The analysis of a conveyor-serviced production station

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This paper gives a queuing analysis of a conveyor-serviced production station using a state-dependent sequential range policy for unloading units from the conveyor into a reserve. The stationary distribution of the number of units in the reserve and the expected delay per unit processed are derived for a Poisson arrival process. The form of the optimal unloading policy, which minimizes the expected delay, will be established.

1. Introduction

Conveyor theory has been principally concerned with two distinctive types of systems: the automatically unloaded, multi-server systems and the worker-loaded–unloaded systems. (For a survey of Conveyor Theory, see White and Muth [6].) In the latter type of system the unloading of units from the conveyor and the processing of units at the work-station are executed by the same person, consequently production is delayed during unloading activity. In a number of articles probabilistic models have been developed for the single worker-unloading production station, see e.g. [1–5]. Research is focused on the development of efficient unloading policies which should minimize unloading delay and consequently increase the productivity of the work station.

In this paper a new unloading policy is introduced and an analysis of this policy is given. This policy is more general then the state-dependent policy investigated by Matsui and Shingu [4], both policies being generalisations of the 'Sequential Range Policy', introduced by Beightler and Crisp [1].

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1.1. Description of the model

Units are transported by means of an irreversible, continuous belt conveyor, moving with constant speed. The units are unloaded from the conveyor by an operator, at a production station situated at some fixed point along the conveyor and stored in a storage, referred to as reserve. After the unloading, the same operator takes one unit from the reserve and performs the necessary operations on the unit. Processed units leave the system. The cycle of the operator's productive activity can be divided into two distinct, sequential operations, i.e., the unloading operation and the actual productive operation. The length of the unloading operation will be referred to as 'delay per unit processed'. All units arriving during the processing of a unit are lost.

The unloading of units, using the newly proposed strategy, proceeds as follows: upon terminating the processing of a unit, the operator turns to the conveyor and inspects a range of c_i time units, i.e., that part of the conveyor that will pass the station within a time c_i, if the number of units in the reservebank equals i. If the range contains one or more units the operator delays and unloads the first unit to arrive and stores it in the reserve after which a new range of length c_{i+1} is inspected, beginning this range at the arrival of the unit just obtained. This procedure continues until either the reserve is filled or a range is inspected that does not contain a unit. The operator then returns to work.

If, upon terminating a processing, the operator finds the reserve empty, he delays until a unit arrives, i.e., $c_0 = \infty$, after which inspection starts with a range c_1.

For a graphical representation of a typical productive cycle see Fig. 1.

The following assumptions are made:

The arrivals of units at the unloading point constitutes a stationary Poisson process with arrivalrate λ.

All handling and walkingtimes are negligible.

The reserve has a finite capacity $K + 1$, i.e., $C_{K+1} = 0$.

The operator's sight along the conveyor, measured in time units, is at least equal to the maximal range.

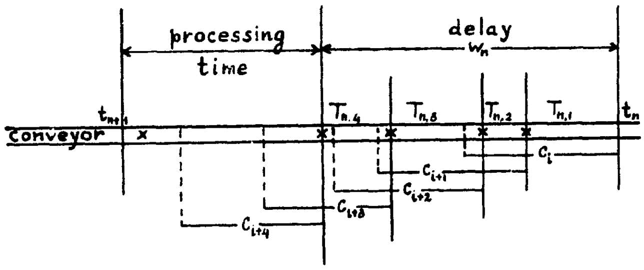

Fig. 1. A typical productive cycle.

Moreover, it will be supposed that the processing-times are always greater than or equal to $\max(c_1, ..., c_K)$. This assumption guarantees that the time interval between the end of a processing and the subsequent arrival of a unit is negative exponentially distributed with mean $1/\lambda$.

2. The content of the reserve

Let the successive completions of processing occur at the moments t_n, $n \geqslant 0$. By x_n, we denote the number of units in the reserve at the time $t_n + 0$. By $u(x_n)$ we denote the number of units unloaded between t_n and t_{n+1}, given the value of x_n. Clearly

$$x_{n+1} = x_n + u(x_n) - 1 , \quad n \geqslant 0 . \tag{2.1}$$

It follows from the assumptions made that the sequence $\{x_n, n \geqslant 0\}$ forms a Markov chain with finite state space $S = \{0, 1, ..., K\}$.

