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A CLASS OF INFINITELY DIVISIBLE MIXTURES

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1. Introduction. In a previous paper [3] it was proved that mixtures of characteristic functions (cf's) of the form

$$(1) \qquad \lambda/(\lambda - it) \qquad (\lambda > 0)$$

are infinitely divisible (inf div). In this paper mixtures of cf's of the more general type

$$\text{I} \qquad \lambda/(\lambda - h(t))$$

are considered. It will be shown that mixtures of cf's of type I are inf div if $h(t)$ is such that $\lambda/(\lambda - h(t))$ is a cf for all $\lambda > 0$. The class of functions $h(t)$ satisfying this condition will be determined.

2. Preliminaries. In our proof we will make use of the Lévy-Khinchine canonical representation: $\phi(t)$ is an inf div cf if and only if

$$(2) \quad \log \phi(t) = ait + \int_{-\infty}^{\infty} \{e^{itx} - 1 - itx/(1 + x^2)\}(1 + x^2)x^{-2}\, d\theta(x),$$

where a is a real constant and $\theta(x)$ is bounded and non-decreasing (see e.g. [2], p. 89).

Further we shall need the well-known fact (cf. [2], p. 203) that a function of the type

$$\text{II} \qquad \lambda/(\lambda + 1 - g(t)) \qquad (g(t) a \text{ cf}; \lambda > 0)$$

is an inf div cf. This is easily seen by writing $\lambda^{1/n}(\lambda + 1 - g(t))^{-1/n}$ as a linear combination of cf's:

$$(3) \quad \lambda^{1/n}(\lambda + 1 - g(t))^{-1/n} = \{\lambda/(\lambda + 1)\}^{1/n} \sum_{k=0}^{\infty} \binom{-1/n}{k}(-1 - \lambda)^{-k}\{g(t)\}^k = \sum_{k=0}^{\infty} C_k^{(n)}\{g(t)\}^k,$$

where $C_k^{(n)}$ can be written as

$$(4) \quad C_k^{(n)} = n^{-1}(1 + n^{-1}) \cdots (k - 1 + n^{-1})(k!)^{-1}\lambda^{1/n}(1 + \lambda)^{-k-1/n} \quad (k \geqq 1).$$

3. Two lemmas.

LEMMA 1. *If* $p_j > 0$, $\sum_1^n p_j = 1$ *and* $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$, *then*

$$\sum_{j=1}^n p_j\lambda_j/(\lambda_j - h) \doteq [\prod_{j=1}^n \lambda_j/(\lambda_j - h)] \prod_{k=1}^{n-1} (\mu_k - h)/\mu_k,$$

where $\lambda_j < \mu_j$ *for* $j = 1, 2, \cdots, n - 1$.

PROOF. See [3].

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LEMMA 2. *If $\theta_\lambda(x)$ is the function $\theta(x)$ in the canonical representation* (2) *corresponding to the* cf $\lambda/(\lambda + 1 - g(t))$ (*of type* II), *then $\theta_\lambda(x) - \theta_\mu(x)$ is non-decreasing for all x if $\lambda \leqq \mu$.*

PROOF. Following Lukacs [2], (p. 89), we have in all continuity points of $\theta_\lambda(x)$

$$(5) \qquad \theta_\lambda(x) = \lim_{n\to\infty} n \int_{-\infty}^{x} y^2/(1 + y^2)\, dF_n(y),$$

where $F_n(y)$ is the distribution function corresponding to $\lambda^{1/n}(\lambda + 1 - g(t))^{-1/n}$. By (3) we have

$$F_n(y) = \sum_{k=0}^{\infty} C_k^{(n)} G^{*k}(y) = \{\lambda/(\lambda + 1)\}^{1/n}\epsilon(y) + \sum_{k=1}^{\infty} C_k^{(n)} G^{*k}(y),$$

where G^{*k} is the distribution function corresponding to g^k and $\epsilon(y)$ is the unit-step function. As $\int_{-\infty}^{x} y^2/(1 + y^2)\, d\epsilon(y) = 0$ it follows from (5) that

$$\theta_\lambda(x) = \lim_{n\to\infty} n \int_{-\infty}^{x} y^2/(1 + y^2)\, d\tilde{F}_n(y),$$

where $\tilde{F}_n(y) = \sum_{k=1}^{\infty} C_k^{(n)} G^{*k}(y)$. By (4) for $k \geqq 1$ we have $\lim_{n\to\infty} nC_k^{(n)} = k^{-1}(\lambda + 1)^{-k}$. Therefore (by uniform convergence)

$$(6) \qquad \lim_{n\to\infty} n\tilde{F}_n(y) = L(y) = \sum_{k=1}^{\infty} k^{-1}(\lambda + 1)^{-k} G^{*k}(y).$$

Hence, by Helly 's second theorem ([2], p. 51),

$$(7) \quad \theta_\lambda(x) = \int_{-\infty}^{x} y^2/(1 + y^2)\, dL(y) = \sum_{k=1}^{\infty} k^{-1}(\lambda + 1)^{-k} \int_{-\infty}^{x} y^2/(1 + y^2)\, dG^{*k}(y).$$

From (7) it is clear that $\theta_\lambda(x) - \theta_\mu(x)$ is non-decreasing if $\lambda \leqq \mu$.

