

# Some remarks on adaptive stabilization of infinite-dimensional systems

Hartmut Logemann

*Institute for Dynamical Systems, University of Bremen,  
Postfach 330440, 2800 Bremen 33, FRG*

Hans Zwart

*Department of Applied Mathematics, University of Twente,  
P.O. Box 217, 7500 AE Enschede, Netherlands*

Received 18 August 1990

**Abstract:** It is the purpose of this note to show that a first-order adaptive controller stabilizes a large class of infinite-dimensional systems described by strongly continuous semigroups. It is assumed that the plant is minimum-phase and has invertible high-frequency gain. Knowledge of the sign of the high-frequency gain is not required.

**Keywords:** Infinite-dimensional systems; adaptive stabilization; strongly continuous semigroups; semilinear evolution equations.

## 1. Introduction

Generalizing a result by Nussbaum [12] Willems and Byrnes [15] constructed a sign-switching high-gain adaptive controller which globally stabilizes any finite-dimensional single-input single-output minimum-phase system with invertible high-frequency gain. In recent years it was shown by several authors (see Dahleh [3], Dahleh and Hopkins [4], Kobayashi [7], Logemann [8] and Logemann and Owens [9]) that the adaptive algorithm presented in [15] stabilizes certain classes of *infinite-dimensional* systems as well. In [3], [4] and [8] the main result of [15] was extended to various classes of retarded systems. Generalizations to distributed parameter systems described by analytic semigroups were given in [7], while an input-output theory of high-gain adaptive stabilization of systems described by non-rational transfer functions was developed in [9].

In the following we shall consider systems of the form

$$\dot{x} = Ax + Bu, \quad y = Cx, \quad (1.1)$$

where  $A$  generates a strongly continuous semigroup  $S(t)$  on a Banach space  $X$  and  $B: \mathbb{R} \rightarrow X$  and  $C: X \rightarrow \mathbb{R}$  are bounded linear operators. Suppose that the system (1.1) has no zeros in  $\text{Re}(s) \geq \alpha$  for some  $\alpha < 0$  and  $CB \neq 0$ . Under these conditions it was shown by Kobayashi [7] that the adaptive control law given in [15] will globally stabilize the system (1.1) provided that

- (i)  $X$  is a Hilbert space,
- (ii)  $A$  is selfadjoint and has a complete orthonormal system of eigenvectors,
- (iii)  $S(t)$  is analytic,
- (iv)  $\text{im } B$  and  $\text{im } C^*$  are contained in the domain of  $A$ .

In this paper we will answer the question posed in [7] whether the conditions (iii) and (iv) are really necessary for adaptive stabilization. We will show that (i)–(iv) can be relaxed considerably. In particular it will turn out that

- (i)–(iii) can be dropped,
- (iv) can be relaxed if (iii) holds.

The paper is organized as follows. Section 2 is devoted to preliminaries concerning the class of systems under consideration. Moreover it contains some technical lemmas which will be used in Section 3 in order to prove the main results of this paper. In the Appendix we prove the existence of a well-defined transfer function for a class of infinite-dimensional systems with unbounded observation operator. This result, which is needed in Section 3, might be of some independent interest.

## Notation

For  $\alpha \in \mathbb{R}$  define

$$\mathbb{C}_\alpha := \{s \in \mathbb{C} \mid \text{Re}(s) > \alpha\}.$$

Let  $H_\alpha^\infty$  denote the algebra of functions which are analytic and bounded on  $\mathbb{C}_\alpha$ .

Let  $X$  and  $Y$  be normed spaces. The vector space of all linear bounded operators from  $X$  to  $Y$  is denoted by  $\mathcal{L}(X, Y)$ .

Let  $A$  be a linear operator. Then we define  $D(A) :=$  domain of  $A$ ,  $\sigma(A) :=$  spectrum of  $A$  and  $\rho(A) :=$  resolvent set of  $A$ .

## 2. Preliminaries and system description

In the following we shall consider systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (2.1a)$$

$$y(t) = Cx(t), \quad t \geq 0, \quad (2.1b)$$

where  $A$  generates a strongly continuous semigroup  $S(t)$  on a real Banach space  $X$ ,  $B \in \mathcal{L}(\mathbb{R}, X)$  and  $C \in \mathcal{L}(X, \mathbb{R})$ . Sometimes it will be necessary to consider the complexifications of  $X$ ,  $A$ ,  $B$ , and  $C$ . For simplicity these will be denoted by  $X$ ,  $A$ ,  $B$ , and  $C$  as well.

The notion of exponential stabilizability will play an important role in the sequel.

**2.1. Definition.** The system (2.1) (or the pair  $(A, B)$ ) is called *exponentially stabilizable* if there exists  $K \in \mathcal{L}(X, \mathbb{R})$  such that the strongly continuous semigroup generated by  $A + BK$  is exponentially stable.

**2.2. Lemma.** *Suppose that the pair  $(A, B)$  is exponentially stabilizable and  $\sigma(A) \subset \mathbb{C} \setminus \mathbb{C}_\alpha$  for some  $\alpha < 0$ . Then the strongly continuous semigroup  $S(t)$  generated by  $A$  will be exponentially stable.*

The proof of the above lemma follows easily from Nefedov and Sholokhovich [11] or Jacobson and Nett [5] (cf. also Curtain [2]).

The following definition will make precise what we mean by a zero of the system (2.1).