Denoting by $q_{ij} = \Pr\{x_{n+1} = j \mid x_n = i\}$ the one-step transition probabilities it can be verified that

$$\begin{aligned} q_{ij} &= e^{-\lambda c_{j+1}} \prod_{k=i}^{j} (1 - e^{-\lambda c_k}) && \text{for } 1 \leqslant i \leqslant K , \quad i-1 \leqslant j \leqslant K-1 , \\ &= \prod_{k=i}^{K} (1 - e^{-\lambda c_k}) && \text{for } 1 \leqslant i \leqslant K , \quad j = K , \\ &= e^{-\lambda c_{j+1}} \prod_{k=1}^{j} (1 - e^{-\lambda c_k}) && \text{for } i = 0 , \quad 0 \leqslant j \leqslant K-1 , \\ &= \prod_{k=1}^{K} (1 - e^{-\lambda c_k}) && \text{for } i = 0 , \quad j = K , \\ &= 0 && \text{elsewhere,} \end{aligned} \tag{2.2}$$

in which the convention is used that $\Pi_1^0(\cdot) = 1$.

From (2.2) it is seen that the chain is time-homogeneous, aperiodic and irreducible. So the chain $\{x_n, n \geqslant 0\}$, having a finite state space, is ergodic. Consequently, it possesses a unique stationary distribution $\{v_i, i = 0, 1, ..., K\}$, which is the solution of the equations

$$v_j = \sum_{i=0}^{K} v_i q_{ij} , \quad j = 0, 1, ..., K , \qquad \text{and} \qquad \sum_{j=0}^{K} v_j = 1 . \tag{2.3}$$

On substituting (2.2) in (2.3) it can be shown that the set of equations (2.3) is equivalent to

$$\begin{aligned} v_1 &= e^{\lambda c_1}(1 - e^{-\lambda c_1})\, v_0 , \\ v_{j+1} &= e^{\lambda c_{j+1}}\, v_j - e^{\lambda c_j}(1 - e^{-\lambda c_j})\, v_{j-1} , \\ &\qquad j = 1, 2, ..., K-1 , \\ \sum_{i=0}^{K} v_i &= 1 . \end{aligned} \tag{2.4}$$

It can be readily verified that the solution is given by

$$v_j = v_0 \prod_{i=1}^{j} (e^{\lambda c_i} - 1) , \quad j = 1, 2, ..., K ,$$

$$v_0 = 1 \Bigg/ \left\{ 1 + \sum_{j=1}^{K} \prod_{i=1}^{j} (e^{\lambda c_i} - 1) \right\} . \tag{2.5}$$

Note that in case $c_i = c$, $i = 1, 2, ..., K$ the model of Crisp [1], [2] is obtained, except that here the arrival process is Poisson instead of Bernoulli. In this case the set of equations (2.4) describes a classical random walk with two reflecting barriers.

From (2.1) it follows, noting that $E\{x_n\}$ tends to a finite limit, that

$$\lim_{n \to \infty} E\{u(x_n)\} = 1 , \tag{2.6}$$

i.e., the expected number of unloaded units during a productive cycle equals one in the stationary state, as indeed it must.

3. The mathematical expectation of the delay per unit processed

Let w_n be defined as the delay of the operator in removing units from the conveyor between t_n and

t_{n+1}, i.e., the length of the unloading operation during the nth productive cycle. The random variable w_n is composed of the successive interarrival times during the unloading period, denoted by $T_{n,i}$, $i = 1, 2, \ldots$.

In Fig. 1 a realisation of a productive cycle, starting with i units in the reserve, is shown.

