A CLASS OF INFINITELY DIVISIBLE MIXTURES

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1. Introduction. In a previous paper [3] it was proved that mixtures of characteristic functions (cf’s) of the form

\[ \frac{\lambda}{\lambda - it} \quad (\lambda > 0) \]

are infinitely divisible (inf div). In this paper mixtures of cf’s of the more general type

\[ \frac{\lambda}{\lambda - h(t)} \]

are considered. It will be shown that mixtures of cf’s of type I are inf div if \( h(t) \) is such that \( \frac{\lambda}{\lambda - h(t)} \) is a cf for all \( \lambda > 0 \). The class of functions \( h(t) \) satisfying this condition will be determined.

2. Preliminaries. In our proof we will make use of the Lévy-Khintchine canonical representation: \( \phi(t) \) is an inf div cf if and only if

\[ \log \phi(t) = a_{it} + \int_{-\infty}^{\infty} \left[ e^{itx} - 1 - itx/(1 + x^2) \right] \frac{1}{(1 + x^2)x^2} d\theta(x), \]

where \( a \) is a real constant and \( \theta(x) \) is bounded and non-decreasing (see e.g. [2], p. 89).

Further we shall need the well-known fact (cf. [2], p. 203) that a function of the type

\[ \frac{\lambda}{\lambda + 1 - g(t)} \quad (g(t) \text{ a cf}; \lambda > 0) \]

is an inf div cf. This is easily seen by writing \( \lambda^{1/n}(\lambda + 1 - g(t))^{-1/n} \) as a linear combination of cf’s:

\[ \lambda^{1/n}(\lambda + 1 - g(t))^{-1/n} = \left\{ \frac{\lambda}{\lambda + 1} \right\}^{1/n} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(-1 - \lambda)^k}{\lambda^{k+1}} g(t)^k = \sum_{k=0}^{\infty} C_k^{(n)} g(t)^k, \]

where \( C_k^{(n)} \) can be written as

\[ C_k^{(n)} = n^{-1}(1 + n^{-1}) \cdots (k - 1 + n^{-1})(k!)^{-1}\lambda^{1/n}(1 + \lambda)^{-k-1/n} \quad (k \geq 1). \]

3. Two lemmas.

Lemma 1. If \( p_j > 0 \), \( \sum_j p_j = 1 \) and \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \), then

\[ \sum_j \frac{p_j \lambda_j}{(\lambda_j - h)} = \left[ \prod_{j=1}^{n} \left( \frac{\lambda_j}{(\lambda_j - h)} \right) \right] \prod_{k=1}^{n-1} \frac{\mu_k - h}{\mu_k}, \]

where \( \lambda_j < \mu_j \) for \( j = 1, 2, \cdots, n - 1 \).

Proof. See [3].

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Lemma 2. If \( \theta_\lambda(x) \) is the function \( \theta(x) \) in the canonical representation (2) corresponding to the cf \( \lambda/(\lambda + 1 - g(t)) \) (of type II), then \( \theta_\lambda(x) - \theta_\mu(x) \) is non-decreasing for all \( x \) if \( \lambda \leq \mu \).

Proof. Following Lukac [2], (p. 89), we have in all continuity points of \( \theta_\lambda(x) \)

\[
(5) \quad \theta_\lambda(x) = \lim_{n \to \infty} n \int_{-\infty}^{\infty} y^2/(1 + y^2) \, dF_n(y),
\]

where \( F_n(y) \) is the distribution function corresponding to \( \lambda^{1/n}(\lambda + 1 - g(t))^{-1/n} \).

By (3) we have

\[
F_n(y) = \sum_{k=0}^{\infty} C_k^{(n)} G^{*k}(y) = \left\{ \frac{\lambda}{\lambda + 1} \right\}^{1/n} \epsilon(y) + \sum_{k=1}^{\infty} C_k^{(n)} G^{*k}(y),
\]

where \( G^{*k} \) is the distribution function corresponding to \( g^k \) and \( \epsilon(y) \) is the unit-step function. As \( \int_{-\infty}^{\infty} y^2/(1 + y^2) \, d\epsilon(y) = 0 \) it follows from (5) that

\[
(6) \quad \theta_\lambda(x) = \lim_{n \to \infty} n \int_{-\infty}^{\infty} y^2/(1 + y^2) \, d\bar{F}_n(y),
\]

where \( \bar{F}_n(y) = \sum_{k=0}^{\infty} C_k^{(n)} G^{*k}(y) \). By (4) for \( k \geq 1 \) we have \( \lim_{n \to \infty} n C_k^{(n)} = k^{-1}(\lambda + 1)^{-k} \). Therefore (by uniform convergence)

\[
\lim_{n \to \infty} n \bar{F}_n(y) = L(y) = \sum_{k=1}^{\infty} k^{-1}(\lambda + 1)^{-k} G^{*k}(y).
\]

Hence, by Helly's second theorem ([2], p. 51),

\[
(7) \quad \theta_\lambda(x) = \int_{-\infty}^{\infty} y^2/(1 + y^2) \, dL(y)
\]

\[
= \sum_{k=1}^{\infty} k^{-1}(\lambda + 1)^{-k} \int_{-\infty}^{\infty} y^2/(1 + y^2) \, dG^{*k}(y).
\]

From (7) it is clear that \( \theta_\lambda(x) - \theta_\mu(x) \) is non-decreasing if \( \lambda \leq \mu \).

