OPTIMAL LINEAR DECISION RULES FOR A WORK FORCE SMOOTHING MODEL WITH
SEASONAL DEMAND

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This paper gives a procedure for the determination of the optimal linear control for a
production-inventory system in steady-state with non-linear cost per period and normally
distributed seasonal demand. In an example this control policy is compared with other
policies.

1. Introduction.

It is very hard, if not impossible, to ob-
tain by using dynamic programming the optimal
policy for a stochastic linear production-in-
ventory system. Therefore simplifications are
necessary. Mostly in practical situations a
certainty-equivalence policy is adopted.
Another approach is that developed by Holt,
Modigliani, Muth and Simon (HMS) 14 1. They
approximate the cost functions by quadratics
and next solve the simplified problem which
results in a linear control.

It is also possible a priori to choose a
linear control which is optimized hereafter
to produce the optimal linear control.
Schneeweiss 111 and Inderfurth 121 exten-
sively studied a stochastic linear production-
inventory problem: the cash balancing pro-
blem, a model with one state and one decision
variable. The authors developed and compared
several policies for solving this problem,
among others the optimal policy, the certainty-
equivalence policy and the optimal linear
policy. It turned out that (for the case
without set-up costs) the certainty-equa-
lence approach is inferior to the optimal li-
near approach. The optimal linear policy can
also be easily determined in the case of
complex demand (input) processes (Gaalman 31).
Therefore one may conclude that this policy
should seriously be taken into consideration.

This conclusion however primarily holds
for the cash balancing model. It is interest-
ing to know whether this conclusion also
holds for more involved stochastic models.
Schneeweiss and Inderfurth already discussed
the cash balancing problem and the pure in-
ventory problem with set-up costs and

concluded that in many cases, depending on
the relative size of the cost parameters and
the variance of the demand, the optimal li-
near policy can still be successfully applied.
In section 3 of this paper the stochastic
work force smoothing model of HMS, a model
with two state and two decision variables,
is treated. Moreover - in contrast with the
constant demand in the earlier mentioned
studies - the stochastic demand is chosen
to have a deterministic seasonal variation
which tremendously increases the size of
the problem. For this large model the qua-
dratic approximation approach of HMS, the
certainty-equivalence approach and the opti-
mal linear approach are compared in section
4 using the cost data of HMS. First in sec-
tion 2 the determination of the optimal linear
control for the general stochastic linear pro-
duction-inventory system with seasonal input
is treated.

2. The optimal linear control for a linear
production-inventory system with seaso-
nal stochastic demand.

This section deals with the determination
of the optimal linear control for the general
linear production-inventory system with
seasonal stochastic input and some simpli-
fying assumptions.

2.1. The linear production-inventory system
with seasonal stochastic input.

Consider the linear discrete-time stochas-
tic difference equation:
\[ x_{t+1} = Ax_t + Bu_t + Cz_{t+1}; \quad t = 1, 2, \ldots \quad (1) \]

where

- \( x_t \) - \( n \) dimensional state vector at time \( t \)
- \( u_t \) - \( m \) dimensional control vector at time \( t \)
- \( z_t \) - \( k \) dimensional normally distributed stochastic input in period \((t-1,t)\)
- \( A \) - \( n \times n \) matrix
- \( B \) - \( n \times m \) matrix
- \( C \) - \( n \times k \) matrix

This system can be seen as a general linear production-inventory system with \( k \) dimensional input (demand). Restrictions on both the state and the control variables are absent. This means that not all (discrete-time) production-inventory systems can be described by (1). However, for some models (including aggregate planning models and cash balancing models) this assumption is not too restrictive.

The costs \( J_t \) in period \((t,t+1)\) associated with this system are usually a non-linear function of \( x_{t+1} \) and \( u_t \). It will be assumed that these costs can be separated into costs associated with the state and costs associated with the control variables. This assumption can generally be met in practice.

The contributions of Schneeweiss, Indefurth and Gaalman concentrate on the optimal linear control of the above-mentioned systems where the stochastic input has a constant mean value. In some cases this assumption is too restrictive.

For example, the aggregate production planning models of the HEMS-type have generally seasonal demands. This paper shows the consequences of a seasonal input.