By assumption it follows for $k = 1, 2, \ldots, j - i + 1$ and $0 \leqslant i \leqslant j \leqslant K$ that

$$\Pr\{T_{n,k} \leqslant t \mid x_n = i, x_{n+1} = j\} = \frac{1 - e^{-\lambda t}}{1 - e^{-\lambda c_{i+k-1}}}, \quad 0 \leqslant t \leqslant c_{i+k-1}, \qquad = 1 \quad t > c_{i+k-1}, \tag{3.1}$$

and

$$E\{T_{n,k} \mid x_n = i, x_{n+1} = j\} = \frac{1}{\lambda} - c_{i+k-1} \frac{e^{-\lambda c_{i+k-1}}}{1 - e^{-\lambda c_{i+k-1}}}. \tag{3.2}$$

By definition, we have

$$E\{w_n \mid x_n = i, x_{n+1} = j\} = \sum_{k=1}^{j-i+1} E\{T_{n,k} \mid x_n = i, x_{n+1} = j\} \quad \text{for } 0 \leqslant i \leqslant j \leqslant K.$$

It then follows from (3.2) that

$$E\{w_n \mid x_n = i, x_{n+1} = j\} = \sum_{k=i}^{j} \left[\frac{1}{\lambda} - c_k \frac{e^{-\lambda c_k}}{1 - e^{-\lambda c_k}}\right], \quad 0 \leqslant i \leqslant j \leqslant K, \tag{3.3}$$

which is obviously zero for $j < i$.

Consider the equality

$$E\{w_n\} = \sum_{i=0}^{K} \sum_{j=0}^{K} \Pr\{x_n = i\} q_{ij} \times E\{w_n \mid x_n = i, x_{n+1} = j\}, \tag{3.4}$$

on substituting in this the expressions from (2.2) and (3.3), we obtain after rearranging terms

$$E\{w_n\} = \frac{1}{\lambda} \sum_{i=0}^{K} \sum_{j=0}^{K} (j - i + 1) \Pr(x_n = i) q_{ij} - \sum_{k=0}^{K} c_k \frac{e^{-\lambda c_k}}{1 - e^{-\lambda c_k}} \times \sum_{i=0}^{k} \sum_{j=k}^{K} \Pr(x_n = i) q_{ij}. \tag{3.5}$$

Hence, from (2.1), (2.2) and noting that $c_0 = \infty$,

$$E\{w_n\} = \frac{1}{\lambda} E\{u(x_n)\} - \sum_{k=1}^{K} c_k \frac{e^{-\lambda c_k}}{1 - e^{-\lambda c_k}} \times \left[\sum_{i=0}^{k} \Pr(x_n = i) - \sum_{i=0}^{k-1} \Pr(x_{n+1} = i)\right] \quad \text{for } n \geqslant 1. \tag{3.6}$$

It will be observed that $E\{u(x_n)\}$ equals the expected number of unloaded units during the nth productive cycle. Let w denote the unloading delay per cycle in the stationary state, then we have, taking the limit for $n \to \infty$, (cf. 2.6),

$$E\{w\} = \frac{1}{\lambda} - \sum_{k=1}^{K} c_k v_k \frac{e^{-\lambda c_k}}{1 - e^{-\lambda c_k}},$$

or (cf. 2.5) equivalently

$$E\{w\} = \frac{1}{\lambda} - \sum_{k=1}^{K} c_k v_{k-1}, \tag{3.7}$$

where the stationary probabilities $\{v_i, i = 0, 1, \ldots, K\}$ are given in (2.5).

For the special case $c_1 = c_2 = \cdots = c_K = c$ one finds

$$E\{w\} = \frac{1}{\lambda} - \frac{c e^{-\lambda c}}{1 - e^{-\lambda c}} (1 - v_0), \tag{3.8}$$

in agreement with a result of Beightler and Crisp [1] adapted to a Poisson arrival process.

4. The optimal unloading policy

A policy which minimizes the average delay per cycle will be called an optimal policy. In this section it will be shown that there exists a unique optimal policy $(c_1^*, \ldots, c_K^*)$ satisfying the following property $c_1^* > c_2^* > \cdots > c_K^*$, i.e., the policy is monotone with respect to the state variable. To this end a number of lemmas will be proved. Writing n for K, the average delay can be written as

$$E\{w\} = \frac{1}{\lambda} - F_n(c_1, \ldots, c_n), \tag{4.1}$$

where the function F_n is defined as, (cf. (2.5) and (3.7)),

$$F_n(c_1, \ldots, c_n) = \frac{G_n(c_1, \ldots, c_n)}{H_n(c_1, \ldots, c_n)},$$

in which

$$G_n(c_1, ..., c_n) = \sum_{i=1}^{n} c_i \prod_{j=1}^{i-1} (e^{\lambda c_j} - 1) , \qquad (4.2)$$

and

$$H_n(c_1, ..., c_n) = \sum_{i=1}^{n+1} \prod_{j=1}^{i-1} (e^{\lambda c_j} - 1) .$$

Obviously, minimizing $E\{w\}$ is equivalent to maximizing F_n. It should be noted that the optimization problem can be formulated as a Markovian decision process. Here, instead, we treat the problem as a classical optimization problem.