**2.3. Definition.** A number  $\lambda \in \mathbb{R}$  is called a *zero* of the system (2.1) if the kernel of the operator

$$\begin{pmatrix} \lambda I - A & B \\ C & 0 \end{pmatrix}: D(A) \oplus \mathbb{C} \rightarrow X \oplus \mathbb{C}$$

is non-trivial.

**2.4. Remark.** Let  $\lambda \in \mathbb{C}$  be a zero of the system (2.1) and suppose that  $\lambda \in \rho(A)$ . Then it is easy to show that  $\lambda$  is a zero of the transfer function

$$G(s) = C(sI - A)^{-1}B$$

of (2.1).

Let us introduce the following assumptions

(A1)  $CB \neq 0$ .

(A2) The system (2.1) has no zeros in  $\mathbb{C}_\alpha$  for some  $\alpha < 0$ .

(A3) The system (2.1) is exponentially stabilizable.

(A4)  $\text{im } B \subset D(A)$ .

(A5)  $\text{im } C^* \subset D(A^*)$ .

(A6)  $\text{im } B \subset D(A^2)$ .

(A7)  $\text{im } C^* \subset D(A^{*2})$ .

The next lemma establishes the existence of a feedback operator which shifts the spectrum of  $A$  into the left half plane.

**2.5. Lemma.** *Let (A1)–(A3) be satisfied and define*

$$F_\gamma := (CB)^{-1}(-CA + \gamma C), \quad (2.2)$$

where  $\gamma < 0$ . Then there exists  $\alpha \in (\gamma, 0)$  such that  $\sigma(A + BF_\gamma) \subset \mathbb{C} \setminus \mathbb{C}_\alpha$ .

**Proof.** By (A3) there exists  $\beta < 0$  such that the spectrum of  $A$  in  $\mathbb{C}_\beta$  consists of isolated eigenvalues with finite multiplicities (see Jacobson and Nett [5] or Curtain [2]). Moreover we have

$$\overline{\mathbb{C}}_0 \cap \rho(A + BF_\gamma) \neq \emptyset$$

by Appendix I. Since  $BF_\gamma$  is an  $A$ -degenerate operator it follows from Theorem 6.2 and Theorem 6.5 in Chapter IV of Kato's book [6] that the spectrum of  $A + BF_\gamma$  in  $\mathbb{C}_\beta$  consists of at most countably many eigenvalues with finite multiplicities. By (A2) there exists a number  $\alpha < 0$  such that the system (2.1) has no zeros in  $\mathbb{C}_\alpha$ . W.l.o.g. we may assume  $\max(\beta, \gamma) < \alpha$ . Suppose that there exists  $\lambda$  in  $\sigma(A + BF_\gamma) \cap \mathbb{C}_\alpha$ . Then  $\lambda$  is an eigenvalue of  $A + BF_\gamma$  and there exists  $x \in X$ ,  $x \neq 0$  such that

$$(\lambda I - A - BF_\gamma)x = 0.$$

Hence

$$(\lambda I - A + B(CB)^{-1}CA)x - \gamma B(CB)^{-1}Cx = 0. \quad (2.3)$$

Applying  $C$  to both sides of the above equation gives  $(\lambda - \gamma)Cx = 0$ . Since  $\gamma < \alpha < \text{Re}(\lambda)$  it follows that  $Cx = 0$ . We obtain using (2.3),

$$\begin{pmatrix} \lambda I - A & B \\ C & 0 \end{pmatrix} \begin{pmatrix} x \\ (CB)^{-1}CAx \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus  $\lambda$  is a zero of (2.1) which is not possible by assumption. Hence we have shown

$$\sigma(A + BF_\gamma) \cap C_\alpha = \emptyset.$$

**2.6. Lemma.** Let  $F_\gamma$  be defined as in (2.2). Suppose that (A1) holds and that  $A + BF_\gamma$  generates a strongly continuous semigroup  $S_\gamma(t)$ . Under these conditions we have

$$S_\gamma(t) \ker C \subset \ker C \quad \forall t \geq 0.$$

**Proof.** Let  $x \in \ker C$ . For  $s \in \rho(A + BF_\gamma)$  we have

$$\begin{aligned} x &= (sI - A - BF_\gamma)(sI - A - BF_\gamma)^{-1}x \\ &= [sI - A - B(CB)^{-1}(-CA + \gamma C)] \\ &\quad \cdot (sI - A - BF_\gamma)^{-1}x. \end{aligned}$$

Applying  $C$  to both sides of the above equation we obtain

$$\begin{aligned} 0 &= (sC - CA + CA - \gamma C)(sI - A - BF_\gamma)^{-1}x \\ &= (s - \gamma)C(sI - A - BF_\gamma)^{-1}x. \end{aligned}$$

Hence we have shown for all  $s \in \rho(A + BF_\gamma)$ ,  $s \neq \gamma$ , that

$$(sI - A - BF_\gamma)^{-1} \ker C \subset \ker C.$$

The claim now follows from Pazy [13], p. 121.

**2.7. Remark.** The feedback law  $F_\gamma$  was introduced by Curtain in [1], Section 8 in the context of disturbance decoupling for infinite-dimensional systems (cf. also Zwart [17]).

### 3. Main results

Let us recall the definition of a Nussbaum gain.

