V. Cepstral/Correlation Matching

Lagunas [9] has shown that under correlation constraints $\{r_i, 0 \le i \le p\}$ and cepstral constraints $\{c_n, 1 \le n \le q\}$, of all the spectra that can match these constraints, the matching (p, q) pole-zero spectrum is the one that maximizes the entropy. Unfortunately, a matching pole-zero spectrum migth not always exist. Unlike the correlation matching problem considered above, here the number of poles is equal to the number of correlations. Hence, there should be no difficulty in finding pole-zero spectra with p poles which can match the given correlations. The difficulty arises in attempting to match the cepstral constraints as well. We show below the source of the difficulty. Note that, if the set of correlations and cepstral constraints had been obtained from a (p, q) pole-zero spectrum, then the model parameters can be retrieved from the given constraints [10].

From the definition of the cepstrum

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log P(\omega) \cos(n\omega)\, d\omega \tag{1}$$

where $P(\omega)$ is a positive spectrum, we see that the values c_n are unconstrained for general $P(\omega)$, i.e., c_n may be any positive or negative real number. In other words, given an arbitrary set of real numbers, they can be valid cepstral coefficients.[5] Now, in attempting to find a pole-zero model that matches these cepstral values, we run into difficulties because a finite-order pole-zero model cannot generate arbitrary cepstral values, as we show below.

Assume that our pole-zero model is minimum phase and that it has poles $\{z_k, 1 \le k \le p\}$ and zeroes $\{w_j, 1 \le j \le q\}$. Then the corresponding cepstrum c_n is symmetric and takes on the following values [11]:

$$c_n = \frac{1}{n}\left[\sum_{k=1}^{p} z_k^n - \sum_{j=1}^{q} w_j^n\right], \qquad n > 0. \tag{2}$$

Because the model is minimum phase, the poles and zeros are inside the unit circle. From (2), it is clear then that c_n is limited to be in the range

$$|c_n| \le \frac{p+q}{n}, \; n > 0. \tag{3}$$

Not all cepstral values that obey (3) are actually attainable. Equation (3) is a necessary but not sufficient condition for cepstral values to be those of a (p, q) pole-zero model. The region of attainable cepstral coefficients is generally nonconvex and is quite complicated for a general (p, q) pole-zero model.

It is clear from the above that if any of the given cepstral values violates (3) or is otherwise outside the range of attainable cepstral values, then no (p, q) pole-zero model can be found that can match the given cepstral coefficients. With $p = 0$, the problem reduces to finding a q-zero model that can match r_0 and the cepstral coefficients $\{c_n, 1 \le n \le q\}$. From the above discussion it follows that such a model might not exist in general.

VI. Conclusions

We have shown that, given a set of correlation constraints, it is not always possible to find a matching pole-zero spectrum if the number of poles is strictly less than the number of correlations. We have also shown that, given an additional set of cepstral constraints, it is not always possible to find a matching pole-zero model of finite order. These results raise the question of the advisability of maximizing the entropy or of matching given correlation and/or cepstral constraints as general methods of spectral modeling. We believe that spectral modeling should be based on minimizing some error criterion instead of requiring exact matching of values. The Itakura–Saito error criterion [12], for example, has been very useful for spectral modeling, including pole-zero modeling [13].

[5]Note that, in constrast, the normalized autocorrelation values corresponding to $P(\omega)$ cannot be greater than 1 in magnitude and are further limited in value by the positive definiteness conditions.

Acknowledgment

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On the Determination of the Position of Extrema of Sampled Correlators

R. Moddemeijer

***Abstract*—The determination of the position of the extremum of a continuous correlator function from its samples, if the spectrum of the correlator is not band limited, is still an unsolved problem. Searching for the position of the extremum by fitting a parabola through three samples around that extremum leads to clustering of the estimates around values corresponding with the sampling moments.**

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I. Introduction

Prof. Lopes da Silva,[1] in private communication, propounded a problem of clustering of delay estimates observed when studying delay estimates obtained from electroencephalogram (EEG) signals. These estimates, obtained by analysis of sampled EEG signals, are clustered around multiples of the sampling interval. In the case of time-delay estimation using a sampled correlator function, such as the cross-covariance function or the mutual information function of Mars *et al.* [1], we investigate the origin of this feature. The accurate and practical determination of the position of the extremum is an unsolved problem; the position of that extremum does not necessarily coincide with the sampling grid and has to be determined using an interpolation strategy. A straightforward parabola fitting method [2], when neglecting any statistical effect, leads to clustering around $n\ \Delta t$ with $n \in \mathbb{Z}$ and Δt the sample distance.