Lemma 1. *The function $F_n(c_1, ..., c_n)$ takes on its maximum in the open set D_n $\{(c_1, ..., c_n) \mid 0 < c_i < \infty$, $i = 1, 2, ..., n\}$.*

Proof. Let the function $F_n(c_1, ..., c_n)$ with $c_i \geq 0$, $i = 1, 2, ..., n$, be transformed into a function $\tilde{F}_n(x_1, ..., x_n)$, with $0 \leq x_i \leq 1, i = 1, 2, ..., n$, using the transformation $x_i = \exp(-\lambda c_i)$. The following properties of the function $\tilde{F}_n, n \geq 2$, are easily verified:

(i) $$\tilde{F}_n(x_1, ..., x_{i-1}, 1, x_{i+1}, ..., x_n) = \tilde{F}_{i-1}(x_1, ..., x_{i-1}) , \qquad i = 2, ..., n,$$

and

$$\tilde{F}_n(1, x_2, ..., x_n) = 0 .$$

(ii) $$\tilde{F}_n(x_1, ..., x_k, ..., x_n) \to F_{n-k}(x_{k+1}, ..., x_n) \quad \text{if } x_k \downarrow 0, \quad k = 1, 2, ..., n-1,$$

and

$$\tilde{F}_n(x_1, ..., x_{n-1}, 0) = 0 .$$

(iii) $$\frac{\partial \tilde{F}_n}{\partial x_n} = -\prod_{i=1}^{n-1} \frac{(1-x_i)}{x_i} \times \left\{\frac{1}{\lambda} - \tilde{F}_{n-1}(x_1, ..., x_{n-1})\right\} \Big/ H_{n-1} , \quad \text{for } x_n \uparrow 1.$$

(iv) $\tilde{F}_n(x_1, ..., x_n)$, considered as a function of x_1 only, takes on its maximum in the open interval (0, 1) for all values of x_i, $i = 2, ..., n$, with $0 \leq x_i \leq 1$, $i = 2, ..., n$.

(v) $$\tilde{F}_n(x_1, ..., x_n) < \frac{1}{\lambda} .$$

This property follows from (4.1) by noting that $w > 0$ almost surely, if $0 < \lambda < \infty$.

The function $\tilde{F}_n$ is a continuous function on a closed and bounded set, therefore it takes on its global maximum on this set, according to a well-known theorem of Weierstrass. Let us denote by $\bar{x}_i(n)$, $i = 1, 2, ..., n$ the values which maximize $\tilde{F}_n$.

From (4.2) we have $\tilde{F}_1(x_1) = -1/\lambda\, x_1 \ln x_1$ and it is readily verified that this function has its maximum at a point of $\tilde{D}_1 = \{x_1 \mid 0 < x_1 < 1\}$. Now consider the function $\tilde{F}_2(x_1, x_2)$. From Property (iv) it follows that $0 < \bar{x}_1(2) < 1$. Moreover from (ii) it follows that $\bar{x}_2(2) > 0$. On combining the Properties (iii) and (v) and noting that $0 < x_1 < 1$ it is seen that $\partial\tilde{F}_2/\partial x_2 < 0$, for $x_2 \uparrow 1$, consequently $0 < \bar{x}_2(2) < 1$.

On taking $x_1 = \bar{x}_1(1)$ and $x_2 = 1$ in (iii) and using Property (i) it follows that $\tilde{F}_1^* < \tilde{F}_2^*$, where $\tilde{F}_i^* = \tilde{F}_i(\bar{x}_1(i), \bar{x}_2(i), ..., \bar{x}_i(i))$, $i = 1, 2, ..., n$. Suppose that the functions $\tilde{F}_j(x_1, ..., x_j)$, $j = 1, 2, ..., k(<n)$, take on their global maximum $\tilde{F}_j^*$ in the open set $\tilde{D}_j = \{(x_1, ..., x_j) \mid 0 < x_i < 1, i = 1, 2, ..., j\}$ and moreover that $\tilde{F}_1^* < \tilde{F}_2^* < \cdots < \tilde{F}_k^*$.