**3.1. Definition.** A measurable locally bounded function  $N: \mathbb{R} \rightarrow \mathbb{R}$  is called a *Nussbaum gain* if for some  $t_0 \in \mathbb{R}$ ,

$$\sup_{t > t_0} \frac{1}{t - t_0} \int_{t_0}^t \tau N(\tau) \, d\tau = +\infty$$

and

$$\inf_{t > t_0} \frac{1}{t - t_0} \int_{t_0}^t \tau N(\tau) \, d\tau = -\infty.$$

**3.2. Example.** A continuously differentiable Nussbaum gain is given by

$$N(\tau) = \cos\left(\frac{1}{2}\pi\tau\right) \exp(\tau^2),$$

cf. Nussbaum [12] or Logemann and Owens [9].

In this section we shall apply the following control law to the system (2.1):

$$\begin{aligned} u(t) &= N(k(t))k(t)y(t), \\ \dot{k}(t) &= y^2(t), \quad k(0) = k_0 \in \mathbb{R}, \end{aligned} \tag{3.1}$$

where  $N$  is a Nussbaum gain. The control law (3.1) has been introduced by Willems and Byrnes [15] for finite-dimensional systems.

Defining

$$A_c: D(A) \times \mathbb{R} \rightarrow X \times \mathbb{R},$$

$$\begin{pmatrix} x \\ k \end{pmatrix} \mapsto \begin{pmatrix} Ax \\ 0 \end{pmatrix};$$

$$f: X \times \mathbb{R} \rightarrow X \times \mathbb{R},$$

$$\begin{pmatrix} x \\ k \end{pmatrix} \mapsto \begin{pmatrix} N(k)kBCx \\ (Cx)^2 \end{pmatrix},$$

and

$$x_c(t) := \begin{pmatrix} x(t) \\ k(t) \end{pmatrix},$$

we can write the closed-loop system as follows:

$$\dot{x}_c(t) = A_c x_c(t) + f(x_c(t)), \quad t \geq 0, \tag{3.2a}$$

$$x_c(0) = \begin{pmatrix} x_0 \\ k_0 \end{pmatrix} \in X \times \mathbb{R}. \tag{3.2b}$$

A continuously differentiable  $D(A_c)$ -valued function which satisfies (3.2) is called a *classical solution* of (3.2). A *mild solution* of (3.2) is a continuous function satisfying

$$x_c(t) = S_c(t)x_c(0) + \int_0^t S_c(t-\tau)f(x_c(\tau)) \, d\tau,$$

where  $S_c(t)$  denotes the strongly continuous semigroup generated by  $A_c$ .

The following lemma shows that (3.2) is well-posed.

**3.3. Lemma.** (i) *If  $N$  satisfies a local Lipschitz condition then for all  $x_c(0) \in X \times \mathbb{R}$ , (3.2) has a unique mild solution which can be continued to the right as long as it remains bounded.*

(ii) *If  $N$  is continuously differentiable then for all  $x_c(0) \in D(A) \times \mathbb{R}$ , (3.2) has a unique classical solution which can be continued to the right as long as it remains bounded.*

**Proof.** (i) It is easy to show that  $f$  satisfies a local Lipschitz condition, i.e. for any  $l > 0$  there exists  $L > 0$  such that

$$\|f(z) - f(z')\| \leq L \|z - z'\|$$

for all  $z, z' \in X \times \mathbb{R}$  satisfying  $\|z\|, \|z'\| \leq l$ , where the norm  $\|\cdot\|$  on  $X \times \mathbb{R}$  is defined by  $\|\cdot\| = \|\cdot\|_X + |\cdot|$ . The claim follows now from Segal [14], Theorem 1 (cf. also Pazy [13], pp. 185).

(ii) It is routine to show that  $f$  is continuously Fréchet-differentiable. Moreover  $f$  satisfies a local Lipschitz condition (notice that this does not follow necessarily from the  $C^1$ -property in the infinite-dimensional case). Application of Theorem 1 and Lemma 3.1 in Segal [14] (cf. also Martin [10], pp. 347) proves the claim.

We are now in the position to state our main results.

**3.4. Theorem.** *Suppose that assumptions (A1)–(A5) are satisfied and that  $N$  is a continuously differentiable Nussbaum gain. The following statements hold true.*

(i) *For all  $(x_0, k_0) \in D(A) \times \mathbb{R}$  the closed-loop system given by (2.1) and (3.1) has a unique globally defined classical solution  $(x(t), k(t))$  with the following properties:*

$$\lim_{t \rightarrow \infty} k(t) \text{ exists and is finite,} \quad (3.3)$$

$$x(\cdot) \in L^2(0, \infty; X) \cap L^\infty(0, \infty; X), \quad (3.4)$$

$$\lim_{t \rightarrow \infty} x(t) = 0. \quad (3.5)$$

(ii) *For all  $(x_0, k_0) \in X \times \mathbb{R}$  the closed-loop system given by (2.1) and (3.1) has a unique globally defined mild solution  $(x(t), k(t))$  satisfying (3.3)–(3.5).*