REMARK. It follows from (6) that $\int_{-\infty}^{\infty} e^{ity}\, dL(y) = -\log\{1 - g(t)/(\lambda + 1)\} = \log(\lambda + 1)\lambda^{-1} + \log\{\lambda/(\lambda + 1 - g(t))\}$. Therefore we have the representation $\log\{\lambda/(\lambda + 1 - g(t))\} = \int_{-\infty}^{\infty} (e^{itx} - 1)\, dL(x)$, as can also be proved directly.

4. Infinitely divisible mixtures. We are now in a position to prove the following theorem:

THEOREM 1. *If $g(t)$ is an arbitrary characteristic function, then $\{\lambda/(\lambda + 1 - g(t));\ \lambda > 0\}$ is a family of* inf div cf '*s with the property that an arbitrary mixture of members of this family*

$$\phi(t) = \int_0^\infty \lambda/(\lambda + 1 - g(t))\, dF(\lambda),$$

where F is a distribution function with $F(+0) = 0$, is inf div.

PROOF. First we restrict ourselves to finite mixtures $\sum_{k=1}^{n} p_k\lambda_k/(\lambda_k + 1 - g(t))$ with $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$. Writing $\phi_\lambda = \lambda/(\lambda + 1 - g)$ by Lemma 1

$$(8) \qquad \sum_1^n p_j\phi_{\lambda_j} = \prod_1^n \phi_{\lambda_j} \prod_1^{n-1} \phi_{\mu_k}^{-1},$$

where $\lambda_j < \mu_j$ for $j = 1, 2, \cdots, n - 1$. From (8) it follows that the function $\theta(x)$ in (2) is in this case given by

$$\theta(x) = \sum_1^n \theta_{\lambda_j}(x) - \sum_1^{n-1} \theta_{\mu_k}(x),$$

which is non-decreasing by Lemma 2. Therefore $\sum_1^n p_j\phi_{\lambda_j}$ has the required representation and is inf div.

Every distribution function F with $F(0+) = 0$ is the weak limit of a sequence of distribution functions $F_n(\lambda)$ of the form

$$F_n(\lambda) = \sum_{k=1}^{n} p_{k,n}\epsilon(\lambda - \lambda_{k,n}),$$

where $\epsilon(\lambda)$ is the unit-step function, $p_{k,n} > 0$, $\sum_1^n p_{k,n} = 1$ and $\lambda_{k,n} > 0$. Therefore by Helly's second theorem

$$\phi(t) = \int_0^\infty \lambda/(\lambda + 1 - g)\, dF = \lim_{n\to\infty} \int_0^\infty \lambda/(\lambda + 1 - g)\, dF_n$$
$$= \lim_{n\to\infty} \sum_{k=1}^{n} p_{k,n}\lambda_{k,n}/(\lambda_{k,n} + 1 - g).$$

It follows, that $\phi(t)$, as a limit of a sequence of inf div cf's, is inf div itself.

In [3] we started from cf's of the form $\lambda/(\lambda - it)$, which are not of type II ($1 + it$ is not a cf). Before considering a more general class of inf div mixtures however it will be shown how Theorem 1 can be used to prove that $\sum p_j\lambda_j/(\lambda_j - it)$ is inf div. If one writes

(9) $$\lambda/(\lambda - it) = [\mu/(\mu - it)]\, \alpha\, \{\alpha + 1 - \mu/(\mu - it)\}^{-1},$$

where $\mu > \lambda$ and $\alpha = \lambda/(\mu - \lambda) > 0$, then $\lambda/(\lambda - it)$ is a product of two inf div cf's, the latter of which is of type II. Now taking $\mu > \max \lambda_j$ it follows that

(10) $$\sum_1^n p_j\lambda_j/(\lambda_j - it) = [\mu/(\mu - it)] \sum_1^n p_j\alpha_j\{\alpha_j + 1 - \mu/(\mu - it)\}^{-1}.$$

The first factor in the right-hand side of (10) is inf div. To the second factor Theorem 1 applies.

