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EXISTENCE OF DOMINATING CYCLES AND PATHS

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A cycle C of a graph G is called *dominating cycle* (*D-cycle*) if every edge of G is incident with at least one vertex of C. A *D-path* is defined analogously. If a graph G contains a D-cycle (D-path), then its edge graph $L(G)$ has a hamiltonian cycle (hamiltonian path). Necessary conditions and sufficient conditions are obtained for graphs to have a D-cycle or D-path. They are analogous to known conditions for the existence of hamiltonian cycles or paths. The notions *edge degree* and *remote edges* arise as analogues of *vertex degree* and *nonadjacent vertices*, respectively. A result of Nash-Williams is improved.

1. Introduction

We use [3] for basic terminology and consider throughout only simple graphs. A trail Q in a graph G is defined to be a *dominating trail* (*D-trail*) of G if every edge of G is incident with at least one vertex of Q. Equivalently, Q is a D-trail if $V(G)-V(Q)$ is an independent set of G.

In [7] Harary and Nash-Williams prove

Theorem A. *The edge graph $L(G)$ of a graph G is hamiltonian if and only if either G has a closed D-trail or G is isomorphic to $K_{1,s}$ for some $s \geq 3$.*

Our concern will be the existence of dominating cycles (D-cycles) and paths (D-paths) in graphs. Graphs containing a D-cycle (D-path) will be called *D-cyclic* (*D-traceable*). Since a D-cycle is a closed D-trail, we have

Corollary A.1. *If G is a D-cyclic graph, then $L(G)$ is hamiltonian.*

If a graph G contains a D-path P with endvertices u and v, then $G+uv$ has the D-cycle $P+uv$, so that $L(G+uv)$ contains a hamiltonian cycle. Since $L(G)$ is obtained from $L(G+uv)$ by deleting the vertex corresponding to the edge uv, $L(G)$ is traceable, i.e. $L(G)$ contains a hamiltonian path.

Corollary A.2. *If G is a D-traceable graph, then $L(G)$ is traceable.*

Recognizing D-cyclic graphs is an NP-complete problem. This is easily seen, using the NP-completeness of the hamiltonian problem.

In Sections 2 up to 6 necessary conditions and sufficient conditions are derived for the existence of D-cycles. The nature of the conditions is reflected by the section titles. There is a nice analogy with known results and proof techniques in hamiltonian graph theory. The conditions for D-cyclicity are weaker than the corresponding ones for hamiltonicity, in accordance with the fact that every hamiltonian graph is D-cyclic whereas the converse is not true. Analogous results on D-traceability of graphs are stated in Section 7.

2. Cut sets

We start with a necessary condition. Theorem 4.2 of [3] states that if a graph G is hamiltonian, then $\omega(G-S)\leqslant|S|$ for every nonempty proper subset S of $V(G)$. Hoede [8] showed that a D-cyclic graph satisfies a similar (but weaker) condition. Let $\omega_1(G)$ denote the number of components of G of order at least two.

Theorem 1. *If a graph G is D-cyclic, then, for every nonempty proper subset S of $V(G)$,*

$$\omega_1(G-S)\leqslant|S|.$$

Proof. Let S be a nonempty proper subset of $V(G)$ and C a D-cycle of G. Then every edge of $G-S$ is incident with a vertex of C. Thus

$$\omega_1(G-S)\leqslant|S\cap V(C)|\leqslant|S|. \qquad \square$$

3. Edge degrees

Ore's theorem [11] asserts that a graph G is hamiltonian if $d(u)+d(v)\geqslant\nu$ for every pair of nonadjacent vertices u and v of G. The existence of a D-cycle is guaranteed by an analogous condition, in which the notions 'edge degree' and 'remote edges' emerge in a natural way.

Two subgraphs H_1 and H_2 of a graph G are said to be *close* in G if they are disjoint and there is an edge of G joining a vertex of H_1 and one of H_2. If H_1 and H_2 are disjoint and not close, then H_1 and H_2 are *remote*. The *degree of an edge e* of G, denoted $d_G(e)$ or $d(e)$, is the number of vertices of G close to e (viewed as a subgraph of G of order two).

Theorem 2. *Let G be a graph other than a tree. If, for every pair of remote edges e and f of G,*

$$d(e)+d(f)\geqslant\nu-2,$$

then G is D-cyclic.

The proof is omitted, since all ingredients needed to prove Theorem 2 can be found in the proof of Theorem 3, stated below.

The condition of Theorem 2 is weaker than Ore's condition. To see this, let e_1 and e_2 be two remote edges of a graph G satisfying Ore's condition. If u_1 and u_2 are vertices incident with e_1 and e_2, respectively, then u_1 and u_2 are nonadjacent and $d(e_i) \geq d(u_i) - 1$ $(i = 1, 2)$. Thus

$$d(e_1) + d(e_2) \geq d(u_1) + d(u_2) - 2 \geq \nu - 2.$$

In the non-D-cyclic graph $K_1 \vee (K_2 + K_{\nu-3})$ $(\nu \geq 5)$ each pair of remote edges has degree-sum $\nu - 3$, showing that Theorem 2 is best possible.