**Proof.** (i) Define the linear bounded operator  $P_1: X \rightarrow X$  by

$$P_1 x = B(CB)^{-1} Cx.$$

Then  $P_1$  is a projection and  $\text{im } P_1 = \text{im } B$ . Moreover set  $P_2 := I - P_1$ . It is obvious that  $\text{im } P_2 = \ker C$  and  $X = \text{im } B \oplus \ker C$ . Let  $(x(t), k(t))$  denote the classical solution of the feedback system given by (2.1) and (3.1) with initial value  $(x_0, k_0) \in D(A) \times \mathbb{R}$  and let  $[0, t_0)$  denote its maximal interval of existence. Realizing that

$$(A + BF_\gamma)(D(A) \cap \ker C) \subset \ker C,$$

$$\text{im } P_1 \subset D(A), \quad P_2(D(A)) \subset D(A)$$

and

$$P_1 A P_2 x = -BF_\gamma P_2 x \quad \forall x \in D(A),$$

we obtain from (2.1),

$$\begin{aligned} P_1 \dot{x}(t) &= P_1 A x(t) + B u(t) \\ &= P_1 A P_1 x(t) - BF_\gamma P_2 x(t) + B u(t) \end{aligned}$$

and

$$\begin{aligned} P_2 \dot{x}(t) &= P_2 (A + BF_\gamma) x(t) \\ &= (A + BF_\gamma) P_2 x(t) + P_2 A P_1 x(t). \end{aligned}$$

Noticing that  $P_1 x(t) = B(CB)^{-1} y(t)$  and setting  $z(t) := P_2 x(t)$  it follows

$$\begin{aligned} B(CB)^{-1} \dot{y}(t) &= B(CB)^{-1} C A B(CB)^{-1} y(t) \\ &\quad + B(u(t) - F_\gamma z(t)), \end{aligned}$$

$$\dot{z}(t) = (A + BF_\gamma) z(t) + P_2 A B(CB)^{-1} y(t).$$

We conclude that the initial value problem given by (2.1) and (3.1) can be written as

$$\dot{y}(t) = C B v_1(t), \quad y(0) = C x_0, \quad (3.6)$$

$$\begin{aligned} \dot{z}(t) &= (A + BF_\gamma) z(t) \\ &\quad + P_2 A B(CB)^{-1} v_2(t), \quad z(0) = P_2 x_0, \end{aligned} \quad (3.7a)$$

$$w(t) = F_\gamma z(t) - (CB)^{-1} C A B(CB)^{-1} v_2(t), \quad (3.7b)$$

$$v_1(t) = u(t) - w(t), \quad v_2(t) = y(t), \quad (3.8)$$

$$\dot{k}(t) = y^2(t), \quad k(0) = k_0, \quad (3.9a)$$

$$u(t) = N(k(t)) k(t) y(t). \quad (3.9b)$$

Hence we have shown that  $(x(t), k(t))$  solves the initial value problem given by (2.1) and (3.1) (where  $x_0 \in D(A)$ ) on  $[0, t_0)$  if and only if

$$x(t) = z(t) + B(CB)^{-1} y(t), \quad (3.10)$$

where  $(z(t), y(t), k(t))$  is a solution of the initial value problem defined by (3.6)–(3.9) on  $[0, t_0)$ .

We obtain from (A4) and (A5) that  $P_2AB(CB)^{-1}$ ,  $F_\gamma$  and  $(CB)^{-1}CAB(CB)^{-1}$  are bounded linear operators. Hence it follows in particular that  $A + BF_\gamma$  generates a strongly continuous semigroup which will be denoted by  $S_\gamma(t)$ . Using Lemma 2.6 we obtain that  $S_\gamma(t)$  is a strongly continuous semigroup on  $\ker C$ . Therefore (3.7) is a well-defined semigroup system on  $\ker C$ . Clearly, by (A3), the pair  $(A + BF_\gamma, B)$  is exponentially stabilizable. Applying Lemma 2.2 and Lemma 2.5 we see that  $S_\gamma(t)$  is exponentially stable. As a consequence the transfer function of (3.7) is in  $H_\alpha^\infty$  for some  $\alpha < 0$ . It now follows from Logemann and Owens [9] that the pair  $(y(t), k(t))$  is bounded on  $[0, t_0)$  which implies via (3.7) and (3.10) that  $(x(t), k(t))$  is bounded on  $[0, t_0)$ . Using Lemma 3.3(ii) we obtain  $t_0 = \infty$ , i.e. the closed-loop system given by (2.1) and (3.1) has a unique globally defined classical solution. Finally it follows again from Logemann and Owens [9] that (3.3)–(3.5) hold with  $x$  replaced by  $y$ , which proves the claim because of (3.10) and the exponential stability of (3.7).

(ii) It follows as in the proof of (i) that  $(y(t), k(t))$  is bounded on  $[0, t_0)$ . Hence, by the exponential stability of (3.7) and Lemma 3.3(i) we have that the mild solution  $(z(t), y(t), k(t))$  of the initial value problem (3.6)–(3.9) is globally defined. Moreover as in the proof of (i) we conclude that (3.3)–(3.5) hold true with  $x$  replaced by  $y$ . In order to prove the claim it is sufficient to show that

$$(z(t) + B(CB)^{-1}y(t), k(t))$$

is the mild solution of the initial value problem given by (2.1) and (3.1). We have already shown in the proof of (i) that this is true if  $x_0 \in D(A)$ . Therefore it remains true in the general case (i.e.  $x_0 \in X$ ), since  $D(A)$  is dense in  $X$  and mild solutions depend continuously on their initial values (cf. Segal [14], Corollary 1.5).

**3.5. Remark.** (i) Notice that in the proof of Theorem 3.4 we have decomposed the original plant (2.1) into a feedback system consisting of an integrator in the forward loop and an (exponentially) stable system in the feedback loop (see (3.6)–(3.8)). Adaptive stabilization of systems ad-

mitting such a decomposition has been investigated by Logemann and Owens [9] using an input–output approach.