5. Generalization. We use a decomposition as in (9) to prove

LEMMA 3. *If the function* $\phi_\lambda(t) = \lambda/(\lambda - h(t))$ *is a* cf *for all* $\lambda > 0$ *then it is infinitely divisible.*

PROOF. Taking $\mu = 2\lambda$, i.e. $\alpha = 1$ (see (9)) one has

$$\phi_\lambda = \phi_{2\lambda}/(2 - \phi_{2\lambda}) = \cdots = \phi_{2^N\lambda} \prod_1^N (2 - \phi_{2^k\lambda})^{-1}.$$

As $\lim_{N\to\infty} \phi_{2^N\lambda}(t) = 1$ for all t it follows that

$$\phi_\lambda = \lim_{N\to\infty} \phi_\lambda/\phi_{2^N\lambda} = \lim_{N\to\infty} \prod_1^N (2 - \phi_{2^k\lambda})^{-1},$$

where $(2 - \phi_{2^k\lambda})^{-1}$ is inf. div (of type II). Therefore ϕ_λ as the limit of a sequence of inf div cf's is inf div.

From a decomposition like (9) we deduce in the same way.

COROLLARY 1.1. *If* $\lambda/(\lambda - h)$ *is a* cf *for* $\lambda = \lambda_0$, *then it is a* cf *for all* λ *with* $0 < \lambda \leqq \lambda_0$. *If it is* inf div *for* $\lambda = \lambda_0$, *then it is* inf div *for* $0 < \lambda \leqq \lambda_0$.

As a special case we have

COROLLARY 1.2. *If* ϕ *is a* cf, *then for* $0 < \lambda \leqq 1$ *the function* $\phi_\lambda = \lambda/(\lambda + \phi^{-1} - 1)$ *is a* cf. *If* ϕ *is* inf div, *then* ϕ_λ *is* inf div *for* $0 < \lambda \leqq 1$.

A characterization of the inf div cf's of type I is given by

LEMMA 4. *A function of the form* $\lambda/(\lambda - h(t))$ *is a* cf *for all* $\lambda > 0$ *if and only if* $\exp h(t)$ *is an* inf div cf.

PROOF. If $\phi_\lambda = \lambda/(\lambda - h)$ is a cf for all $\lambda > 0$, then by Lemma 3 it is inf div.

Therefore $\phi_n^n = \{n/(n - h)\}^n$ is an inf div cf for all $n > 0$. By the continuity theorem $\lim_{n\to\infty} \phi_n^n = \exp h(t)$ is a cf, which by the closure property is inf div as well.

If, conversely, $\exp h(t)$ is an inf div cf, then

$$\lambda/(\lambda - h(t)) = \int_0^\infty e^{-s} \exp [(s/\lambda)h(t)]\, ds,$$

as a mixture of cf's of the form $\exp \mu h(t)$, is a cf for all $\lambda > 0$. More constructively, Lemma 4 can be expressed as follows:

LEMMA 4′. *$\lambda/(\lambda - h(t))$ is a* cf *for all $\lambda > 0$ if and only if $h(t)$ has the form $h(t) = \log f(t)$, where $f(t)$ is an* inf div cf.

REMARK. For distributions on $[0, \infty)$ a necessary and sufficient condition is that $-(d/d\tau)h(i\tau)$ is completely monotone (cf. [1], p. 425).

Theorem 1 can now be generalized as follows:

THEOREM 2. *If $h(t)$ is the logarithm of an arbitrary* inf div *characteristic function, then $\{\lambda/(\lambda - h(t)); \lambda > 0\}$ is a family of* inf div cf's *with the property that an arbitrary mixture of members of this family*

$$\phi(t) = \int_0^\infty \lambda/(\lambda - h(t))\, dF(\lambda),$$

with $F(+0) = 0$, is inf div.

PROOF. As in the proof of Theorem 1 we start with a finite mixture. Taking $\mu > \max \lambda_j$ and using a decomposition as in (10) we have

$$\sum p_j\phi_{\lambda_j} = \phi_\mu \sum p_j\alpha_j/(\alpha_j + 1 - \phi_\mu), \tag{11}$$

where ϕ_μ is inf div by Lemma 3 and $\sum p_j\alpha_j/(\alpha_j + 1 - \phi_\mu)$ by Theorem 1. The generalization to arbitrary mixtures parallels that in the proof of Theorem 1.

We find in the same way

COROLLARY 2.1. *If $\phi_\lambda(t)$ of type* I *is* inf div *for $\lambda \leqq \lambda_0$, then mixtures of functions ϕ_λ with $\lambda \leqq \lambda_0$ are* inf div.