2.2 Simplifying assumptions.

Throughout this paper the following simplifying assumptions are made:

(i) The stochastic input is normally distributed with

\[ z_t = z_t + E_t; \quad t = 1, 2, \ldots \quad (2) \]

where

\[ z_t = z_t \mod \tau; \quad t = 1, 2, \ldots, \tau \quad (\tau \text{ is cycle length}) \]

\( E_t \) is an uncorrelated normally distributed zero-mean variable with variance matrix

\[ E_t \mod \tau; \quad t = 1, 2, \ldots, \tau \]

For more involved normally distributed input processes a procedure used by Gaalman \( [3] \) can be adopted.

(ii) The system is studied after an infinite operating time \((t \to \infty)\) and moreover the system is in steady state. This assumption corresponds with that of Schneeweiss \( [1] \) and generally means a considerable reduction of the problem size because the variables possess now quasi stationary probability distributions with cycle length \( \tau \). (The consequence of a non-steady state approach is discussed in \( [3]/ \).

Considering the steady state we can split system (1) into:

a. a deterministic part:

\[ x_{j+1} = Ax_j + Bu_j + Cz_j; \quad j = 1, 2, \ldots, \tau \quad (3) \]

where

\[ x_j = E(x_j), \quad u_j = E(u_j) \]

b. a stochastic part:

\[ x_{j+1} = Ax_j + Bu_j + Cz_{j+1} \quad (4) \]

where

\[ x_j = x_j - x_j; \quad u_j = u_j - u_j \]

2.3 The optimal linear control.

We adopt the following (suboptimal) linear control policy

\[ u_j = u_j + u_j; \quad j = 1, 2, \ldots, \tau \quad (5) \]

where

\[ u_j = u_j \mod \tau; \quad F_j = F_j \mod \tau \quad \text{and } \quad F_j \text{ is a } m \times n \text{ matrix} \]

This policy maintains the cyclic character of the system. The linearity of this policy and the normality of the demand cause the normality of the variables \( x_j \) and \( u_j \) with mean values \( x_j \) and \( u_j \) and with variance matrices \( V_j \) and \( V_j \). The relation between the mean values is described by (3). The relation between the variance matrices can be obtained from the stochastic part (4) and the feedback part of (5):
\[ V_{j+1} = (A - BF_j) V_j (A - BF_j)^T + CE_j + j \]
\[ j = 1, 2, \ldots, \tau \]  
\[ V_{T+1} = V_1 \]  
\[ V_{uu} = F_j V_j F_j^T \]  
\[ \text{where} \]

\[ T \text{ means transposed} \]

\[ V_j = E(x^o_j \ x^o_j^T), \ V_{uu} = E(u^o_j u^o_j^T) \]  

The expected value of the costs per cycle - if the system is in steady state -

\[ J = \sum_{j=1}^{\tau} E(J_j) \]  

is now a non-linear function of the mean values and variance matrices:

\[ J = J(x, V, u), V_{uu} = E(x u) \]  

So by the introduction of the linear control policy (5) the stochastic optimization problem of minimizing (8) subject to (1) in the steady state is converted into a deterministic optimization problem of minimizing (9) subject to (3) and (6). From this problem the optimal linear control results.

It is possible to derive necessary conditions which might be used to determine the optimal \( F_j \) and \( u_j \). Since (9) is not always continuously differentiable the optimal \( F_j \) and \( u_j \) will be determined by means of a search method which does not use gradients. The method used for solving the problem in section 3 is a pattern search routine written by W.M. Taubert [7].

The values of \( F_j \) and \( u_j \) should be chosen in such a way that the begin/end conditions of (6) and (3), which assure the cyclic performance, are always satisfied. The variance matrix \( V_{uu} \) can be expressed, by successively substituting of \( V_j \), as a function of \( V_j \). From this relation the so-called Lyapunov-equation:

\[ V_{T+1} = V_1 D V_1 D^T + CE_1 C^T \]  

where

\[ D = G_0 (A - BF_j) \]

results. This equation has a solution if \( D \) is asymptotically stable. If the pair \( (A, B) \) is stabilizable one can always find such \( D \). Equation (10) can be solved numerically. The other \( V_{*,*} \) follow from (6).

Analogously \( x_{T+1} \) can be expressed as a function of \( x_j \) giving:

\[ x_{T+1} = (A - I) x_T + \sum_{j=0}^{\tau} A (B u - C) \]  

This relation consists of \( n \) equations with \( n + m \) variables. If \( I - A \) is invertible \( x \), can be expressed as a function of the \( u_j \)'s. If not, the stabilizability of the pair \( (A, B) \) guarantees that always \( n \) variables can be expressed as a function of \( n \) remaining variables.