(ii) Kobayashi [7] proved a result similar to Theorem 3.4. However he had to assume that  $X$  is a Hilbert space and that  $A$  is a selfadjoint operator on  $X$  having complete orthonormal system of eigenvectors and generating an analytic semigroup. In particular Theorem 3.4 gives an affirmative answer to the question posed in [7] whether the assumption on the analyticity of the semigroup can be relaxed.

**3.6. Corollary.** *Suppose that assumptions (A1)–(A3) and (A6) are satisfied and that  $N$  is a continuously differentiable Nussbaum gain. Under these conditions statement (i) of Theorem 3.4 holds true.*

**Proof.** Let  $\lambda \in \rho(A)$  and define a new state-space system  $(\tilde{A}, \tilde{B}, \tilde{C})$  by  $\tilde{A} := A$ ,  $\tilde{B} := (\lambda I - A)B$  and  $\tilde{C} := C(\lambda I - A)^{-1}$ . Notice that the transfer functions of  $(A, B, C)$  and  $(\tilde{A}, \tilde{B}, \tilde{C})$  are the same. It is clear that  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{C}$  satisfy (A1), (A2), (A4) and (A5). Moreover it follows from Jacobson and Nett [5] or Curtain [2] via (A3) that  $(\tilde{A}, \tilde{B})$  is exponentially stabilizable. Let  $x_0 \in D(A)$  and denote the mild solution of

$$\begin{aligned} \dot{x}(t) &= \tilde{A}x(t) + \tilde{B}u(t), & x(0) &= (\lambda I - A)x_0, \\ y(t) &= \tilde{C}x(t), \\ u(t) &= N(k(t))k(t)y(t), \\ \dot{k}(t) &= y^2(t), & k(0) &= k_0, \end{aligned}$$

by  $(\tilde{x}(t), \tilde{k}(t))$ . It follows from Theorem 3.4(ii) that  $(\tilde{x}(t), \tilde{k}(t))$  is globally defined and satisfies (3.3)–(3.5) with  $x$  and  $k$  replaced by  $\tilde{x}$  and  $\tilde{k}$ . Finally notice that the pair  $(x(t), k(t))$  defined by

$$x(t) := (\lambda I - A)^{-1}\tilde{x}(t) \quad \text{and} \quad k(t) := \tilde{k}(t)$$

is a classical solution of the initial value problem given by (2.1) and (3.1).

**3.7. Corollary.** *Suppose that the assumptions (A1)–(A3) and (A7) are satisfied and that  $N$  is a continuously differentiable Nussbaum gain. Under these conditions the statements (i) and (ii) of Theorem 3.4 hold true.*

**Proof.** Let  $\lambda \in \rho(A)$  and define  $\tilde{A} := A$ ,  $\tilde{B} := (\lambda I - A)^{-1}B$  and  $\tilde{C} := C(\lambda I - A)$ . As in the proof of

Corollary 3.6 we have that the system given by  $(\tilde{A}, \tilde{B}, \tilde{C})$  satisfies (A1)–(A5). Let  $x_0 \in X$  and denote the classical solution of

$$\dot{\tilde{x}}(t) = \tilde{A}\tilde{x}(t) + \tilde{B}u(t), \quad \tilde{x}(0) = (\lambda I - A)^{-1}x_0, \quad (3.11a)$$

$$y(t) = \tilde{C}\tilde{x}(t), \quad (3.11b)$$

$$u(t) = N(k(t))k(t)y(t), \quad (3.11c)$$

$$\dot{k}(t) = y^2(t), \quad k(0) = k_0, \quad (3.11d)$$

by  $(\tilde{x}(t), \tilde{k}(t))$ . By Theorem 3.4(i),  $(\tilde{x}(t), \tilde{k}(t))$  is globally defined and satisfies (3.3)–(3.5) with  $x$  and  $k$  replaced by  $\tilde{x}$  and  $\tilde{k}$ . Notice that the pair  $(x(t), k(t))$  defined by

$$x(t) := (\lambda I - A)\tilde{x}(t) \quad \text{and} \quad k(t) := \tilde{k}(t)$$

is a mild solution of the initial value problem given by (2.1) and (3.1). It will be a solution in the classical sense if  $x_0 \in D(A)$ . We have

$$\tilde{x}(t) = S(t)\tilde{x}_0 + \int_0^t S(t-\tau)\tilde{B}\tilde{u}(\tau) \, d\tau, \quad (3.12)$$

where  $\tilde{x}_0 := (\lambda I - A)^{-1}x_0$  and

$$\tilde{u}(t) := N(\tilde{k}(t))\tilde{k}(t)C\tilde{x}(t).$$

Since the pair  $(\tilde{A}, \tilde{B})$  is exponentially stabilizable, there exist closed subspaces  $X_s$  and  $X_u$  of  $X$  such that

- $X = X_s \oplus X_u$ ,  $X_u$  is finite-dimensional and  $X_u \subset D(\tilde{A}) = D(A)$ ;

- the projections  $P_s: X \rightarrow X_s$  and  $P_u: X \rightarrow X_u$  commute with  $S(t)$  and  $\tilde{A} = A$ ;

- the strongly continuous semigroup  $S_s(t) := S(t)|_{X_s}$  on  $X_s$  is exponentially stable.

(See Jacobson and Nett [5] or Curtain [2].)