Remark. It follows from (6) that \( \int_{-\infty}^{\infty} e^{ity} \, dL(y) = -\log \{ 1 - g(t)/(\lambda + 1) \} = \

\log (\lambda + 1)\lambda^{-1} + \log \{ \lambda/(\lambda + 1 - g(t)) \}. \) Therefore we have the representation

\[
\log \{ \lambda/(\lambda + 1 - g(t)) \} = \int_{-\infty}^{\infty} (e^{ity} - 1) \, dL(x),
\]

as can also be proved directly.

4. Infinitely divisible mixtures. We are now in a position to prove the following theorem:

Theorem 1. If \( g(t) \) is an arbitrary characteristic function, then \( \{ \lambda/(\lambda + 1 - g(t)); \lambda > 0 \} \) is a family of inf div cf's with the property that an arbitrary mixture of members of this family

\[
\phi(t) = \int_0^\infty \lambda/(\lambda + 1 - g(t)) \, dF(\lambda),
\]

where \( F \) is a distribution function with \( F(+0) = 0 \), is inf div.

Proof. First we restrict ourselves to finite mixtures \( \sum_{k=1}^{n} p_k \lambda_k/(\lambda_k + 1 - g(t)) \) with \( 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \). Writing \( \phi_\lambda = \lambda/(\lambda + 1 - g) \) by Lemma 1

\[
(8) \quad \sum_{i=1}^{n} p_i \phi_{\lambda_i} = \prod_{i}^{n} \phi_{\lambda_i} \prod_{i=1}^{n-1} \phi_{\lambda_i^{-1}},
\]

where \( \lambda_j < \mu_j \) for \( j = 1, 2, \cdots, n - 1 \). From (8) it follows that the function \( \theta(x) \) in (2) is in this case given by

\[
\theta(x) = \sum_{i=1}^{n} \theta_{\lambda_i}(x) - \sum_{i=1}^{n-1} \theta_{\mu_i}(x),
\]

which is non-decreasing by Lemma 2. Therefore \( \sum_{i=1}^{n} p_i \phi_{\lambda_i} \) has the required representation and is inf div.
Every distribution function $F$ with $F(0+) = 0$ is the weak limit of a sequence of distribution functions $F_n(\lambda)$ of the form

$$F_n(\lambda) = \sum_{k=1}^{\infty} p_{k,n} \epsilon(\lambda - \lambda_{k,n}),$$

where $\epsilon(\lambda)$ is the unit-step function, $p_{k,n} > 0$, $\sum_{k=1}^{\infty} p_{k,n} = 1$ and $\lambda_{k,n} > 0$. Therefore by Helly's second theorem

$$\phi(t) = \int_0^\infty \lambda/(\lambda + 1 - g) \, dF = \lim_{n \to \infty} \int_0^\infty \lambda/(\lambda + 1 - g) \, dF_n$$

$$= \lim_{n \to \infty} \sum_{k=1}^{\infty} p_{k,n} \lambda_{k,n}/(\lambda_{k,n} + 1 - g).$$

It follows, that $\phi(t)$, as a limit of a sequence of inf div cf's, is inf div itself.

In [3] we started from cf's of the form $\lambda/(\lambda - it)$, which are not of type II ($1 + it$ is not a cf). Before considering a more general class of inf div mixtures however it will be shown how Theorem 1 can be used to prove that $\sum \lambda_{ij}/(\lambda_j - it)$ is inf div. If one writes

$$(9) \quad \lambda/(\lambda - it) = [\mu/(\mu - it)] \alpha \{\alpha + 1 - \mu/(\mu - it)\}^{-1},$$

where $\mu > \lambda$ and $\alpha = \lambda/(\mu - \lambda) > 0$, then $\lambda/(\lambda - it)$ is a product of two inf div cf's, the latter of which is of type II. Now taking $\mu > \max \lambda_j$ it follows that

$$(10) \quad \sum \lambda_{ij}/(\lambda - it) = [\mu/(\mu - it)] \sum \alpha_{ij} \{\alpha_j + 1 - \mu/(\mu - it)\}^{-1}.$$  

The first factor in the right-hand side of (10) is inf div. To the second factor Theorem 1 applies.

