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OPTIMAL LINEAR DECISION RULES FOR A WORK FORCE SMOOTHING MODEL WITH SEASONAL DEMAND

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This paper gives a procedure for the determination of the optimal linear control for a production-inventory system in steady-state with non-linear costs per period and normally distributed seasonal demand. In an example this control policy is compared with other policies.

1. Introduction.

It is very hard, if not impossible, to obtain by using dynamic programming the optimal policy for a stochastic linear production-inventory system. Therefore simplifications are necessary. Mostly in practical situations a certainty-equivalence policy is adopted. Another approach is that developed by Holt, Modigliani, Muth and Simon (HMMS) |4|. They approximate the cost functions by quadratics and next solve the simplified problem which results in a linear control.

It is also possible a priori to choose a linear control which is optimized hereafter to produce the optimal linear control. Schneeweiss |1| and Inderfurth |2| extensively studied a stochastic linear production-inventory problem: the cash balancing problem, a model with one state and one decision variable. The authors developed and compared several policies for solving this problem, among others the optimal policy, the certainty-equivalence policy and the optimal linear policy. It turned out that (for the case without set-up costs) the certainty-equivalence approach is inferior to the optimal linear approach. The optimal linear policy can also be easily determined in the case of complex demand (input) processes (Gaalman |3|). Therefore one may conclude that this policy should seriously be taken into consideration.
This conclusion however primarily holds for the cash balancing model. It is interesting to know whether this conclusion also holds for more involved stochastic models. Schneeweiss and Inderfurth already discussed the cash balancing problem and the pure inventory problem with set-up costs and concluded that in many cases, depending on the relative size of the cost parameters and the variance of the demand, the optimal linear policy can still be successfully applied. In section 3 of this paper the stochastic work force smoothing model of HMMS, a model with two state and two decision variables, is treated. Moreover - in contrast with the constant demand in the earlier mentioned studies - the stochastic demand is chosen to have a deterministic seasonal variation which tremendously increases the size of the problem. For this large model the quadratic approximation approach of HMMS, the certainty-equivalence approach and the optimal linear approach are compared in section 4 using the cost data of HMMS. First in section 2 the determination of the optimal linear control for the general stochastic linear production-inventory system with seasonal input is treated.

2. The optimal linear control for a linear production-inventory system with seasonal stochastic demand.

This section deals with the determination of the optimal linear control for the general linear production-inventory system with seasonal stochastic input and some simplifying assumptions.

2.1 The linear production-inventory system with seasonal stochastic input.

Consider the linear discrete-time stochastic difference equation:

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$$x_{t+1}=Ax_t+Bu_t+Cz_{t+1}; \quad t=1,2,... \qquad (1)$$

where

- x_t - n dimensional state vector at time t where x_1 is known
- u_t - m dimensional control vector at time t
- z_t - k dimensional normally distributed stochastic input in period (t-1,t)
- A - n x n matrix
- B - n x m matrix
- C - n x k matrix

This system can be seen as a general linear production-inventory system with k dimensional input (demand). Restrictions on both the state and the control variables are absent. This means that not all (discrete-time) production-inventory systems can be described by (1). However, for some models (including aggregate planning models and cash balancing models) this assumption is not too restrictive.

The costs J_t in period (t,t+1) associated with this system are usually a non-linear function of x_{t+1} and u_t. It will be assumed that these costs can be separated in costs associated with the state and costs associated with the control variables. This assumption can generally be met in practice.

The contributions of Schneeweiss, Inderfurth and Gaalman concentrate on the optimal linear control of the above-mentioned system where the stochastic input has a constant mean value. In some cases this assumption is too restrictive.
For example, the aggregate production planning models of the HMMS-type have generally seasonal demands. This paper shows the consequences of a seasonal input.

2.2 Simplifying assumptions.

Throughout this paper the following simplifying assumptions are made:

(i) The stochastic input is normally distributed with

$$z_t=z'_t+\varepsilon_t \qquad ; \ t=1,2,... \qquad (2)$$

where

$$z'_t=z'_{t \bmod \tau}=E(z_{t \bmod \tau}); \ t=1,2,...,\tau$$

(τ is cycle length)

ε_t is an uncorrelated normally distributed zero-mean variable with variance matrix $E^0_{t \bmod \tau}$; $t=1,2,...,\tau$

For more involved normally distributed input processes a procedure used by Gaalman |3| can be adopted.