For blocks of order at least 3 we now prove a more general result, using the following additional notation: if H is an oriented cycle or path in a graph and u and v are vertices of H, then $u\vec{H}v$ and $v\overleftarrow{H}u$ denote, respectively, the segment of H from u to v and the reverse segment from v to u.

Theorem 3. *Let G be a k-connected graph $(k \geq 2)$ such that, for every $k+1$ mutually remote edges $e_0, e_1, \ldots, e_k$ of G,*

$$\sum_{i=0}^{k} d(e_i) > \tfrac{1}{2}k(\nu - k).$$

Then G is D-cyclic.

Proof. By contraposition. Let G be a k-connected non-D-cyclic graph $(k \geq 2)$. We will find a set of $k+1$ mutually remote edges with degree-sum at most $\frac{1}{2}k(\nu - k)$.

Let C be a longest cycle among all cycles C' of G for which $\varepsilon(G - V(C'))$ is minimum. Fix an orientation on C. By Menger's theorem, two consecutive vertices w_1 and w_2 of C are connected by at least k internally-disjoint paths. One of these paths may be the edge w_1w_2; each of the other paths contains at least one vertex of C as an internal vertex, otherwise C could be enlarged to a cycle C_1 with $\varepsilon(G - V(C_1)) \leq \varepsilon(G - V(C))$ and $|V(C_1)| > |V(C)|$, contrary to hypothesis. Hence C has length at least $k+1$.

Let $u_{01}u_{02}$ be an edge of $G - V(C)$ and $\mathcal{P} = \{P_1, P_2, \ldots, P_m\}$ a collection of paths with the following properties:

(1) P_i has origin u_{01} and terminus on C $(i = 1, 2, \ldots, m)$,
(2) two distinct paths of $\mathcal{P}$ have only u_{01} in common,
(3) no internal vertex of P_i is on C $(i = 1, 2, \ldots, m)$,
(4) m is maximum, i.e. there exists no collection of more than m paths satisfying (1), (2) and (3).

Since $|V(C)| \geq k$, a variation on Menger's theorem asserts that $m \geq k$. Note that one of the paths in $\mathcal{P}$ may contain u_{02}. Let v_i be the terminus of P_i $(i = 1, 2, \ldots, m)$ and assume that $v_1, v_2, \ldots, v_m$ occur on C in the order of their indices. From the choice of C it follows that v_i and v_{i+1} are not consecutive vertices of C ($i=1,2,\ldots,m$, indices mod m). Define

$$V_1=\bigcup_{i=1}^{m} V(P_i)\cup V(C)\cup\{u_{02}\},\qquad V_2=V(G)-V_1$$

and let u_{i1} be the immediate successor of v_i on C ($i=1,2,\ldots,m$).

We make the following crucial observations:

(I) u_{i1} is neither adjacent to one of the vertices u_{01} and u_{02} nor to an internal vertex of one of the paths in $\mathcal{P}$ ($i=1,2,\ldots,m$). Assuming the contrary, v_i and u_{i1} would be connected by a path none of whose internal vertices is on C. Then C could be enlarged to a cycle C_2 satisfying $\varepsilon(G-V(C_2))<\varepsilon(G-V(C))$, contradicting the choice of C.

(II) u_{i1} and u_{j1} are nonadjacent ($i\neq j$). If $u_{i1}u_{j1}$ were an edge of G, then the cycle

$$C_3=v_i\overleftarrow{P}_iu_{01}\vec{P}_jv_j\overleftarrow{C}u_{i1}u_{j1}\vec{C}v_i$$

would contradict the choice of C, since $\varepsilon(G-V(C_3))<\varepsilon(G-V(C))$. For the same reason the neighbour sets $N(u_{i1})$ and $N(u_{j1})$ have no vertex of V_2 in common.

(III) If $N(u_{i1})\subset V(C)$, then u_{i1} and v_{i+1} are not consecutive vertices of C, otherwise the cycle $v_i\overleftarrow{P}_iu_{01}\vec{P}_{i+1}v_{i+1}\vec{C}v_i$ would contradict the way C was chosen.