3. An example: the HMMS work force smoothing model.

A well known example of a linear production-inventory system is the work force smoothing model of a paint company developed by Holt, Modigliani, Muth and Simon [4]. In this company many items are produced and kept in stock. The model is on an aggregate level, individual variables are added to one variable using a common unit. The (aggregate) demand exhibits a seasonal character. HMMS approximated the relevant variable costs by quadratics and were, by this, able to derive a linear decision rule. They also reported cost savings of this rule with respect to the original company decisions.

The HMMS-model consists of two linear difference equations, an inventory balance equation and a work force equation:

\[ i_{t+1} = i_t + p_t - d_t, \quad i_1 = i_1 \]  

\[ w_{t+1} = w_t + r_t, \quad w_1 = w_1 \]  

where

\[ i_t \quad \text{net inventory state variable at the end of period (t, t+1) and \( i_1 \) given,} \]

\[ p_t \quad \text{production decision variable at time \( t, \)} \]

\[ r_t \quad \text{hiring and layoff decision variable at time \( t, \)} \]

\[ w_t \quad \text{work force state variable at the end of period (t, t+1) and \( w_1 \) given,} \]

\[ d_t \quad \text{stochastic demand in period (t, t+1).} \]
HVMS assume that the changing manpower decision \( r_t \) at \( t \) is immediately realized. By this the fork force \( w_{t+1} \) is available during period \((t, t+1)\), upon which the production \( p \) can be based. The production variable consists of two terms, the regular production and the overtime-idle time production:

\[
p_t = b w_{t+1} + a_t
\]

where

- \( b \) - average worker productivity per period,
- \( a_t \) - overtime and idle time decision variable at time \( t \).

The general linear difference equation (1) corresponds now with the above equations (12)-(12') if:

\[
A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 \\ 0 \end{pmatrix}
\]

\[
x_t = \begin{pmatrix} i_t \\ w_t \end{pmatrix}, \quad u_t = \begin{pmatrix} a_t \\ r_t \end{pmatrix}, \quad z_t = d_t
\]

and \( n = m = 2, \ k = 1 \).

The variable costs in period \((t, t+1)\) are

(i) regular payroll costs, \( C_{w,t+1} = gw_{t+1} \),

(ii) overtime costs,

\[
C_{a,t} = \begin{cases} fa_t, & \text{if } a_t \geq 0 \\
0, & \text{if } a_t < 0
\end{cases}
\]

(iii) hiring and layoff costs,

\[
C_{r,t} = \begin{cases} pr_t, & \text{if } r_t \geq 0 \\
-q r_t, & \text{if } r_t < 0
\end{cases}
\]

(iv) net inventory costs, \( C_{i,t+1} = \)

\[
\begin{cases}
  h_2(i^2_{t+1} - i^2_t - i^{2'}_t), & \text{if } i_{t+1} \geq 0 \\
  h_1(i^2_{t+1} - i^2_t - i^{2'}_t), & \text{if } i_{t+1} < 0
\end{cases}
\]

Note: The form of the net inventory costs (iv) is not reported by HVMS. They give only the approximated quadratic inventory costs.

The total costs in period \((t, t+1)\) amount to:

\[
J_t = C_{w,t+1} + C_{i,t+1} + C_{a,t} + C_{r,t} \tag{13}
\]

The expected costs per cycle are

\[
J' = \frac{1}{T} \sum_{j=1}^{T} gw_j + \left( \sqrt{h_j} + v_j \right) \sqrt{\text{I}} (a_j \phi(a_j) + \phi(a_j)) + \phi(a_j) + \phi(a_j) + \phi(a_j) + \phi(a_j) + \phi(a_j) + \phi(a_j)
\]

\[
+ (v_2 - v_1) \sqrt{\text{I}} (a_j \phi(a_j) + \phi(a_j))
\]

\[
+ (v_2 - v_1) \sqrt{\text{I}} (a_j \phi(a_j) + \phi(a_j)) + \phi(a_j)
\]

\[
+ (p+q) \sqrt{\text{I}} (a_j \phi(a_j) + \phi(a_j)) - qr_j
\]

where

\[
\phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} \exp(-1/2a^2) da, \quad \phi(a) = d\phi(a)/da,
\]

\[
a_1 = (i^{2'}_t - i^2_t)/\sqrt{\text{I}}, \quad a_2 = (i^{2'}_t - i^2_t)/\sqrt{\text{I}}, \quad a_3 = (i^{2'}_t - i^2_t)/\sqrt{\text{I}}, \quad a_4 = a_t/\sqrt{\text{I}}, \quad a_5 = r_t/\sqrt{\text{I}}
\]
The cost coefficients given by HMMs for the paint company are used:

\[ g = 340.2, \quad p = 180, \quad q = 360, \quad f = 90. \]