(ii)The system is studied after an infinite operating time ($t\to\infty$) and moreover the system is in steady state. This assumption corresponds with that of Schneeweiss |1| and generally means a considerable reduction of the problem size because the variables possess now quasi stationary probability distributions with cycle length τ.(The consequence of a non-steady state approach is discussed in |3'|).

Considering the steady state we can split system (1) into:

a. a deterministic part:

$$x'_{j+1}=Ax'_j+Bu'_j+C'_{,+1} \ ; \ j=1,2,...,\tau \qquad (3)$$

where

$$x'_j=E(x_j), \ u'_j=E(u_j)$$

b. a stochastic part:

$$x^0_{j+1}=Ax^0_j+Bu^0_j+C\varepsilon_{j+1} \qquad (4)$$

where

$$x^0_j=x_j-x'_j; \ u^0_j=u_j-u'_j$$

2.3 The optimal linear control.

We adopt the following (suboptimal) linear control policy

$$u_j=u'_j+u^0_j=u'_j-F_jx^0_j \ ; \ j=1,2,...,\tau \qquad (5)$$

where

$u'_j=u'_{j \bmod \tau}$, $F_j=F_{j \bmod \tau}$ and F_j is a mxn matrix

This policy maintains the cyclic character of the system. The linearity of this policy and the normality of the demand cause the normality of the variables x_j and u_j with mean values x_j and u_j and with variance matrices V_j and V^{uu}_j. The relation between the mean values is described by (3). The relation between the variance matrices can be obtained from the stochastic part (4) and the feedback part of (5):

$$V_{j+1}=(A-BF_j)V_j(A-BF_j)^T+CE^o_{j+1}C^T;$$

$$j=1,2,\ldots,\tau \qquad (6)$$

$$V_{\tau+1}=V_1$$

$$V^{uu}_j=F_jV_jF^T_j \qquad (7)$$

where

T means transposed

$$V_j=E(x^o_j\,x^{o\,T}_j)\ ,\ V^{uu}_j=E(u^o_ju^{o\,T}_j)$$

The expected value of the costs per cycle - if the system is in steady state -

$$J'_\tau=\sum_{j=1}^{\tau} E\{J_j\} \qquad (8)$$

is now a non-linear function of the mean values and variance matrices :

$$J'_\tau=J_\tau(x'_j,V_j,u'_j,V^{uu}_j,\ j=1,\ldots,\tau) \qquad (9)$$

So by the introduction of the linear control policy (5) the stochastic optimization problem of minimizing (8) subject to (1) in the steady state is converted into a deterministic optimization problem of minimizing (9) subject to (3) and (6). From this problem the optimal linear control results, that is to say optimal values of F_j and u'_j are obtained.

It is possible to derive necessary conditions which might be used to determine the optimal F_j and u'_j. Since (9) is not always continuously differentiable the optimal F_j and u'_j will be determined by means of a search method which does not use gradients. The method used for solving the problem in section 3. is a pattern search routine written by W.M. Taubert |7|.

The values of F_j and u'_j should be choosen in such a way that the begin/end conditions of (6) and (3), which assure the cyclic performance, are always satisfied. The variance matrix $V_{\tau+1}$ can be expressed, by successively substituting of V_j, as a function of V_1. From this relation the so-called Lyapunov-equation:

$$V_{\tau+1}=V_1=D_1V_1D_1^T+\sum_{\ell=2}^{\tau}D_\ell CE^o_\ell(D_\ell C)^T+CE^o_1C^T \qquad (10)$$

where

$$D_\ell=\prod_{\theta=0}^{\tau-\ell}(A-BF_{\tau-\theta})$$

results. This equation has a solutio.. if D_1 is asymptotically stable. If the pair {A,B} is stabilizable one can always find such D_1. Equation (10) can be solved numerically. The other V_j's follow from (6). Analogously $x'_{\tau+1}$ can be expressed as a function of x'_1 giving:

$$x'_{\tau+1}=x'_1=A^\tau x'_1+\sum_{\theta=0}^{\tau-1}A^\theta(Bu'_{\tau-\theta}+Cz'_{\tau+1-\theta}) \qquad (11)$$

This relation consists of n equations with n+τm variables. If $I-A^\tau$ is invertable x'_1 can be expressed as a function of the u'_j's. If not, the stabilizability of the pair {A,B} guarantees that always n variables can be expressed as a function of τm remaining variables.

3. An example: the HMMS work force smoothing model.

A well known example of a linear production-inventory system is the work force smoothing model of a paint company developed by Holt, Modigliani, Muth and Simon |4| . In this company many items are produced and kept in stock. The model is on an aggregate level. Individual variables are added to one variable using a common unit. The (aggregate) demand exhibits a seasonal character. HMMS approximated the relevant variable costs by quadratics and were, by this, able to derive a linear decision rule. They also reported cost savings of this rule with respect to the original company decisions.