5. Generalization. We use a decomposition as in (9) to prove

**Lemma 3.** If the function $\phi(t) = \lambda/(\lambda - h(t))$ is a cf for all $\lambda > 0$ then it is infinitely divisible.

**Proof.** Taking $\mu = 2\lambda$, i.e. $\alpha = 1$ (see (9)) one has

$$\phi_h = \phi_{2\lambda}/(2 - \phi_{2\lambda}) = \cdots = \phi_{2^n\lambda} \prod (2 - \phi_{2^m\lambda})^{-1}. $$

As $\lim_{N \to \infty} \phi_{2^n\lambda}(t) = 1$ for all $t$ it follows that

$$\phi_h = \lim_{N \to \infty} \phi_h/\phi_{2^n\lambda} = \lim_{N \to \infty} \prod (2 - \phi_{2^m\lambda})^{-1},$$

where $(2 - \phi_{2^m\lambda})^{-1}$ is inf. div (of type II). Therefore $\phi_h$ as the limit of a sequence of inf div cf's is inf div.

From a decomposition like (9) we deduce in the same way.

**Corollary 1.1.** If $\lambda/(\lambda - h)$ is a cf for $\lambda = \lambda_0$, then it is a cf for all $\lambda$ with $0 < \lambda \leq \lambda_0$. If $it$ is inf div for $\lambda = \lambda_0$, then it is inf div for $0 < \lambda \leq \lambda_0$.

As a special case we have

**Corollary 1.2.** If $\phi$ is a cf, then for $0 < \lambda \leq 1$ the function $\phi_h = \lambda/(\lambda + \phi^{-1} - 1)$ is a cf. If $\phi$ is inf div, then $\phi_h$ is inf div for $0 < \lambda \leq 1$.

A characterization of the inf div cf's of type I is given by

**Lemma 4.** A function of the form $\lambda/(\lambda - h(t))$ is a cf for all $\lambda > 0$ if and only if $\exp h(t)$ is an inf div cf.

**Proof.** If $\phi_h = \lambda/(\lambda - h)$ is a cf for all $\lambda > 0$, then by Lemma 3 it is inf div.
Therefore \( \phi_n = \{n/(n - h)\}^n \) is an inf div cf for all \( n > 0 \). By the continuity theorem \( \lim_{n \to \infty} \phi_n = \exp h(t) \) is a cf, which by the closure property is inf div as well.

If, conversely, \( \exp h(t) \) is an inf div cf, then

\[
\lambda/(\lambda - h(t)) = \int_0^\infty e^{-s} \exp [(s/\lambda)h(t)] \, ds,
\]
as a mixture of cf's of the form \( \exp \mu h(t) \), is a cf for all \( \lambda > 0 \). More constructively, Lemma 4 can be expressed as follows:

**Lemma 4**. \( \lambda/(\lambda - h(t)) \) is a cf for all \( \lambda > 0 \) if and only if \( h(t) \) has the form \( h(t) = \log f(t) \), where \( f(t) \) is an inf div cf.

**Remark**. For distributions on \([0, \infty)\) a necessary and sufficient condition is that \(- (d/dx)h(x)\) is completely monotone (cf. [1], p. 425).

Theorem 1 can now be generalized as follows:

**Theorem 2**. If \( h(t) \) is the logarithm of an arbitrary inf div characteristic function, then \( \{\lambda/(\lambda - h(t)); \lambda > 0\} \) is a family of inf div cf's with the property that an arbitrary mixture of members of this family

\[
\phi(t) = \int_0^\infty \frac{\lambda}{\lambda - h(t)} \, dF(\lambda),
\]

with \( F(0) = 0 \), is inf div.

**Proof.** As in the proof of Theorem 1 we start with a finite mixture. Taking \( \mu > \max \lambda_j \) and using a decomposition as in (10) we have

\[
\sum p_j \phi_{\lambda_j} = \phi_\mu \sum p_j \alpha_j / (\alpha_j + 1 - \phi_\mu),
\]

where \( \phi_\mu \) is inf div by Lemma 3 and \( \sum p_j \alpha_j / (\alpha_j + 1 - \phi_\mu) \) by Theorem 1. The generalization to arbitrary mixtures parallels that in the proof of Theorem 1.

We find in the same way

**Corollary 2.1**. If \( \phi(x) \) of type I is inf div for \( \lambda \leq \lambda_0 \), then mixtures of functions \( \phi \) with \( \lambda \leq \lambda_0 \) are inf div.