The data of the piece-wise linear net inventory costs (iv) are calculated in [5] using the same data as HMMs:

\[ h_1 = 3.1, \quad h_2 = 20, \quad v_1 = 26.7, \quad v_2 = 69.9, \]

\[ i^1 = 362, \quad i^2 = 701, \quad i^3 = 234. \]

Moreover, it is assumed that the demand is seasonal with \( \tau = 12, \{d_{j+1}\} = \{770, 696.7, 623.3, 476.7, 403.3, 330, 403.3, 476.7, 550, 623.3, 696.7\} \) and the, in each period constant, variance \( \sigma^2 = 12100. \) Using the pattern search procedure of Taubert the optimal \( F_j \) and \( u_j \) are determined. The results are given in table 1. The payroll cost to satisfy the mean demand \( (\sum_{j=1}^{\tau} d_j) \) are subtracted. From this we can conclude that the net inventory shows a remarkable seasonal pattern. Work force changes are relatively small. Overtime is only performed in the first two periods when the peak in the demand occurs. Idle time is not used.

<table>
<thead>
<tr>
<th>( j )</th>
<th>( d_j )</th>
<th>( i'_j )</th>
<th>( w_j )</th>
<th>( a_j )</th>
<th>( r_j )</th>
<th>( F_j )</th>
<th>( v_{ii} )</th>
<th>( v_{ww} )</th>
<th>( v_{ir} )</th>
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<td>1</td>
<td>696.7</td>
<td>579.9</td>
<td>107.0</td>
<td>63.7</td>
<td>0.1</td>
<td>0.43</td>
<td>4.72</td>
<td>27.6</td>
<td>61.7</td>
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<td>2</td>
<td>770</td>
<td>480.8</td>
<td>107.1</td>
<td>42.1</td>
<td>-0.5</td>
<td>0.34</td>
<td>2.88</td>
<td>19.8</td>
<td>3.5</td>
</tr>
<tr>
<td>3</td>
<td>696.7</td>
<td>430.3</td>
<td>106.5</td>
<td>0.1</td>
<td>-5.4</td>
<td>0.00</td>
<td>0.01</td>
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<td>52.2</td>
</tr>
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<td>4</td>
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<td>380.8</td>
<td>101.2</td>
<td>-0.0</td>
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<td>0.00</td>
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<td>550</td>
<td>370.5</td>
<td>95.2</td>
<td>-0.0</td>
<td>-7.8</td>
<td>0.00</td>
<td>0.00</td>
<td>20.9</td>
<td>87.6</td>
</tr>
<tr>
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<td>389.2</td>
<td>87.4</td>
<td>0.0</td>
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<td>70.5</td>
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<tr>
<td>7</td>
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<td>468.1</td>
<td>85.1</td>
<td>-0.1</td>
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<td>0.00</td>
<td>0.00</td>
<td>26.8</td>
<td>52.3</td>
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<tr>
<td>8</td>
<td>330</td>
<td>613.9</td>
<td>83.9</td>
<td>0.0</td>
<td>0.8</td>
<td>0.00</td>
<td>0.00</td>
<td>26.8</td>
<td>52.3</td>
</tr>
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<td>9</td>
<td>403.3</td>
<td>691.0</td>
<td>84.7</td>
<td>0.0</td>
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<td>0.00</td>
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<tr>
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<td>104.5</td>
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<td>2.5</td>
<td>0.00</td>
<td>0.00</td>
<td>27.4</td>
<td>72.0</td>
</tr>
</tbody>
</table>

Optimal linear decision rule results with costs \( J^* = \sum_{j=1}^{\tau} d_j = 40360. \)

Table 1.
Expected costs per cycle for:

(0) - the optimal linear decision rule
(i) - the optimal time-invariant linear decision rule
(ii) - the linear decision rule of HMMs

Table 2.