Due to this feature, the delay estimates of Mars [1] will, in the case of first-order autoregressively filtered white noise which he used, cluster around $n\ \Delta t$. Therefore, his simulation results using a zero delay system are better than can be expected in any realistic delay estimation situation. Consequently, in his conclusions he is too optimistic about the performance of his delay estimation method.

II. The Method of Extremum Localization

Without loss of generality we assume the extremum is a maximum. We determine a (local) peak of the correlator function by fitting a parabola through three samples around the peak. The position of the peak is the delay estimate.

The signal model we propose is given in Fig. 1, $\underline{\mu}$ and $\underline{\nu}$ are normal mutually independent noise sources and $\underline{\mu}$ is white. The system is a first- or second-order (time-continuous) autoregressive filter and τ is the delay of y with respect to x. The cross-covariance function of x and y is, apart from a delay, equal to the autocovariance function of x.

Assume a correlator function $c(t)$ has a local maximum for $t = \tau$. In an interval with $|t - \tau| < \epsilon$ the function $c(t)$ is an increasing function for $t < \tau$ and is a decreasing function for $t > \tau$. This surrounding should be large enough: $\epsilon \geq 3/2\ \Delta t$. We consider $|\tau| \leq 1/2\ \Delta t$, which can be generalized by shifting the correlator function an integer number of sampling intervals. We determine the parabola $\hat{c}(t)$ passing through three samples surrounding the maximum (at $-\Delta t$, 0, and Δt):

$$\hat{c}(t) = at^2 + bt + c \tag{2.1}$$

with the coefficients

$$a = \frac{c(-\Delta t) - 2c(0) + c(\Delta t)}{2\ \Delta t^2} \qquad (a < 0)$$

$$b = \frac{c(\Delta t) - c(-\Delta t)}{2\ \Delta t}$$

$$c = c(0). \tag{2.2}$$

The position of the maximum equals $t = \hat{\tau} = -b/(2a)$. This leads to the delay estimate

$$\hat{\tau} = \frac{c(-\Delta t) - c(\Delta t)}{c(-\Delta t) - 2c(0) + c(\Delta t)} \frac{\Delta t}{2}. \tag{2.3}$$

Without any statistical effect this deterministic procedure causes bias due to the procedure of the peak localization. In the next section we will show that in general $\tau \neq \hat{\tau}$.

III. Applications of the Peak Localization

We apply our peak localization method to two correlator functions: the cross-covariance function and the mutual information function, and we do this for two different types of signals: first- and second-order filtered noise. We derive and discuss an expression for $\hat{\tau}$ as a function of τ.

[1]Prof. Lopes da Silva, Department of Gen. Zoology, University of Amsterdam, Amsterdam, The Netherlands.

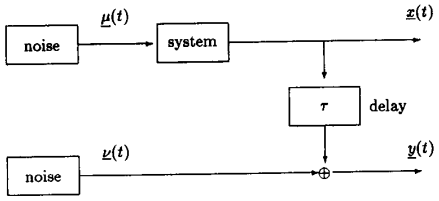

Fig. 1. The proposed signal model.

In the case of a first-order system, due to our model, the cross-covariance function is, apart from a delay, equal to the autocovariance function of first-order filtered white noise

$$c(t) = \sigma_x^2 \rho^{|t' - \tau'|} \qquad 0 < \rho < 1 \tag{3.1}$$

with $\sigma_x^2 = \text{var}\{x(t)\}$, $t' = t/\Delta t$ and $\tau' = \tau/\Delta t$. The x signal is not strictly band limited, but has an effective bandwidth $-1/(2\Delta t)$ ln ρ (the variance divided by the integral of the autocovariance function), which equals $0.6/\Delta t$ for $\rho = 0.3$. Substitution of (3.1) into (2.3) provides us with an expression for $\hat{\tau}$. Considering the symmetry of (3.1) we restrict ourselves to the interval $0 \leq \tau \leq 1/2\ \Delta t$

$$\hat{\tau} = \frac{\rho^{1+\tau'} - \rho^{1-\tau'}}{\rho^{1+\tau'} - 2\rho^{\tau'} + \rho^{1-\tau'}} \frac{\Delta t}{2}. \tag{3.2}$$