From this assumption we will now prove that the same is true for $j = 1, 2, ..., k+1$. If one or more variables of $\tilde{F}_{k+1}$ equal zero or one, then the function $\tilde{F}_{k+1}$ is reduced to a function $\tilde{F}_j(x_1, ..., x_j)$, with $j \leq k$, according to the Properties (i) and (ii). From the assumption made it follows that in case $\tilde{F}_{k+1}$ is optimal j will not be less than k. The only cases to be considered in which a variable is on its boundary are $x_1 = 0, 0 < x_i < 1, i = 2, ..., k+1$ and $0 < x_i < 1$, $i = 1, 2, ..., k$, $x_{k+1} = 1$. The first case cannot give a maximum taking into account Property (iv). On combining the Properties (iii) and (v) one finds that $\partial\tilde{F}_{k+1}/\partial x_{k+1} < 0$ for $x_{k+1} = 1$ so that the second case can equally not give a maximum, which finally implies that $\tilde{F}_{k+1}$ must take on its global maximum in the open set $\tilde{D}_{k+1}$. Because $\tilde{F}_k^* = \tilde{F}_{k+1}(\bar{x}_1(k), ..., \bar{x}_k(k), 1)$ it also follows from the Properties (iii) and (v) that $\tilde{F}_k^* < \tilde{F}_{k+1}^*$.

Consequently if the assumption made for the functions $\tilde{F}_j$, $j = 1, 2, ..., k$ holds, it also holds for $1 \leq j \leq k+1$. Noting that the assumption holds for $k = 2$ it follows by mathematical induction that the functions $\tilde{F}_j$, $j = 1, 2, ..., n$, have their global maximum in the sets $\tilde{D}_j$, $j = 1, 2, ..., n$. Now it should be observed that F_n has a maximum in $(\bar{c}_1, ..., \bar{c}_n)$, with $0 < \bar{c}_i < \infty$, $i = 1, 2, ..., n$ if and only if $\tilde{F}_n$ has a maximum in $(\bar{x}_1, ..., \bar{x}_n)$ with $0 < \bar{x}_i < 1$ and $\bar{x}_i = \exp(-\lambda\bar{c}_i)$, $i = 1, 2, ..., n$, due to the monotony of the transformation. Moreover, from $\bar{x}_i(n) > 0$ we have $\bar{c}_i(n) < \infty$.

So F_n is optimal in $D_n = \{ (c_1, ..., c_n) \mid 0 < c_i < \infty, i = 1, 2, ..., n\}$.

The maximum of the function F, dropping the subscript, satisfies the necessary conditions $\partial F/\partial c_i = 0$, $i = 1, 2, ..., n$.

The solutions of these equations are called the stationary points of F. Noting that F has continuous first order derivatives on D_n, Lemma 1 implies that there exists at least one stationary point.

Lemma 2. *The stationary points of the function F in the set D_n satisfy the equations*

$$\lambda c_i = \lambda c_{i+1} + e^{-\lambda c_{i+1}} - e^{-\lambda c_{i-1}}, \quad i = 1, 2, ..., n, \tag{4.3}$$

with $c_0 = \infty$ and $c_{n+1} = 0$.

Proof. The stationary points of F satisfy the equations

$$H \frac{\partial G}{\partial c_i} - G \frac{\partial H}{\partial c_i} = 0, \quad i = 1, 2, ..., n,$$

because $H > 0$ for all $c_i \geq 0$. This leads to

$$H \left\{ 1 + \lambda e^{\lambda c_i} \sum_{j=i+1}^{n} c_j \prod_{k=i+1}^{j-1} (\exp(\lambda c_k) - 1) \right\} - \lambda G e^{\lambda c_i} \sum_{j=i+1}^{n} \prod_{k=i+1}^{j-1} (\exp(\lambda c_k) - 1) = 0, \tag{4.4}$$

considering only solutions in D_n.