The HMMS-model consists of two linear difference equations, an inv balance equation and a work force eq

$$i_{t+1}=i_t+p_t-d_{t+1},\quad i_1=\bar{i}_1 \qquad t=1,2,\ldots \qquad (12)$$

$$w_{t+1}=w_t+r_t,\quad w_1=\bar{w}_1$$

where

- i_{t+1} -net inventory state variable at the end of period (t,t+1) and i_1 given,
- p_t -production decision variable at time t,
- r_t -hiring and layoff decision variable at time t,
- w_{t+1} -work force state variable at the end of period (t,t+1) and w_1 given,
- d_{t+1} -stochastic demand in period (t,t+1).

HMMS assume that the changing manpower decision r_t at t is immediately realized. By this the work force w_{t+1} is available during period (t,t+1), upon which the production p_t can be based. The production variable consists of two terms, the regular production and the overtime-idle time production:

$$p_t=bw_{t+1}+a_t \qquad (12')$$

where

b – average worker productivity per period,

a_t – overtime and idle time decision variable at time t.

The general linear difference equation (1) corresponds now with the above equations (12)-(12') if:

$$A=\begin{pmatrix}1 & b\\0 & 1\end{pmatrix},\ B=\begin{pmatrix}1 & b\\0 & 1\end{pmatrix},\ C=\begin{pmatrix}-1\\0\end{pmatrix}$$

$$x_t=\begin{pmatrix}i_t\\w_t\end{pmatrix},\ u_t=\begin{pmatrix}a_t\\r_t\end{pmatrix},\ z_t=d_t$$

and n=m=2, k=1.

The variable costs in period (t,t+1) are

(i) regular payroll costs, $C_{w,t+1}=gw_{t+1}$,

(ii) overtime costs ,

$$C_{a,t}=\begin{cases}fa_t, & \text{if } a_t\geq o\\ o, & \text{if } a_t<o\end{cases}$$

(iii) hiring and layoff costs,

$$C_{r,t}=\begin{cases}pr_t, & \text{if } r_t\geq o\\ -qr_t, & \text{if } r_t<o\end{cases}$$

(iv) net inventory costs, $C_{i,t+1}=$

$$\begin{cases}h_2(i_{t+1}-\hat{\imath}^2)+h_1(\hat{\imath}^2-\hat{\imath}^1), & \text{if } i_{t+1}\geq\hat{\imath}^2\\ h_1(i_{t+1}-\hat{\imath}^1), & \text{if } \hat{\imath}^1\leq i_{t+1}<\hat{\imath}^2\\ -v_1(i_{t+1}-\hat{\imath}^1), & \text{if } \hat{\imath}^3\leq i_{t+1}<\hat{\imath}^1\\ -v_2(i_{t+1}-\hat{\imath}^3)-v_1(\hat{\imath}^2-\hat{\imath}^1), & \text{if } i_{t+1}<\hat{\imath}^3\end{cases}$$

Note: The form of the net inventory costs (iv) is not reported by HMMS. They give only the approximated quadratic inventory costs.

The total costs in period (t,t+1) amount to:

$$J_t=C_{w,t+1}+C_{i,t+1}+C_{a,t}+C_{r,t} \qquad (13)$$

The expected costs per cycle are

$$\begin{aligned}J'_\tau=\sum_{j=1}^{\tau} gw'_j+\{&(h_1+v_1)\sqrt{V_j^{ii}}(\alpha_1\Phi(\alpha_1)+\phi(\alpha_1))+\\ &+(h_2-h_1)\sqrt{V_j^{ii}}(\alpha_2\Phi(\alpha_2)+\phi(\alpha_2))+\\ &+(v_2-v_1)\sqrt{V_j^{ii}}(\alpha_3\Phi(\alpha_3)+\phi(\alpha_3))+\\ &-v_2(i'_t-\hat{\imath}^1)+(v_1-v_2)(\hat{\imath}^1-\hat{\imath}^3)\}+\\ &+f\sqrt{V_j^{aa}}(\alpha_4\Phi(\alpha_4)+\phi(\alpha_4))+\\ &+(p+q)\sqrt{V_j^{rr}}(\alpha_5\Phi(\alpha_5)+\phi(\alpha_5))-qr'_j\end{aligned} \qquad (14)$$

where

$$\Phi(\alpha)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha}\exp(-1/2\theta^2)d\theta,\ \phi(\alpha)=d\Phi(\alpha)/d\alpha,$$

$$\alpha_1=(i'_t-\hat{\imath}^1)/\sqrt{V_j^{ii}},\ \alpha_2=(i'_t-\hat{\imath}^2)/\sqrt{V_j^{ii}},$$

$$\alpha_3=(i'_t-\hat{\imath}^3)/\sqrt{V_j^{ii}},\ \alpha_4=a'_t/\sqrt{V_j^{aa}},\ \alpha_5=r'_t/\sqrt{V_j^{rr}}$$