If $N(u_{i1})\subset V(C)$, then define u_{i2} as the successor of u_{i1} on C; from (III) it follows that u_{i2} does not coincide with v_{i+1} ($i=1,2,\ldots,m$). If $N(u_{i1})\not\subset V(C)$, then $N(u_{i1})\cap V_2$ is nonempty by (I); in that case let u_{i2} be an arbitrary vertex of $N(u_{i1})\cap V_2$. Put $e_i=u_{i1}u_{i2}$ ($i=0,1,\ldots,m$) and $F=\{e_i\mid 0\leq i\leq m\}$. By (II), the edges in F are mutually nonadjacent. In fact, by the way C was chosen, they are mutually remote. Suppose, for example, that u_{i2} and u_{j2} were adjacent ($i<j$). Then, additionally assuming that u_{i2} is on C while u_{j2} is not, the cycle

$$C_4=v_i\overleftarrow{P}_iu_{01}\vec{P}_jv_j\overleftarrow{C}u_{i2}u_{j2}u_{j1}\vec{C}v_i$$

would satisfy $\varepsilon(G-V(C_4))<\varepsilon(G-V(C))$, thus contradicting the choice of C. The other nonadjacency conditions for e_i and e_j to be remote are checked similarly.

Let $U=\{u_{ij}\mid 1\leq i\leq m,\ 1\leq j\leq 2\}$. To every vertex $v\in V(C)\cup U$ we define the vertex $v^+\in V(C)\cup U$:

(i) If $v\in V(C)-U$, then v^+ is the successor of v on C.

(ii) If $v=u_{i1}$, then $v^+=u_{i2}$ ($i=1,2,\ldots,m$).

(iii) If $v=u_{i2}$ and $u_{i2}\in V(C)$, then v^+ is the successor of u_{i2} on C ($i=1,2,\ldots,m$).

(iv) If $v=u_{i2}$ and $u_{i2}\notin V(C)$, then v^+ is the successor of u_{i1} on C ($i=1,2,\ldots,m$).

We now determine an upper bound on $d(e_0)+d(e_i)+d(e_j)$, where e_i and e_j are arbitrary edges of $F-\{e_0\}$. Let S be the vertex set of the segment $u_{i1}\vec{C}v_j$ of C and define

$$A_1 = S \cup \{v^+ \mid v \in S - \{v_j\}\},$$
$$A_2 = (V(C) \cup U) - A_1,$$
$$A_3 = V(G) - (V(C) \cup U),$$
$$I_1 = \{v \in A_1 \mid e_i \text{ and } v^+ \text{ are close}\},$$
$$J_1 = \{v \in A_1 \mid e_j \text{ and } v \text{ are close}\},$$
$$I_2 = \{v \in A_2 \mid e_i \text{ and } v \text{ are close}\},$$
$$J_2 = \{v \in A_2 \mid e_j \text{ and } v^+ \text{ are close}\},$$
$$I_3 = \{v \in A_3 \mid e_i \text{ and } v \text{ are close}\},$$
$$J_3 = \{v \in A_3 \mid e_j \text{ and } v \text{ are close}\},$$
$$Z = \{v \in A_3 \mid e_0 \text{ and } v \text{ are close}\}.$$

From the choice of the collection $\mathscr{P}$ it follows that no vertex of $(V(C) \cup U) - \{v_1, v_2, \ldots, v_m\}$ is close to e_0. Thus

$$d(e_0) \leq |Z| + m.$$

Since $v \to v^+$ is a one-to-one correspondence of $V(C) \cup U$ onto itself, we have

$$d(e_i) = |I_1| + |I_2| + |I_3|,$$
$$d(e_j) = |J_1| + |J_2| + |J_3|.$$

The sets $I_1, J_1, I_2, J_2, I_3, J_3$ and Z are mutually disjoint. Again a contradiction with the choice of C arises if the opposite is assumed. As an illustration, suppose that $I_1 \cap J_1$ contains the vertex w, say that w^+ is adjacent to u_{i1} while w is adjacent to u_{j2}. Then, assuming $u_{j2} \in V(C)$, the cycle

$$C_5 = v_i \overleftarrow{P}_i u_{01} \overrightarrow{P}_j v_j \overleftarrow{C} w^+ u_{i1} \overleftarrow{C} w u_{j2} \overrightarrow{C} v_i$$

satisfies $\varepsilon(G - V(C_5)) < \varepsilon(G - V(C))$, a contradiction.

Since the vertices $u_{01}, u_{02}, u_{11}, u_{21}, \ldots, u_{m1}$ are in none of the above-mentioned sets, we have

$$d(e_0) + d(e_i) + d(e_j) \leq |Z| + m + |I_1| + |I_2| + |I_3| + |J_1| + |J_2| + |J_3|$$
$$\leq \nu - m - 2 + m = \nu - 2.$$