Setting

$$z_s(t) := S_s(t)P_s x_0 + \int_0^t S_s(t-\tau)P_s B\tilde{u}(\tau) \, d\tau$$

and

$$z_u(t) := S_u(t)P_u \tilde{x}_0 + \int_0^t S_u(t-\tau)P_u \tilde{B}\tilde{u}(\tau) \, d\tau,$$

where  $S_u(t) := S(t)|_{X_u}$ , we obtain from (3.12),

$$\tilde{x}(t) = z_u(t) + (\lambda I - A)^{-1}z_s(t).$$

Since  $\tilde{x}, z_s \in L^2(0, \infty; X) \cap L^\infty(0, \infty; X)$  and

$$\lim_{t \rightarrow \infty} \tilde{x}(t) = \lim_{t \rightarrow \infty} z_s(t) = 0,$$

the same is true for  $z_u(t)$ . Realizing that  $(\lambda I - A)|_{X_u}$  is a bounded operator ( $X_u \subset D(A)$  is finite-dimensional) it follows from

$$x(t) = (\lambda I - A)\tilde{x}(t) = z_s(t) + (\lambda I - A)|_{X_u}z_u(t)$$

that  $x \in L^2(0, \infty; X) \cap L^\infty(0, \infty; X)$  and

$$\lim_{t \rightarrow \infty} x(t) = 0.$$

Hence the pair  $(x(t), k(t))$  satisfies (3.3)–(3.5).

In Theorem 3.4 it was required that (A4) and (A5) hold. The next two results show that either (A4) or (A5) become superfluous provided that the semigroup  $S(t)$  generated by  $A$  is analytic.

**3.8. Theorem.** *If (A1)–(A4) are satisfied,  $N$  is a continuously differentiable Nussbaum gain and the semigroup  $S(t)$  generated by  $A$  is analytic, then statement (i) of Theorem 3.4 holds true.*

**Proof.** As in the proof of Theorem 3.4 we can show that the closed-loop system given by (2.1) and (3.1) is equivalent to the system (3.6)–(3.9). Since  $BF_\gamma$  is an  $A$ -degenerate operator it follows from Zabczyk [16] that  $A + BF_\gamma$  generates an analytic semigroup  $S_\gamma(t)$ . Now analytic semigroups satisfy the spectrum determined growth assumption and hence  $S_\gamma(t)$  is exponentially stable by Lemma 2.5. The stability result follows from [9] as in the proof of Theorem 3.4 provided that

- (i) the transfer function  $H$  of (3.7) belongs to  $H_\alpha^\infty$  for some  $\alpha < 0$ , and

- (ii) the function  $f(t) := F_\gamma S_\gamma(t)P_2 x_0$  produced by the initial condition is in  $L^2(0, \infty)$ .

Notice that (i) and (ii) do not follow trivially because  $F_\gamma$  is unbounded. Define

$$R := A + BF_\gamma, \quad D := -(CB)^{-1}CAB(CB)^{-1}$$

and

$$E := P_2 AB(CB)^{-1}.$$

It follows from Appendix II that the transfer function  $H$  of (3.7) is given by

$$H(s) = F_\gamma(sI - R)^{-1}E + D.$$

Using the fact that  $0 \in \rho(R)$  we obtain

$$\begin{aligned} H(s) &= F_\gamma R^{-1}R(sI - R)^{-1}E + D \\ &= F_\gamma R^{-1}(s(sI - R)^{-1} - I)E + D \\ &= sF_\gamma R^{-1}(sI - R)^{-1}E - F_\gamma R^{-1}E + D. \end{aligned}$$

Now realizing that  $F_\gamma R^{-1}$  is a bounded operator (by the closed-graph theorem) and using that  $R$  generates an exponentially stable analytic semigroup it follows that there exist  $\beta < 0$  and  $M > 0$  such that  $H$  is holomorphic on  $\mathbb{C}_\beta$  and

$$\|(sI - R)^{-1}\| \leq \frac{M}{|s - \beta|} \quad \text{for all } s \in \mathbb{C}_\beta.$$

Hence  $H \in H_\alpha^\infty$  for all  $\alpha > \beta$ , which shows that (i) holds true.

In order to prove (ii), write

$$f(t) = F_\gamma R^{-1} R S_\gamma(t) P_2 x_0 = F_\gamma R^{-1} S_\gamma(t) R P_2 x_0$$

where we have used that  $P_2 x_0 \in D(A)$  which is true because  $x_0 \in D(A)$  and  $\text{im } P_1 \subset D(A)$ .

**3.9. Corollary.** *If (A1)–(A3) and (A5) are satisfied,  $N$  is a continuously differentiable Nussbaum gain and the semigroup generated by  $A$  is analytic then the statements (i) and (ii) of Theorem 3.4 hold.*

**Proof.** Define  $\tilde{A}$ ,  $\tilde{B}$  and  $\tilde{C}$  as in the proof of Corollary 3.7 and verify that the system given by  $(\tilde{A}, \tilde{B}, \tilde{C})$  fulfils (A1)–(A4). Application of Theorem 3.8 gives that for  $x_0 \in X$  the solution  $(\tilde{x}(t), \tilde{k}(t))$  of the initial value problem (3.11) satisfies (3.3)–(3.5) with  $x$  and  $k$  replaced by  $\tilde{x}$  and  $\tilde{k}$ . Now proceed as in the proof of Corollary 3.7.

**3.10. Remark.** Notice that Theorem 3.8 and Corollary 3.9 improve the result by Kobayashi [7]. They give an affirmative answer to the question raised in [7] whether the assumption that both (A4) and (A5) are satisfied can be relaxed.