The cost coefficients given by HMMS for the paint company are used:

$g=340.2$, $p=180$, $q=360$, $f=90$.

The data of the piecewise linear net inventory costs (iv) are calculated in |5| using the same data as HMMS:

$h_1=3.1$, $h_2=20$, $v_1=26.7$, $v_2=69.9$,

$\hat{i}^1=362$, $\hat{i}^2=701$, $\hat{i}^3=234$.

Moreover, it is assumed that the demand is seasonal with $\tau=12, \{d_{j+1}\}=\{770, 696.7, 623.3, 550, 476.7, 403.3, 330, 403.3, 476.7, 550, 623.3, 696.7\}$ and the, in each period constant, variance $E^o=12100$. Using the pattern search procedure of Taubert the optimal F_j and u_j' are determined. The results are given in table 1. The payroll cost to satisfy the mean demand ($\sum_{j=1}^{\tau} \frac{g}{b} d_j'$) are substracted from J_τ'.
From this we can conclude that the net inventory shows a remarkable seasonal pattern. Work force changes are relatively small. Overtime is only performed in the first two periods when the peak in the demand occurs. Idle time is not used.

j	d_j'	i_j'	w_j'	a_j'	r_j'	F_j		V_j^{ii} $(x10^3)$	V_j^{ww}	V_j^{ii} $(x10^2)$	V_j^{rr}
1	696.7	579.9	107.0	63.7	0.1	0.43	4.72	27.6	61.7	30.3	[illegible]
						0.00	0.06				
2	770	480.8	107.1	42.1	-0.5	0.34	2.88	19.8	3.5	18.5	
						0.01	0.20				
3	696.7	430.3	106.5	0.1	-5.4	0.00	0.01	18.2	52.2	0.0	50.6
						0.05	0.61				
4	623.3	380.8	101.2	-0.0	-6.0	0.00	0.00	20.5	68.0	0.0	34.4
						0.05	0.59				
5	550	370.5	95.2	-0.0	-7.8	0.00	0.00	20.9	87.6	0.0	41.6
						0.05	0.76				
6	476.7	389.2	87.4	0.0	-2.3	0.00	0.00	21.9	70.5	0.0	14.1
						0.03	0.52				
7	403.3	468.1	85.1	-0.1	-1.2	0.00	0.00	24.2	57.4	0.0	10.3
						0.02	0.43				
8	330	613.9	83.9	0.0	0.8	0.00	0.00	26.8	52.3	0.0	5.5
						0.02	0.27				
9	403.3	691.0	84.7	0.0	1.6	0.00	0.00	29.6	54.0	0.0	3.8
						0.01	0.18				
10	476.7	703.7	86.3	0.0	10.2	0.00	0.00	32.8	56.2	0.0	50.6
						0.04	0.66				
11	550	700.8	96.5	0.0	8.0	0.00	0.00	29.2	86.5	0.0	28.1
						0.04	0.78				
12	623.3	669.8	104.5	0.1	2.5	0.00	0.00	27.4	72.0	0.0	11.1
						0.03	0.53				

Optimal linear decision rule results with costs $J_\tau' - \sum_{j=1}^{\tau} 60d_j' = 40360$.

Table 1.

4. Comparison with alternative policies.

To determine how good this linear decision rule is the results (i.e. the expected costs per cycle) can be compared with the results of alternative policies.
For this purpose the following three policies are chosen:

(i) A time-invariant linear control. Instead of the linear control of section 3 the time-invariant linear control, $F_j=F$, $j=1,\ldots,\tau$ is adopted. By this a considerable reduction in the number of independent variables is obtained; $(\tau-1)nm=44$. The expected costs per cycle are calculated using the same procedure. The result is given in column (i) of table 2.

	(0)	(i)	(ii)
$J'_\tau - \sum_{j=1}^{\tau} 60d'_j$	40360	41942	52579
%	100	104	130
CPU-times (approx.)	15 min.	5 min.	1 sec.