Furthermore $d(e_0) \geq m \geq k$, so that

$$d(e_i) + d(e_j) \leq \nu - k - 2.$$

The above inequalities hold for arbitrary i and j. Thus

$$(k-1) \sum_{i=1}^{k} d(e_i) + \binom{k}{2} d(e_0) = \sum_{1 \leq i < j \leq k} (d(e_0) + d(e_i) + d(e_j))$$
$$\leq \binom{k}{2} (\nu - 2) \qquad (*)$$

and

$$(k-1)\sum_{i=1}^{k} d(e_i) = \sum_{1\leq i<j\leq k} (d(e_i)+d(e_j)) \leq \binom{k}{2}(\nu-k-2),$$

or equivalently

$$\tfrac{1}{2}(k-1)(k-2)\sum_{i=1}^{k} d(e_i) \leq \frac{1}{2}\binom{k}{2}(k-2)(\nu-k-2). \qquad (**)$$

Summing $(*)$ and $(**)$ yields

$$\binom{k}{2}\sum_{i=0}^{k} d(e_i) \leq \binom{k}{2}(\nu-2+\tfrac{1}{2}(k-2)(\nu-k-2)),$$

so that

$$\sum_{i=0}^{k} d(e_i) \leq \tfrac{1}{2}k\,(\nu-k). \qquad \square$$

For each $k \geq 2$ there exists a non-D-cyclic k-connected graph containing a set of $k+1$ mutually remote edges with degree-sum exactly $[\frac{1}{2}k(\nu-k)]$. Denoting by nG the union of n disjoint copies of a graph G, the graph $K_k \vee (k+1)K_2$ has these properties. It seems unlikely that for each $k \geq 2$ there exist infinitely many such graphs. In fact, the following proposition, stronger than Theorem 3, might hold.

Conjecture 1. If G is a k-connected graph $(k \geq 2)$ such that, for every $k+1$ mutually remote edges $e_0, e_1, \ldots, e_k$ of G,

$$\sum_{i=0}^{k} d(e_i) > \tfrac{1}{3}(k+1)(\nu-2),$$

then G is D-cyclic.

Referring to the proof of Theorem 3, the truth of Conjecture 1 would be established if it was shown that the degree-sum of every three edges of F is at most $\nu-2$. Conjecture 1, if true, improves Theorem 3 for $\nu \geq 3k+2$. On the other hand it is implicit in the proof of Theorem 3 that every k-connected graph with less than $3k+2$ vertices is D-cyclic, so that we do not need Conjecture 1 for $\nu < 3k+2$. For each $k \geq 2$ the collection $\{K_t \vee (t+1)K_2 \mid t \geq k\}$ consists of infinitely many k-connected non-D-cyclic graphs having a set of $k+1$ mutually remote edges with degree-sum $\frac{1}{3}(k+1)(\nu-2)$. Thus Conjecture 1 would, in a sense, be best possible.

Although Theorem 3 is probably not the best one can do, it has a number of best possible corollaries. First note that for $k \geq 2$ a k-connected graph containing no set of $k+1$ mutually remote edges trivially satisfies the condition of Theorem 3. Similarly Theorem 2 trivially applies to connected graphs, other than trees, having no pair of remote edges. For a graph G, define $\iota(G)$ to be the maximum cardinality of a set of mutually remote edges of G.

Corollary 3.1. *Let G be a k-connected graph ($k \geq 1$) other than a tree. If $\iota \leq k$, then G is D-cyclic.*

For $k \geq 1$ and $\nu \geq 3k+2$ the k-connected graph $K_k \vee (kK_2 + K_{\nu-3k})$ is non-D-cyclic while

$$\iota(K_k \vee (kK_2 + K_{\nu-3k}) = k+1,$$

showing that the result is sharp.

Chvátal and Erdös [4] showed that a k-connected graph with independence number α is hamiltonian if $\alpha \leq k$. Corollary 3.1 is analogous to this result. Since every graph satisfies $\iota \leq \alpha$, the condition of Corollary 3.1 is met whenever the condition of Chvátal and Erdös is.

For $k = 2$ Theorem 3 and Conjecture 1 coincide in

Corollary 3.2. *Let G be a 2-connected graph. If the degree-sum of every three mutually remote edges of G is at least $\nu - 1$, then G is D-cyclic.*

Since it is a special case of Conjecture 1, Corollary 3.2 is best possible. There even exist extremal graphs for all $\nu \geq 8$: in the non-D-cyclic graph $K_2 \vee (2K_2 + K_{\nu-6})$ all triples of mutually remote edges have degree-sum $\nu - 2$.

Corollary 3.2 enables us to prove a result more general than the following, due to Nash-Williams [9].