#### 4. Appendices

##### Appendix I

In the proof of Lemma 2.5 we have made use of the following result:

**4.1. Lemma.** *If the operator  $F_\gamma$  is given by (2.2) then*

$$\bar{\mathbb{C}}_0 \cap \rho(A + BF_\gamma) \neq \emptyset.$$

**Proof.** Set  $G_\gamma(s) := F_\gamma(sI - A)^{-1}B$ . Since  $A$  generates a strongly continuous semigroup there exists  $\alpha \in [0, \infty)$  such that  $\mathbb{C}_\alpha \subset \rho(A)$ . Hence  $G_\gamma(s)$  is well defined for all  $s \in \mathbb{C}_\alpha$ .

*Step 1:* We claim that  $s \in \rho(A + BF_\gamma)$  if  $s \in \mathbb{C}_\alpha$  and  $G_\gamma(s) \neq 1$ . Notice that for  $s \in \mathbb{C}_\alpha$ ,

$$I = (sI - A - BF_\gamma)(sI - A)^{-1} + BF_\gamma(sI - A)^{-1} \quad (4.1)$$

so that

$$B(1 - G_\gamma(s)) = (sI - A - BF_\gamma)(sI - A)^{-1}B. \quad (4.2)$$

For  $s \in \mathbb{C}_\alpha$  satisfying  $G_\gamma(s) \neq 1$  we obtain

$$B = (sI - A - BF_\gamma)(sI - A)^{-1}B(1 - G_\gamma(s))^{-1}. \quad (4.3)$$

Substituting (4.3) into (4.1) gives

$$I = (sI - A - BF_\gamma)(sI - A)^{-1} \cdot \left[ I + B(1 - G_\gamma(s))^{-1}F_\gamma(sI - A)^{-1} \right]. \quad (4.4)$$

We obtain from the definition of  $F_\gamma$  that the operator

$$H_\gamma(s) := (sI - A)^{-1} \cdot \left[ I + B(1 - G_\gamma(s))^{-1}F_\gamma(sI - A)^{-1} \right]$$

is bounded. Equation (4.4) shows that  $H_\gamma(s)$  is a right inverse of  $sI - A - BF_\gamma$ . The claim now follows since it is not difficult to show that  $H_\gamma(s)$  is a left inverse of  $sI - A - BF_\gamma$  as well.

*Step 2:* It remains to show that there exists  $\xi \in \mathbb{C}_\alpha$  satisfying  $G_\gamma(\xi) \neq 1$ . We will prove that

$$\lim_{\lambda \rightarrow \infty} G_\gamma(\lambda) = 0 \quad (4.5)$$

where  $\lambda$  is a real variable. Since  $A$  generates a strongly continuous semigroup there exist real numbers  $M$  and  $\beta$  such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{\lambda - \beta} \quad \text{for all } \lambda > \beta. \quad (4.6)$$

In order to prove that (4.5) holds true it is sufficient to show

$$\lim_{\lambda \rightarrow \infty} \|A(\lambda I - A)^{-1}B\| = 0. \quad (4.7)$$

Notice that

$$\begin{aligned} \|A(\lambda I - A)^{-1}\| &= \|\lambda(\lambda I - A)^{-1} - I\| \\ &\leq 1 + \frac{M\lambda}{\lambda - \beta} \quad \forall \lambda > \min(0, \beta). \end{aligned}$$

Thus there exists  $\lambda_0 > \max(0, \beta)$  such that

$$\|A(\lambda I - A)^{-1}\| \leq 1 + 2M \quad \forall \lambda > \lambda_0. \quad (4.8)$$

Let  $\varepsilon > 0$  be given. Set  $x := B(1)$  and choose  $z \in D(A)$  satisfying

$$\|x - z\| \leq \frac{\varepsilon}{2(1 + 2M)}. \quad (4.9)$$

Moreover let  $\lambda_1 \geq \lambda_0$  be such that

$$\frac{M}{\lambda - \beta} \|Az\| \leq \frac{1}{2}\varepsilon \quad \forall \lambda > \lambda_1. \quad (4.10)$$

Then it follows from (4.6) and (4.8)–(4.10),

$$\begin{aligned} \|A(\lambda I - A)^{-1}B\| &= \|A(\lambda I - A)^{-1}x\| \\ &\leq \|A(\lambda I - A)^{-1}\| \|x - z\| \\ &\quad + \|(\lambda I - A)^{-1}\| \|Az\| \\ &\leq \varepsilon \quad \forall \lambda > \lambda_1, \end{aligned}$$

which proves (4.7).

## Appendix II

Consider the system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0, \quad (4.11a)$$

$$y(t) = Cx(t), \quad t \geq 0, \quad (4.11b)$$

where  $A$  generates a strongly continuous semigroup  $S(t)$  on a Banach space  $X$ ,  $B \in \mathcal{L}(\mathbb{R}, X)$  and  $C: D(C) \rightarrow \mathbb{R}$  is an  $A$ -bounded linear operator. If  $x_0 \in D(A)$  and  $u \in C^1(0, \infty; \mathbb{R})$  there exists a unique classical solution  $x(t) \in D(A)$  ( $\forall t \geq 0$ ) and hence the output  $y$  is well defined.

In the following let  $\lambda$  denote the exponential growth constant of  $S(t)$ . As usual the Laplace transformation is denoted by the superscript  $\hat{\cdot}$ .