Expected costs per cycle for:

(0) - the optimal linear decision rule
(i) - the optimal time-invariant linear decision rule
(ii)- the linear decision rule of HMMS

Table 2.

(ii) The HMMS-policy. HMMS approximate the cost functions by quadratics and then derive the optimal decision rule. This rule appears also to be a linear decision rule with the structure:

$$u_t = -Fx_t + \sum_{\theta=1}^{\infty} \gamma_\theta \hat{d}_{t+\theta,t} + c$$

where $\hat{d}_{t+\theta,t}$ is the conditional expectation of $d_{t+\theta}$ at t. For the given data, the values of F, γ_θ and c can be obtained from HMMS |4|. However, through a better quadratic approximation of the net inventory costs this rule was improved in |5| which changes only c. By applying the latter rule on (3), (6) and (7) for the given model it is not difficult to find analytically x', u', V_j, V^{uu}_j for j=1,2,...,12 and finally J'_τ. These costs are shown in column (ii) of table 2. Note: In fact it is not completely fair to compare this decision rule with the other ones because this rule is derived for the original paint company data while in our case the demand data are different.

Also in table 2 the CPU-times, consumed to obtain the reported results, are given. Of course the given times for (0) and (i) depend on the starting solution for the optimization procedure.

(iii) A certainty-equivalence policy proposed by Thomas and McClain |6| . A brief introduction:

a) First the steady state deterministic problem ($d_j=d'_j$; j=1,2,...,τ and $E^0=0$) for one cycle is written as an LP-problem.

b) At time t, the initial state $(i_t,w_t)^T$ is known. A horizon length Δ is chosen (Δ=5 appears to be suitable in this case). The demand d_t during the horizon is considered to be equal to d'_t. For the end state condition the corresponding stationary value is used (see a). Next the LP-problem is solved and the decisions a_t and r_t are carried out.

c) This procedure is repeated at time t+1, etc.

Since in this case it is not possible to calculate the expected costs per cycle in the steady state we rely on simulation. The certainty-equivalence rule is compared with the optimal linear decision rule. A simulation run of 396x12 periods with a CPU-time of 25 minutes gave the results depicted in table 3.

	(iii)	(iv)
$J'_\tau - \sum_{j=1}^{\tau} 60d'_j$	39781	-122
95% confidence interval	(37403, 42158)	(-832,589)

Estimated costs per cycle and confidence intervals for:

(iii) - the certainty-equivalence rule
(iv) - the certainty-equivalence rule minus the optimal linear decision rule.

Table 3.

We can now conclude with a 95% confidence that the two policies differ with respect to the costs per cycle no more than 2%. A more concrete result can be obtained by longer simulation runs. However a four times longer run reduces the confidence interval only by a factor two.
Note: The certainty-equivalence policy described above can be improved by introducing a safety-stock.

Comparing the different policies we notice that the optimal linear decision rule and the certainty-equivalence policy give the best results with respect to the costs per cycle, which are about the same for both policies.

However the determination of the first one is very time consuming which is due to both the high number of variables which have to be optimized (72) and the flat objective function near the optimum. This latter appears from the fact that the value of $J'_{\tau}-\sum_{j=1}^{12} 60d'_j$ found after 2½ minutes CPU-time differs only 4% from the value given in table 2, found after 15 minutes. The CPU-time (= 3 sec) consumed for obtaining the certainty-equivalence policy is much shorter. This also holds for the determination of the results of the time-invariant optimal linear decision rule, but this one involves 4% higher expected costs per cycle than those for the optimal linear decision rule. The CPU-time needed to determine the HMMS-policy results is very short, but on the other hand the expected costs per cycle are 30% higher than those for the optimal linear decision rule.

5. Conclusions.

It is demonstrated that the determination of the optimal linear decision rule for the treated HMMS-model, where the stochastic demand is chosen to have a seasonal pattern, is very time consuming because of the large number of variables to be optimized. For this model we found that, in contrast with more simple models, it seems more reasonable to adopt alternative rules which need less time to derive them and involve even about the same or only somewhat higher costs. Of course it is not possible to conclude from this study of a single model that this is generally true for complex production-inventory systems. The outcome of the comparisons between the different policies will depend on several factors: e.g. relative sizes of the cost coefficients and the variances of the demands. So other models need to be investigated to yield general conclusions. Also it may be possible to reduce - by using other optimization methods- the CPU-time consumed to determine the optimal linear decision rule. Whether the CPU-time of the certainty-equivalence policy can be approximated is doubtful.

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