Theorem B. *Let G be a 2-connected graph. If $\delta \geq \max(\alpha, \frac{1}{3}(\nu+2))$, then G is hamiltonian.*

Implicit in the proof of Theorem B is

Theorem C. *Let G be a 2-connected graph. If $\delta \geq \frac{1}{3}(\nu+2)$, then every longest cycle of G is a D-cycle.*

The connection between Theorems B and C is expressed by

Lemma 1. *If G is a D-cyclic graph such that $\delta \geq \alpha$, then G is hamiltonian.*

Proof. Let C be a longest D-cycle of G. Assuming that C is not a hamiltonian cycle, there exists a vertex w not on C. Since C is a D-cycle, all neighbours of w are on C. Let $v_1, v_2, \ldots, v_{d(w)}$ be the vertices of C adjacent to w and denote by u_i the immediate successor of v_i on C ($i = 1, 2, \ldots, d(w)$), C being arbitrarily oriented. Now $I = \{w, u_1, \ldots, u_{d(w)}\}$ is an independent set, otherwise there would exist a D-cycle longer than C. But then $\alpha \geq |I| = d(w) + 1 \geq \delta + 1$, a contradiction. □

Bondy [2] improved Theorem C to

Theorem D. *Let G be a 2-connected graph. If the degree-sum of every three independent vertices is at least $\nu+2$, then every longest cycle of G is a D-cycle.*

Combination of Lemma 1 and Theorem D yields a parallel improvement of Theorem B. However, a more general result is obtained by combining Lemma 1 with Corollary 3.2, since the condition of Corollary 3.2 in turn is weaker than Bondy's condition.

Corollary 3.2.1. *Let G be a 2-connected graph such that the degree-sum of every three mutually remote edges is at least $\nu-1$. If $\delta \geqslant \alpha$, then G is hamiltonian.*

Note that the condition of Corollary 3.2 does not guarantee that every longest cycle of G is a D-cycle. However, in proving Corollary 3.2.1 this is irrelevant.

4. Size

A consequence of Theorem 2 is

Corollary 2.1. *If a graph G has at least $\binom{\nu-2}{2}+4$ edges, then G is D-cyclic. Moreover, the only non-D-cyclic graph with ν vertices and $\binom{\nu-2}{2}+3$ edges is $K_1 \vee (K_2 + K_{\nu-3})$ $(\nu \geqslant 5)$.*

Proof. By contraposition. Let G be a non-D-cyclic graph. If G is a tree, then $\varepsilon = \nu - 1 \leqslant \binom{\nu-2}{2}+3$. Assuming now that G is not a tree, it follows from Theorem 2 that G contains two remote edges e and f with degree-sum at most $\nu-3$. Thus

$$\varepsilon \leqslant \binom{\nu-4}{2} + 2(d(e)+d(f))+2$$

$$\leqslant \binom{\nu-4}{2} + 2(\nu-3)+2 = \binom{\nu-2}{2}+3.$$

It is easily checked that there exist no disconnected non-D-cyclic graphs with $\binom{\nu-2}{2}+3$ edges and $K_1 \vee (K_2 + K_{\nu-3})$ is the only extremal graph of order ν that is connected and has a cut vertex $(\nu \geqslant 5)$. As shown later on (Corollary 3.3.1), 2-connected non-D-cyclic graphs of order ν have at most $\binom{\nu-4}{2}+10$ edges. The proof is completed by noting that

$$\binom{\nu-4}{2}+10<\binom{\nu-2}{2}+3 \quad \text{for} \quad \nu\geqslant 8,$$

while for $\nu<8$ there exist no 2-connected non-D-cyclic graphs of order ν. □

The first part of Corollary 2.1 can also be deduced from Bondy's result [1] that a graph G of order ν and size at least $\binom{\nu-2}{2}+4$ has a cycle of length $\nu-1$, since cycles of length $\nu-1$ are D-cycles of G.

From Theorem 3 one deduces

Corollary 3.3. *Let G be a k-connected graph ($k\geqslant 2$). If*

$$\varepsilon\geqslant\binom{\nu-2k}{2}+(k-2)(\nu-k+3)+11,$$

then G is D-cyclic.

The proof is omitted, since it is analogous to the proof of the first part of Corollary 2.1.

Probably the result is best possible only for $k=2$; it then reads

Corollary 3.3.1. *If G is a 2-connected graph with $\varepsilon\geqslant\binom{\nu-4}{2}+11$, then G is D-cyclic.*

For $\nu\geqslant 8$ the graph $K_2\vee(2K_2+K_{\nu-6})$ is non-D-cyclic and has $\binom{\nu-4}{2}+10$ edges, showing that Corollary 3.3.1 is sharp.

5. Forbidden subgraphs

A number of results in hamiltonian graph theory assert that a graph is hamiltonian if it does not contain certain subgraphs or induced subgraphs. As an example, Goodman and Hedetniemi prove in [6] that a 2-connected graph is hamiltonian if it contains no induced subgraph isomorphic to either $K_{1,3}$ or $K_{1,3}+e$. Denoting by $\tau_1, \tau_2, \tau_3, \tau_4$ the graphs depicted in Fig. 1 we deduce a similar result on D-cyclic graphs.