**4.2. Proposition.** *Suppose  $x_0 = 0$  and let  $u \in C^1(0, \infty; \mathbb{R})$  be Laplace transformable such that  $\hat{u}(s)$  exists on  $\mathbb{C}_\alpha$  for some  $\alpha \in \mathbb{R}$ . Then the Laplace transform  $\hat{y}(s)$  of the output of (4.11) exists for all  $s \in \mathbb{C}$  satisfying  $\operatorname{Re}(s) > \max(\alpha, \lambda)$  and is given by*

$$\hat{y}(s) = C(sI - A)^{-1}B\hat{u}(s).$$

Moreover the expression  $C(sI - A)^{-1}B$  is analytic in  $\mathbb{C}_\lambda$ .

**4.3. Remark.** The above proposition says that there exists a transfer function for the system (4.11) and that it is given by  $C(sI - A)^{-1}B$ . This seems like a trivial fact. However, since  $C$  is unbounded, we have to *prove* that  $C$  can be taken out of the Laplace integral.

**Proof of Proposition 4.2.** W.l.o.g. we may assume that  $\lambda < 0$  and hence  $A^{-1} \in \mathcal{L}(X, X)$ . It is well known from semigroup theory that

$$\frac{d}{d\tau}(A^{-1}T(\tau)B) \Big|_{\tau=t} = T(t)B. \quad (4.12)$$

Using (4.12), the variation-of-constants formula and partial integration we obtain

$$\begin{aligned} x(t) &= - \int_0^t d_\tau(A^{-1}T(t-\tau)B)u(\tau) d\tau \\ &= \int_0^t A^{-1}T(t-\tau)Bu'(\tau) d\tau \\ &\quad - A^{-1}Bu(t) + A^{-1}T(t)Bu(0). \end{aligned}$$

Applying  $C$  to both sides of the equation, using the fact that  $CA^{-1}$  is bounded and taking Laplace transforms gives

$$\begin{aligned} \hat{y}(s) &= CA^{-1}(sI - A)^{-1}B(s\hat{u}(s) - u(0)) \\ &\quad - CA^{-1}B\hat{u}(s) + CA^{-1}(sI - A)^{-1}Bu(0) \\ &= CA^{-1}\{s(sI - A)^{-1} - I\}B\hat{u}(s) \\ &= CA^{-1}\{A(sI - A)^{-1}\}B\hat{u}(s) \\ &= C(sI - A)^{-1}B\hat{u}(s). \end{aligned}$$

It is clear that the above equations hold for all  $s \in \mathbb{C}$  satisfying  $\operatorname{Re}(s) > \max(\alpha, \lambda)$ . Moreover it follows from the identity

$$C(sI - A)^{-1}B = CA^{-1}\{s(sI - A)^{-1} - I\}B$$

that  $C(sI - A)^{-1}B$  is analytic in  $\mathbb{C}_\lambda$ .

## References

- [1] R.F. Curtain, Invariance concepts in infinite dimensions, *SIAM J. Control Optim.* **24** (1986) 1009–1031.
- [2] R.F. Curtain, Equivalence of input–output stability and exponential stability for infinite-dimensional systems, *Math. Systems Theory* **21** (1988) 14–48.



- [3] M. Dahleh, Generalization of Tychonov's theorem with applications to adaptive control of SISO delay systems, *Systems Control Lett.* **13** (1989) 421–427.
- [4] M. Dahleh and W.E. Hopkins, Jr., Adaptive stabilization of single-input single-output delay systems, *IEEE Trans. Automat. Control* **31** (1986) 577–579.
- [5] C.A. Jacobson and C.N. Nett, Linear state-space systems in infinite-dimensional space: the role and characterization of joint stabilizability/detectability, *IEEE Trans. Automat. Control* **33** (1988) 541–549.
- [6] T. Kato, *Perturbation Theory for Linear Operators* (Springer, Berlin, 2nd ed., 1976).
- [7] T. Kobayashi, Global adaptive stabilization of infinite-dimensional systems, *Systems Control Lett.* **9** (1987) 215–223.
- [8] H. Logemann, Adaptive exponential stabilization for a class of nonlinear retarded processes, *Math. Control Signals Systems* **3** (1990) 255–269.
- [9] H. Logemann and D.H. Owens, Input–output theory of high-gain adaptive stabilization of infinite-dimensional systems with non-linearities, *Internat. J. Adaptive Control and Signal Process.* **2** (1988) 193–216.
- [10] R.H. Martin, Jr., *Nonlinear Operators and Differential Equations in Banach Spaces* (J. Wiley, New York, 1976).
- [11] S.A. Nefedov and F.A. Sholokhovich, A criterion for the stabilizability of dynamical systems with finite-dimensional input, *Differentsial'nye Uravneniya* **22** (1986) 163–166.
- [12] R.D. Nussbaum, Some remarks on a conjecture in parameter adaptive control, *Systems Control Lett.* **3** (1983) 243–246.
- [13] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations* (Springer, New York, 1983).
- [14] I. Segal, Non-linear semigroups, *Ann. of Math.* **78** (1963) 339–364.
- [15] J.C. Willems and C.I. Byrnes, Global adaptive stabilization in the absence of information on the sign of the high-frequency gain, in: *Proc. INRIA Conf. on Analysis and Optimization of Systems*, Lecture Notes in Control and Information Sciences, No. 62 (Springer, New York, 1984) 49–57.
- [16] J. Zabczyk, On decomposition of generators, *SIAM J. Control Optim.* **16** (1978) 523–534.
- [17] H.J. Zwart, Equivalence between open-loop and closed-loop invariance for infinite-dimensional systems: a frequency-domain approach, *SIAM J. Control Optim.* **26** (1988) 1175–1199.