REMARK. Theorem 2 (and therefore Theorem 1) can be slightly generalized such as to include mixtures with a component $\phi_\infty(t) \equiv 1$. These mixtures can then be rewritten in the form

$$\int_0^\infty \{1 - xh(t)\}^{-1}\, dF(x),$$

where $F(x)$ may have an atom in $x = 0$ (cf. [3], p. 1305).

For Laplace transforms of distributions on $[0, \infty)$ Theorem 2 and the first assertion of Lemma 4′ follow from the infinite divisibility of the Laplace transform $\sum p_j\lambda_j/(\lambda_j + \tau)$ as proved in [3]. More generally, if $\gamma_1(\tau)$ and $\gamma_2(\tau)$ are inf div Laplace transforms, then $\gamma(\tau) = \gamma_1(-\log \gamma_2(\tau))$ is an inf div Laplace transform. This can be proved as follows: $\gamma_i(\tau) = \exp(-\psi_i(\tau))$, where $\psi_i'(\tau)$ is completely monotone (see e.g. [1], p. 425). Now it follows that $\gamma(\tau) = \exp(-\psi(\tau))$, with $\psi(\tau) = \psi_1(\psi_2(\tau))$. Using criteria 1 and 2 of [1], (p. 417), it is easily seen that $\psi'(\tau)$ is completely monotone. Characteristic functions do not in general have this property.[1] For instance, taking $\phi_1(t) = \phi_2(t) = \exp(-t^2)$ we get $\phi_1(\log \phi_2(t)) = \exp(-t^4)$, which is not a cf.

[1] See however [1] p. 538.

5. Examples.

(A) Mixtures of the following cf's are inf div:

(a) $\lambda/(\lambda - it)$ (exponential)

(b) $\lambda/(\lambda + t^2)$ (Laplace)

(c) $\lambda/(\lambda + 1 - \exp it)$ (geometric; type II)

(d) $\lambda/(\lambda + \sin^2 t)$ (type II)

(e) $\lambda/(\lambda + \log(1 - it))$

(f) $\lambda/\{\lambda - it + [(1 - it)^2 - 1]^{\frac{1}{2}}\}$ (cf. Remark following Lemma 4′).

The cf given in (f), has density function $\lambda x^{-1}e^{-x}\sum_1^\infty (1 - \lambda)^{n-1} n I_n(x)$, where I_n denotes the modified Bessel function of the first kind. For $\lambda = 1$ we find the cf $1 - it - [(1 - it)^2 - 1]^{\frac{1}{2}}$ with density function $x^{-1}e^{-x}I_1(x)$ as discussed in [1].

(B) Examples of inf div mixtures are

(a) $$\int_0^1 \{1 - xh(t)\}^{-1}\, dx = -\{h(t)\}^{-1} \log(1 - h(t)),$$

(b) $$6\pi^{-2}\sum_1^\infty 1/(n^2 + t^2) = 6\pi^{-2}\sum_1^\infty n^{-2}n^2/(n^2 + t^2)$$
$$= 6(\pi t)^{-1}\{(\exp 2\pi t)^{-1} + \tfrac{1}{2} - (2\pi t)^{-1}\}$$

(see e.g. [4], p. 113). The density function corresponding to the mixture of Laplace-type cf's in (b) is (as can be seen by inverting term by term) $-3\pi^{-2}\log\{1 - \exp(-|x|)\}$.

(C) An example of a function of type I, which is a cf for $0 < \lambda \leqq 1$ but not for any $\lambda > 1$ (as then $|\phi_\lambda| > 1$) is provided by $\phi_\lambda(t) = \lambda/(\lambda + \exp it - 1)$, which is of the form $\lambda/(\lambda + \phi^{-1} - 1)$. As ϕ_λ is inf div for $0 < \lambda \leqq 1$ it follows that Lemma 3 can not be reversed. A class of functions of the form $\lambda/(\lambda + \phi^{-1} - 1)$, which are cf's for *all* $\lambda > 0$ is obtained by taking $\phi = \{\mu/(\mu - it)\}^\alpha$ for $0 < \alpha \leqq 1$: it is easily verified that $(1 + \tau/\mu)^\alpha$ has a completely monotone derivative for $0 < \alpha \leqq 1$. The density function corresponding to $\lambda/\{\lambda + (1 + \tau/\mu)^\alpha - 1\}$ is

$$\lambda\mu(\mu x)^{\alpha-1}e^{-\mu x}\sum_0^\infty (1 - \lambda)^n(\mu x)^{\alpha n}/\Gamma(\alpha n + n).$$

Another example of this kind is the function (f) given in (A).

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