Theorem 4. *Let G be a connected graph, other than a tree, containing no induced subgraph isomorphic to τ_3 or τ_4. If, moreover, at most one of the graphs τ_1 and τ_2 is an induced subgraph of G, then G is D-cyclic.*

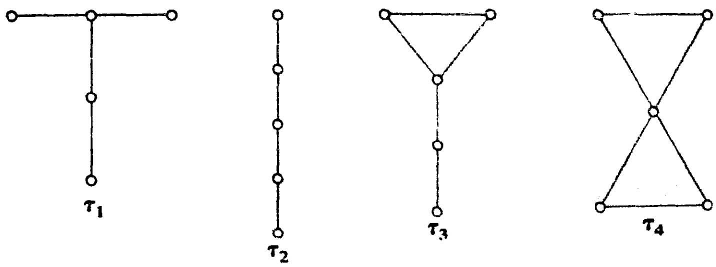

Fig. 1.

Proof. By contraposition. Let G be a connected non-D-cyclic graph other than a tree and C a cycle of G for which $\varepsilon(G-V(C))$ is minimum. Then there exists an edge $e=u_1u_2$ such that $u_1, u_2 \notin V(C)$ and one of the vertices incident with e, say u_2, is adjacent to a vertex v of C. Fixing an orientation on C, let w_1 and w_2 be the immediate predecessor and successor, respectively, of v on C. By the way C was chosen the edge e and the vertex w_1 are remote; so are e and w_2.

Let w_3 be the successor of w_2 on C if $N(w_2) \subset V(C)$ and an arbitrary vertex of $N(w_2) \cap V(G-V(C))$ otherwise. Note that w_3 may coincide with w_1. Putting $f=w_2w_3$, the edges e and f are remote by the choice of C.

Denoting by H_1 and H_2 the subgraphs of G induced by $\{u_1, u_2, v, w_1, w_2\}$ and $\{u_1, u_2, v, w_2, w_3\}$, respectively, it follows that H_2 is isomorphic to one of the graphs τ_2, τ_3 and τ_4. If G contains neither τ_3 nor τ_4 as an induced subgraph, then $H_1 \cong \tau_1$ and $H_2 \cong \tau_2$, completing the proof. □

Note that the condition of Theorem 4 is weaker than Goodman and Hedetniemi's condition, since the graphs τ_1, τ_3 and τ_4 all contain either $K_{1,3}$ or $K_{1,3}+e$ as an induced subgraph.

Within the class of 2-connected graphs a weaker condition is sufficient for the existence of a D-cycle. Denote by T the graph of Fig. 2 and write $G_1 \leq G_2$ if G_1 is a spanning subgraph of G_2.

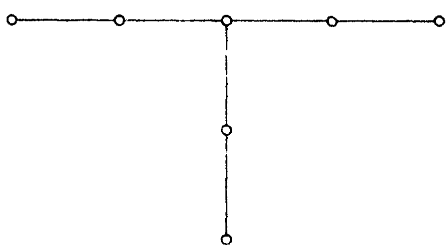

Fig. 2. T.

Theorem 5. *If a 2-connected graph G contains no element of $\mathcal{T}=\{H \mid T \leq H \leq K_1 \vee (K_2+K_4)\}$ as an induced subgraph, then G is D-cyclic.*

The proof resembles that of Theorem 4 and is omitted. Referring to the proof of Theorem 4, the essential difference is that in a 2-connected non-D-cyclic graph the cycle C can be chosen to have length at least 5.

Theorem 5 together with Corollary A.1 yield a sufficient condition for hamiltonicity of edge graphs.

Corollary 5.1. *If the edge graph $L(G)$ of a 2-connected graph G contains no induced subgraph isomorphic to $L(T)$, then $L(G)$ is hamiltonian.*

Proof. Let G be a 2-connected graph and suppose that $L(G)$ is nonhamiltonian. Then Corollary A.1 implies that G is non-D-cyclic. Consequently, G contains T as a subgraph. Thus $L(G)$ contains $L(T)$ as an induced subgraph. □

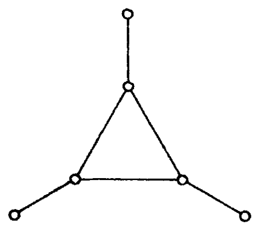

Fig. 3. $L(T)$.

Since the edge graph of a 2-connected graph is 2-connected and no edge graph contains an induced $K_{1,3}$, Corollary 5.1 is a special case of the following result, recently proved by Duffus, Gould and Jacobson [5].

Theorem E. *A 2-connected graph containing no induced subgraph isomorphic to $K_{1,3}$ or $L(T)$ is hamiltonian.*

Note that Theorem E sharpens Goodman and Hedetniemi's result.

6. Contractibility

We state two results in terms of contractibility. Only one of them is proved, since their proof involve the same arguments. In describing series of contractions and graphs resulting from them we omit loops and identify multiple edges whenever they occur.