On taking $i = n$ in (4.4) gives

$$H - \lambda e^{\lambda c_n} G = 0. \tag{4.5}$$

Considering $i = n - 1$ in (4.4), we have

$$(1 + \lambda c_n e^{\lambda c_{n-1}}) H - \lambda G e^{\lambda c_n + \lambda c_{n-1}} = 0. \tag{4.6}$$

Substituting (4.5) in (4.6) yields

$$\lambda c_n = 1 - e^{-\lambda c_{n-1}}. \tag{4.7}$$

Again, considering (4.4) for $i = n - 2$ and using (4.5) and (4.6) leads, after simplification, to

$$\lambda c_{n-1} = 1 - e^{-\lambda c_{n-1}} - e^{-\lambda c_{n-2}} + e^{-\lambda c_n}. \tag{4.8}$$

In proceeding this way for $i = n - 3, n - 4, ..., 1$ one finds the following set of equations

$$H - \lambda G e^{\lambda c_n} = 0,$$

$$\lambda c_i = 1 - e^{-\lambda c_i} - e^{-\lambda c_{i-1}} + e^{-\lambda c_n}, \quad i = 2, 3, ..., n. \tag{4.9}$$

Now it can be deduced that the first equation is equivalent to

$$\lambda c_1 = 1 - e^{-\lambda c_1} + e^{-\lambda c_n}, \tag{4.10}$$

by substituting the equations for $i = 2, 3, ..., n$ in (4.9) into the first. Then from (4.9) and (4.10) it is seen that

$$\lambda c_i = \lambda c_{i+1} + e^{-\lambda c_{i+1}} - e^{-\lambda c_{i-1}}, \quad i = 1, 2, ..., n,$$

with $c_0 = \infty$ and $c_{n+1} = 0$, which proves the lemma.

Lemma 3. *Any finite positive solution $(c_1, ..., c_n)$ of the equations (4.3) satisfies $c_1 > c_2 > \cdots > c_n > 0$.*

Proof. The first equation in (4.3) reads

$$\lambda c_1 = \lambda c_2 + e^{-\lambda c_2},$$

hence $c_1 > c_2$.

Adding the equations for $i = 1, 2, ..., j - 1, j > i$, in (4.3) gives

$$\lambda c_1 = \lambda c_j - e^{-\lambda c_1} + e^{-\lambda c_j} + e^{-\lambda c_{j-1}}. \tag{4.11}$$

Suppose $c_1 > c_k$, for $k = 2, 3, ..., j - 1$, then from (4.11) we have $c_1 > c_j$; thus, $c_1 > c_2$ being true, it follows by mathematical induction that $c_1 > c_k$, $k = 2, ..., n$. Now suppose that

$$c_k > c_j, \text{ for } j = k + 1, ..., n \text{ and } k = 1, 2, ..., i < n. \tag{4.12}$$

From (cf. (4.3))

$$\lambda c_{i+1} = \lambda c_{i+2} + e^{-\lambda c_{i+2}} - e^{-\lambda c_i},$$

it is easily seen that $c_{i+1} > c_{i+2}$, because, by assumption $c_i > c_{i+2}$. Moreover, suppose $c_{i+1} > c_m$ holds for $m = i + 2, ..., r$. From (4.11) we have

$$\lambda c_{i+1} = \lambda c_{r+1} - e^{-\lambda c_i} + e^{-\lambda c_{r+1}} - e^{-\lambda c_{i+1}} + e^{-\lambda c_r}.$$

Then it is easily seen that $c_{i+1} > c_{r+1}$, having assumed that $c_i > c_{r+1}$ and $c_{i+1} > c_r$, consequently if assumption (4.12) holds up to i, it holds for $i + 1$ also. Now, noting that (4.12) is true for $i = 1$, it is true for $i = 1, 2, ..., n$ by induction.

We have seen that the function F possesses at least one stationary point. We will prove now that this is the only stationary point in D_n.