(i) The HMMs-policy. HMMs approximate the cost functions by quadratics and then derive the optimal decision rule. This rule appears also to be a linear decision rule with the structure:

\[ u_t = -Fx_t + \sum_{\kappa \leq t} \hat{d}_{t+\kappa} + c \]

where \( \hat{d}_{t+\kappa} \) is the conditional expectation of \( d_{t+\kappa} \) at \( t \). For the given data, the values of \( F, \gamma_0 \) and \( c \) can be obtained from HMMs [4]. However, through a better quadratic approximation of the net inventory costs this rule was improved in [5] which changes only \( c \). By applying the latter rule on (3), (6) and (7) for the given model it is not difficult to find analytically \( x, u_j, V_j \) for \( j=1,2,\ldots,12 \) and finally \( J \). These costs are shown in column (ii) of table 2. Note: In fact it is not completely fair to compare this decision rule with the other ones because this rule is derived for the original paint company data while in our case the demand data are different.

Also in table 2 the CPU-times, consumed to obtain the reported results, are given. Of course the given times for (0) and (i) depend on the starting solution for the optimization procedure.

(iii) A certainty-equivalence policy proposed by Thomas and McClain [6]. A brief introduction:

a) First the steady state deterministic problem \((d_j=d; j=1,2,\ldots,T \text{ and } E^0=0)\) for one cycle is written as an LP-problem.

b) At time \( t \), the initial state \((i_0,w_0)^T\) is known. A horizon length \( \Delta \) is chosen (\( \Delta=5 \) appears to be suitable in this case). The demand \( d_j \) during the horizon is considered to be equal to \( d_j \).

For the end state condition the corresponding stationary value is used (see a). Next the LP-problem is solved and the decisions \( a_t \) and \( r_t \) are carried out.

c) This procedure is repeated at time \( t+1 \), etc.

Since in this case it is not possible to calculate the expected costs per cycle in the steady state we rely on simulation. The certainty-equivalence rule is compared with the optimal linear decision rule. A simulation run of 396x12 periods with a CPU-time of 25 minutes gave the results depicted in table 3.

<table>
<thead>
<tr>
<th>(iii)</th>
<th>(iv)</th>
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</thead>
<tbody>
<tr>
<td>( J_t )</td>
<td>39781</td>
</tr>
<tr>
<td>95% confidence intervals</td>
<td>(37403, (-832,589))</td>
</tr>
</tbody>
</table>

Estimated costs per cycle and confidence intervals for:

(iii) - the certainty-equivalence rule
(iv) - the certainty-equivalence rule minus the optimal linear decision rule.

Table 3.

We can now conclude with a 95% confidence that the two policies differ with respect to the costs per cycle no more than 2%. A more concrete result can be obtained by longer simulation runs. However a four times longer run reduces the confidence interval only by a factor two.

Note: The certainty-equivalence policy described above can be improved by introducing a safety-stock.

Comparing the different policies we notice that the optimal linear decision rule and the certainty-equivalence policy give the best results with respect to the costs per cycle, which are about the same for both policies.
However the determination of the first one is very time consuming which is due to both the high number of variables which have to be optimized (72) and the flat objective function near the optimum. This latter appears from the fact that the value of
\[
j^{*} = \arg\min_{j} 60 d(j)
\]
found after 2\(\frac{1}{2}\) minutes CPU-time differs only 4% from the value given in table 2, found after 15 minutes. The CPU-time (= 3 sec) consumed for obtaining the certainty-equivalence policy is much shorter. This also holds for the determination of the results of the time-invariant optimal linear decision rule, but this one involves 4% higher expected costs per cycle than those for the optimal linear decision rule. The CPU-time needed to determine the HMM-policy results is very short, but on the other hand the expected costs per cycle are 30% higher than those for the optimal linear decision rule.

5. Conclusions.

It is demonstrated that the determination of the optimal linear decision rule for the treated HMM-model, where the stochastic demand is chosen to have a seasonal pattern, is very time consuming because of the large number of variables to be optimized. For this model we found that, in contrast with more simple models, it seems more reasonable to adopt alternative rules which need less time to derive them and involve even about the same or only somewhat higher costs. Of course it is not possible to conclude from this study of a single model that this is generally true for complex production-inventory systems. The outcome of the comparisons between the different policies will depend on several factors: e.g. relative sizes of the cost coefficients and the variances of the demands. So other models need to be investigated to yield general conclusions. Also it may be possible to reduce - by using other optimization methods - the CPU-time consumed to determine the optimal linear decision rule. Whether the CPU-time of the certainty-equivalence policy can be approximated is doubtful.

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