Theorem 6. *Let G be a connected graph other than a tree. If G is not contractible to τ_2, τ_3 or τ_4, then G is D-cyclic.*

D-cyclicity of 2-connected graphs is guaranteed by a weaker condition. Let Θ be the graph of Fig. 4.

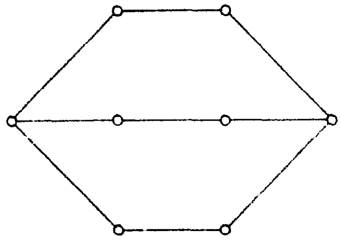

Fig. 4. Θ.

Theorem 7. *If a 2-connected graph G is not contractible to an element of* $\mathscr{A}=\{H \mid \Theta \leq H \leq K_2 \vee 3K_2\}$, *then G is D-cyclic.*

Proof. By contraposition. In a 2-connected non-D-cyclic graph G, let C be a cycle for which $\varepsilon(G-V(C))$ is minimum and let $e_0=u_{01}u_{02}$ be an edge of $G-V(C)$. Fix an orientation on C.

From Menger's theorem it is easily deduced that e_0 belongs to a path P that connects two vertices v_1 and v_2 of C and is internally-disjoint from C. The choice of C implies that v_1 and v_2 are not consecutive vertices of C.

Let u_{11} and u_{21} be the successors on C of v_1 and v_2, respectively and denote by S the set of vertices of G not belonging to C or P. As in the proof of Theorem 3, the vertex u_{i2} is defined to be the successor of u_{i1} on C if $N(u_{i1}) \subset V(C)$, while otherwise u_{i2} is arbitrarily selected from $N(u_{i1}) \cap S$ $(i=1,2)$. Putting $e_i=u_{i1}u_{i2}$ $(i=1,2)$ it follows that the edges of $F=\{e_0, e_1, e_2\}$ are mutually remote. Hence, if x and y are vertices incident with distinct edges of F, then x and y are nonadjacent. Similarly x and y are not connected by a path that is internally-disjoint from $C \cup P$. The rest of the proof is based on this essential remark.

If $u_{i2} \in S$, then, since G is 2-connected, there exists a path P_i, internally-disjoint from C, connecting u_{i2} to a vertex of C other than u_{i1} $(i=1,2)$; by the way C was chosen, no vertex of P_i is incident with e_{3-i} and P_i is internally-disjoint from P. Moreover, if both u_{12} and u_{22} belong to S, then P_1 and P_2 are internally-disjoint on account of the choice of C.

Let H be the subgraph induced by $V(C) \cup V(P) \cup V(P_1) \cup V(P_2)$, where $\emptyset$ is to be substituted for $V(P_i)$ if u_{i2} is on C $(i=1,2)$. Denote by m the number of components of $G-V(H)$.

We carry out the following series of contractions.

First the components of $G-V(H)$ are contracted to single vertices $c_1, c_2, \ldots, c_m$. Since G is 2-connected, each c_i has degree ≥ 2 in the resulting graph G_1. Furthermore it follows from the choice of C that at most one of the edges of F is incident with vertices of $N_{G_1}(c_i)$ $(i=1,2,\ldots,m)$. If $N_{G_1}(c_i)=\{u_{j1}, u_{j2}\}$ for some $j \in \{0,1,2\}$, then c_i is contracted onto one of the vertices u_{j1} and u_{j2}; otherwise c_i is contracted onto a vertex not incident with one of the edges of F $(i=1,2,\ldots,m)$. In the graph resulting from this first set of contractions the edges of F are still mutually remote.

Next all edges of the path P not incident with u_{01} or u_{02} are contracted. If $u_{i2} \in S$, then the edges of P_i not incident with u_{i2} are also contracted ($i = 1, 2$). Again no two edges of F are close in the resulting graph.

Finally all edges of C not equal or adjacent to e_1 or e_2 are contracted. Denoting the resulting graph by G_2, the graph Θ is a spanning subgraph of G_2. Since the edges of F are still mutually remote, it follows that $G_2 \leq K_2 \vee 3K_2$, completing the proof. □

Note that none of the graphs in $\mathscr{A} \cup \{\tau_2, \tau_3, \tau_4\}$ satisfies the necessary condition of Theorem 1.