Lemma 4. *The finite positive solution of the equations (4.3) is unique.*

Proof. Suppose that the equations (4.3) admit two bounded solutions $(c_1, ..., c_n)$ and $(d_1, ..., d_n)$. Then both solutions satisfy Lemma 3. Assume that $c_1 > d_1$. Then from (4.3) it follows that

$$e^{-\lambda c_n} - e^{-\lambda d_n} = (\lambda c_1 + e^{-\lambda c_1}) - (\lambda d_1 + e^{-\lambda d_1}) > 0 ,$$

noting that $c_1 > d_1 > 0$ and that the function between the brackets is strictly increasing. Consequently, we have $c_n < d_n$.

The following equality is easily established

$$(\lambda c_{j+1} + e^{-\lambda c_{j+1}}) - (\lambda d_{j+1} + e^{-\lambda d_{j+1}})$$
$$= (\lambda c_1 + e^{-\lambda c_1}) - (\lambda d_1 + e^{-\lambda d_1}) + (e^{-\lambda d_j} - e^{-\lambda c_j}) .$$

From this it is readily verified by an inductive argument that if $c_1 > d_1$ then $c_i > d_i$ for $i = 1, 2, ..., n$, which apparently leads to a contradiction. Hence $c_1 > d_1$ is impossible. Starting with the assumption $c_1 < d_1$, again a contradiction is found, consequently one must have $c_1 = d_1$. Then, using (4.3) it is seen that $c_i = d_i$, $i = 1, 2, ..., n$.

Lemma 5. *The optimal value of* $E\{w\}$ *satisfies*

$$\lambda E\{w\} = 1 - e^{-\lambda c_n^*} . \tag{4.13}$$

Proof. This immediately follows from the relations (4.1), (4.2) and (4.5).

Combining the results of the five lemmas we finally have

Theorem 1. *There exists a unique optimal policy* $(c_1^* \cdots c_n^*)$ *minimizing the average delay per unit processed, with the following properties:*

(i) The ranges c_i^*, $i = 1, 2, ..., n$ are the unique positive bounded solution of the equations

$$\lambda c_i^* = \lambda c_{i+1}^* + e^{-\lambda c_{i+1}^*} - e^{-\lambda c_{i-1}^*} ,$$
$$i = 1, 2, ..., n,$$

with $c_0^* = \infty$ and $c_{n+1}^* = 0$

(ii) $c_1^* > c_2^* > \cdots > c_n^* > c_{n+1}^* = 0$.

(iii) $\lambda E\{w\} = 1 - e^{-\lambda c_n^*}$ for $c_i = c_i^*, i = 1, ..., n$.

Up to now no consideration has been given to the influence of the reserve capacity on the unloading policy. Let $(c_1(n), ..., c_n(n))$ denote the optimal unloading policy for $K = n$, $n \geq 1$. The influence of the reservecapacity on the unloading policy is given by the following:

Theorem 2. *The optimal unloading ranges* $c_i(n)$, $i = 1, 2, ..., n$ *are monotone increasing functions of the reservecapacity* n, *i.e.,* $c_i(n-1) < c_i(n)$, $i = 1, 2, ..., n$.

Proof. The ranges $c_i(n)$, $i = 1, 2, ..., n$ and $c_i(n-1)$, $i = 1, ..., n$ satisfy the equation (cf. (4.3))

$$[\lambda c_{j+1}(n-1) + \exp\{-\lambda c_{j+1}(n-1)\}]$$
$$- [\lambda c_{j+1}(n) + \exp\{-\lambda c_{j+1}(n)\}]$$
$$= [\lambda c_1(n-1) + \exp\{-\lambda c_1(n-1)\}]$$
$$- [\lambda c_1(n) + \exp\{-\lambda c_1(n)\}]$$
$$+ \exp\{-\lambda c_j(n)\} - \exp\{-\lambda c_j(n-1)\} ,$$
$$j = 1, 2, ..., n.$$

Suppose $c_1(n-1) \geq c_1(n)$, analogous to the proof of Lemma 4, it follows by mathematical induction that $c_i(n-1) \geq c_i(n)$ for $i = 1, 2, ..., n$. However, noting that $c_n(n-1) = 0$, this implies $c_n(n) \leq 0$, which, according to Theorem 1, is not true. Therefore we must have $c_1(n-1) < c_1(n)$. Again, using mathematical induction it then follows that $c_i(n-1) < c_i(n)$, $i = 1, 2, ..., n$.