**Remark.** Theorem 2 (and therefore Theorem 1) can be slightly generalized such as to include mixtures with a component \( \phi_\mu(t) \equiv 1 \). These mixtures can then be rewritten in the form

\[
\int_0^\infty \{1 - x h(t)\}^{-1} \, dF(x),
\]

where \( F(x) \) may have an atom in \( x = 0 \) (cf. [3], p. 1305).

For Laplace transforms of distributions on \([0, \infty)\) Theorem 2 and the first assertion of Lemma 4 follow from the infinite divisibility of the Laplace transform \( \sum p_j \lambda_j / (\lambda_j + \tau) \) as proved in [3]. More generally, if \( \gamma_1(\tau) \) and \( \gamma_2(\tau) \) are inf div Laplace transforms, then \( \gamma(\tau) = \gamma_1(- \log \gamma_2(\tau)) \) is an inf div Laplace transform. This can be proved as follows: \( \gamma_i(\tau) = \exp (-\psi_i(\tau)) \), where \( \psi'_i(\tau) \) is completely monotone (see e.g. [1], p. 425). Now it follows that \( \gamma(\tau) = \exp (-\psi(\tau)) \), with \( \psi(\tau) = \psi_1(\psi_2(\tau)) \). Using criteria 1 and 2 of [1], (p. 417), it is easily seen that \( \psi'(\tau) \) is completely monotone. Characteristic functions do not in general have this property.\(^1\) For instance, taking \( \phi_1(t) = \phi_\mu(t) = \exp (-\hat{\theta}) \) we get \( \phi_1(\log \phi_2(t)) = \exp (-\hat{\theta} \hat{\mu}) \), which is not a cf.

\(^1\) See however [1] p. 538.
5. Examples.

(A) Mixtures of the following cf’s are inf div:

(a) \( \lambda/(\lambda - it) \)  
    (exponential)

(b) \( \lambda/(\lambda + t^2) \)  
    (Laplace)

(c) \( \lambda/(\lambda + 1 - \exp it) \)  
    (geometric; type II)

(d) \( \lambda/(\lambda + \sin^2 t) \)  
    (type II)

(e) \( \lambda/(\lambda + \log (1 - it)) \)  

(f) \( \lambda/[(1 - it) + [(1 - it)^2 - 1]^3] \)  
    (cf. Remark following Lemma 4’).

The cf given in (f), has density function \( \lambda x^{-1}e^{-x} \sum_1^\infty (1 - \lambda)^{n-1}nI_n(x) \), where \( I_n \) denotes the modified Bessel function of the first kind. For \( \lambda = 1 \) we find the cf \( [1 - it - [(1 - it)^2 - 1]^3 \) with density function \( x^{-1}e^{-x}I_1(x) \) as discussed in [1].

(B) Examples of inf div mixtures are

(a) \( \int_0^1 \{1 - xh(t)\}^{-1} dx = -\{h(t)\}^{-1} \log (1 - h(t)), \)

(b) \( 6\pi^{-2} \sum_1^\infty 1/(n^2 + t^2) = 6\pi^{-2} \sum_1^\infty n^{-2}n^2/(n^2 + t^2) \)

\( = 6(\pi t)^{-1}\{\exp 2\pi t\}^{-1} + \frac{1}{2} - (2\pi t)^{-1} \)

(see e.g. [4], p. 113). The density function corresponding to the mixture of Laplace-type cf’s in (b) is (as can be seen by inverting term by term)

\( -3\pi^{-2} \log \{1 - \exp (-|x|)\}. \)

(C) An example of a function of type I, which is a cf for \( 0 < \lambda \leq 1 \) but not for any \( \lambda > 1 \) (as then \( |\phi_\lambda| > 1 \), provided \( \phi_\lambda(t) = \lambda/(\lambda + \exp it - 1) \), which is of the form \( \lambda/(\lambda + \phi^{-1} - 1) \). As \( \phi_\lambda \) is inf div for \( 0 < \lambda \leq 1 \) it follows that Lemma 3 can not be reversed. A class of functions of the form \( \lambda/(\lambda + \phi^{-1} - 1) \), which are cf’s for all \( \lambda > 0 \) is obtained by taking \( \phi = \{\mu/(\mu - it)\}^{-\alpha} \) for \( 0 < \alpha \leq 1 \): it is easily verified that \( (1 + \tau/\mu)^{-\alpha} \) has a completely monotone derivative for \( 0 < \alpha \leq 1 \). The density function corresponding to \( \lambda/(\lambda + (1 + \tau/\mu)^{-\alpha} - 1) \) is

\( \lambda\mu(\mu x)^{-\alpha-1}e^{-\mu x} \sum_0^\infty (1 - \lambda)^{n}(\mu x)^{\alpha n}/\Gamma(\alpha n + n). \)

Another example of this kind is the function (f) given in (A).

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REFERENCES