Again there is a nice analogy with a theorem in hamiltonian graph theory. In [8] Hoede and Veldman proved that a 2-connected graph is hamiltonian if it is not contractible to one of the graphs θ and θ^* depicted in Fig. 5. As expected, the

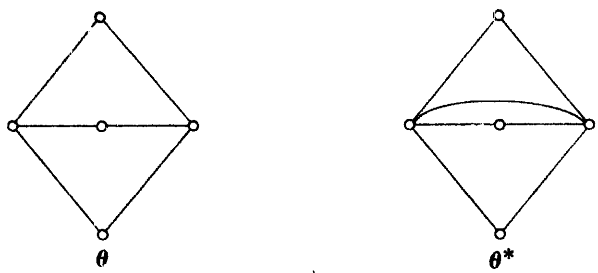

Fig. 5.

condition of Theorem 7 is weaker than this condition, since all graphs in $\mathscr{A}$ are contractible to θ or to θ^*. A related analogy is the one between the graph invariants α and ι, as emerged in Section 3. This may be clarified by noting that

$$\{\theta, \theta^*\} = \{H \mid \theta \leq H \leq K_2 \vee 3K_1\}.$$

Without proof we mention that Theorem 7 is more general than Theorem 5. Since, in a way, Theorem 6 and Theorem 4 parallel Theorem 7 and Theorem 5, respectively, one might expect Theorem 6 to be more general than Theorem 4. This is, however, not true: the D-cyclic graph of Fig. 6(a) satisfies the condition of Theorem 4, whereas the condition of Theorem 6 is not met; on the other hand, Theorem 6 applies to the graph of Fig. 6(b) while Theorem 4 does not.

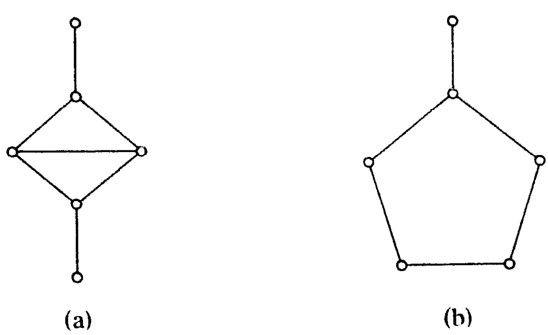

Fig. 6.

7. D-paths

The conditions for D-cyclicity derived in preceding sections have analogues for D-traceability. We start with an obvious lemma.

Lemma 2. *A graph G is D-traceable if and only if $G \vee K_1$ is D-cyclic.*

Most of the results stated below can be proved by combining the corresponding results on D-cyclicity with Lemma 2.

Theorem 8. *If a graph G is D-traceable, then, for every non-empty proper subset S of V(G),*

$$\omega_1(G-S) \leq |S|+1.$$

Theorem 9. *Let G be a k-connected graph ($k \geq 1$) such that, for every $k+2$ mutually remote edges $e_0, e_1, \ldots, e_{k+1}$ of G,*

$$\sum_{i=0}^{k+1} d(e_i) > \frac{1}{2}(k+1)(\nu-k-2)-1.$$

Then G is D-traceable.

Theorem 9 might be improved to

Conjecture 2. If G is a k-connected graph ($k \geq 1$) such that, for every $k+2$ mutually remote edges $e_0, e_1, \ldots, e_{k+1}$ of G,

$$\sum_{i=0}^{k+1} d(e_i) > \frac{1}{3}(k+2)(\nu-4),$$

then G is D-traceable.

For $k=1$ Theorem 9 and Conjecture 2 coincide in

Corollary 9.1. *Let G be a connected graph. If the degree-sum of every three mutually remote edges of G is at least $\nu-3$, then G is D-traceable.*

Corollary 9.2. *Let G be a k-connected graph ($k \geq 1$). If $\iota \leq k+1$, then G is D-traceable.*

Corollary 9.3. *Let G be a k-connected graph ($k \geq 1$). If*

$$\varepsilon \geq \binom{\nu-2k-2}{2}+(k-1)(\nu-k+1)+7,$$

then G is D-traceable.

This result is probably sharp only for $k=1$. It then becomes

Corollary 9.3.1. *A connected graph with* $\varepsilon \geq \binom{\nu-4}{2}+7$ *is D-traceable.*

Theorem 10. *If a connected graph G contains no element of* $\mathcal{T}$ *as an induced subgraph, then G is D-traceable.*

Theorem 11. *If a connected graph G is not contractible to an element of* $\{H \mid T \leq H \leq K_1 \vee 3K_2\}$, *then G is D-traceable.*

Theorem 11 is more general than Theorem 10.

As a final remark, sufficient conditions for the existence of Hamilton paths can be obtained by combining sufficient conditions for D-traceability with

Lemma 3. *If G is a D-traceable graph with* $\delta \geq \alpha - 1$, *then G is traceable.*

Note added in proof

If G is a graph other than a tree and

$$\min\{d(e) \mid e \in E(G)\} \geq \frac{1}{2}\nu - 1,$$

then $L(G)$ is hamiltonian. For $G \not\cong K_{1,\nu-1}$ this consequence of Corollary A.1 and Theorem 2 generalizes Theorem 2 of a recent paper by Brualdi and Shanny (J. Graph Theory 5 (1981) 307–314).

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