5. Numerical considerations

In order to obtain the solution of the equations (4.3) the following successive approximation procedure was used:

$$\lambda c_1^{(n+1)} = \lambda c_2^{(n)} + \exp\{-\lambda c_2^{(n)}\} ,$$
$$\lambda c_i^{(n+1)} = \lambda c_{i+1}^{(n)} + \exp\{-\lambda c_{i+1}^{(n)}\}$$
$$- \exp\{-\lambda c_{i-1}^{(n+1)}\} , \quad i = 2, ..., K-1,$$
$$\lambda c_K^{(n+1)} = 1 - \exp\{-\lambda c_{K-1}^{(n)}\} ,$$

starting with $c_i^{(1)} = 0, i = 1, 2, ..., K$.

Table 1 gives the values of the optimal ranges for several values of the reserve capacity. Note the monotonicity properties in accordance with Theorems 1 and 2.

In Table 2 an illustrative comparison is given between the policies of Crisp [2], Matsui and Shingu [4] and our policy. In the second column the optimal value of λc for Crisp's policy is given. The third and fourth columns show the largest and smallest optimal ranges of our policy. The last three columns give the optimal mean delay for the three policies (for the policy of Matsui and Shingu the only two optimal values which could be found are given).

The policy presented here is the best of the three, which will be evident keeping in mind that Crisp's policy corresponds to the case $c_1 = c_2 = \cdots = c_K$, and

Table 1
Optimal ranges

K	λc_1	λc_2	λc_3	λc_4	λc_5	λc_6	λc_7	λc_8	λc_9	λc_{10}
2	1.20	0.70								
3	1.28	0.86	0.58							
4	1.33	0.95	0.72	0.52						
6	1.39	1.03	0.85	0.72	0.60	0.45				
8	1.41	1.07	0.91	0.80	0.72	0.64	0.55	0.42		
10	1.42	1.09	0.94	0.84	0.77	0.71	0.66	0.60	0.52	0.40

Table 2
Comparison with other policies

K	λc	c_1^*	c_K^*	Crisp	M&S	$\lambda E\{w\}$
2	0.90	1.20	0.70	0.5177		0.5021
3	0.85	1.28	0.58	0.4625	0.4521	0.4388
4	0.85	1.33	0.52	0.4305		0.4026
6	0.80	1.39	0.45	0.3939		0.3644
7	0.80	1.40	0.44	0.3832	0.3629	0.3534
8	0.75	1.41	0.42	0.3746		0.3453
10	0.75	1.42	0.40	0.3616		0.3345
15	0.75	1.44	0.39	0.3447		0.3213
20	0.75	1.45	0.38	0.3371		0.3156
30	0.70	1.45	0.37	0.3275		0.3109
50	0.70	1.46	0.37	0.3189		0.3078

the fact that Matsui and Shingu use the same range during the production cycle, which depends only on the number of usables in the reserve just after the termination of the service.

The relative gain over Crisp's policy in using our policy is at most 8%. In case Morris' single unit policy is used, for which $c_1 = c_2 = \cdots = c_K = \infty$, the mean delay equals λ^{-1}.

Taking this policy as level of reference the relative gain over Crisp's policy is only 3%.

6. Conclusion

Beightler and Crisp [1] and Crisp [2], introduced the so called 'Sequential Range Policy', (SRP). This policy is a special case of the policy considered here and can be obtained by putting $c_1 = c_2 = \cdots = c_K = c$. In his thesis Crisp suggested to generalize SRP in that the unloading ranges should be dependent on the current number of units in the reserve. Independently, Matsui and Shingu [4], investigated a state-dependent sequential range policy, which strongly resembles the policy considered here. The difference is that instead of setting the size of the range every moment an unloading have taken place, they use the same range during one production cycle, the size of which depends only on the number of units in the reserve just after the termination of the processing. From a mathematical point of view it seems that our policy is a more natural generalisation of the SRP. Comparing the state dependent policies with SRP, see also [4], shows that the gain over SRP is marginal, taking the case $K = 0$ as level of reference. Moreover the state dependent policies are more difficult to implement, which is a practical drawback. To overcome this drawback one could introduce simplified policies, using a smaller number of ranges, see [4].

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