with at most $2b$ breaks. Consider the induced home–away assignment for teams $T_1, T_2, \ldots, T_n$ in the first r rounds and the induced home–away assignment for the teams $T'_1, T'_2, \ldots, T'_n$ in the first r rounds. Since one of these two induced assignments must contain at most b breaks, we derive a corresponding assignment for S with at most b breaks.

Corollary 10. *Break minimization in partial opponent schedules with n teams and a fixed number $r \geq 4$ of rounds is NP-hard.* □

5. Conclusion

We have derived the first non-trivial upper and lower bounds on $b(n)$, and we conjecture that, for all n, the upper bound can be improved to $\frac{1}{6}n(n - 1)$, so that it matches our lower bound construction in Section 2. Note that the results of [8] (see Fig. 1) are in accordance with this conjecture. As a first step towards getting a better understanding of $b(n)$, it might be interesting to close some of the gaps in Fig. 1.

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