The resulting estimates $\hat{\tau}$ as a function of τ are presented in Fig. 2(a) for several values of ρ. Ideally, the curves of Fig. 2(a) follow the line $\hat{\tau} = \tau$. The delay estimate $\hat{\tau}$ tends to deviate towards $\hat{\tau} = 0$, which results in clustering of the estimates around $\hat{\tau} = 0$. Generalizing this property for $\tau \in \mathbb{R}$: $\hat{\tau}$ tends to deviate into the direction of the nearest multiple of Δt. If $\rho \geq 0.7$ the deviations are small; for $\tau = 0.25\ \Delta t$ they are smaller than approximately $0.1\ \Delta t$.

The deviations increase if the mutual information function is taken instead of the cross-covariance function:

$$c(t) = -\tfrac{1}{2} \log (1 - \lambda^2 \rho^{2|t' - \tau'|}). \tag{3.3}$$

The constant λ depends on the signal-to-noise ratio of the y signal: $\lambda^2 = s_n/(s_n + 1)$ and $s_n = \text{var}\{x(t)\}/\text{var}\{v(t)\}$. For the mutual information function the deviations of $\hat{\tau}$ from τ depend on λ, see Fig. 2(b). For $\lambda = 1$ and $\tau < 0.25\ \Delta t$ the deviations are in the same order as the delay, which leads to severe clustering. This clustering is caused by the bad matching of the sharp peak of the mutual information function with the parabola. Larger deviations of the correlator function from the parabolic shape can result in larger errors. If the signal-to-noise ratio is small, $c(t)$ in (3.3) is approximately proportional to the squared cross-covariance function. Consequently, the peak of the mutual information function for $\lambda = 1/2$ is not so sharp as for $\lambda = 1$, which leads to better results (Fig. 2(c)).

Now we study a case in which the cross-covariance function has a more parabolic shape around the peak; we take second-order autoregressively filtered white noise.

$$\frac{1}{\omega^2} \frac{d^2}{dt^2} x(t) - \frac{\zeta}{\omega} \frac{d}{dt} x(t) + x(t) = \mu(t) \qquad \text{with } \zeta = \cos\theta. \tag{3.4}$$

According to Jenkins and Watts [3, p. 166] the cross-covariance function equals

$$c(t) = \sigma_x^2 \frac{e^{-\zeta\omega|t-\tau|} \sin\left(\theta + \omega \cdot |t - \tau| \cdot \sin\theta\right)}{\sin\theta}. \tag{3.5}$$

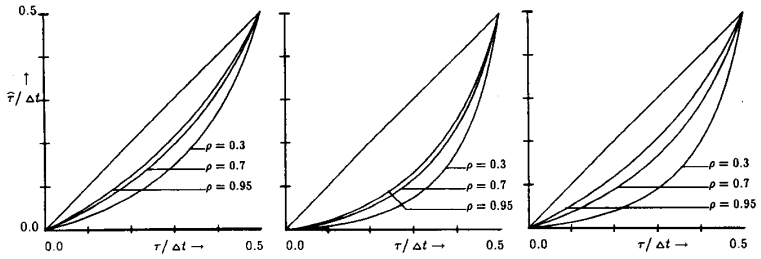

Fig. 2. The delay estimate $\hat{\tau}$ as a function of τ for the cross-covariance function (a) and the mutual information function (b) ($\lambda = 1.0$) and (c) ($\lambda = 0.5$). The delay estimates are calculated for the signal model of Fig. 1 with a first-order autoregressive filter with parameter ρ.

TABLE I
The Parameters for the Models Used in Fig. 3(a)–(c)

	1)	2)	3)	4)	5)	6)
$\omega\,\Delta t$	1.57	1.57	1.57	0.79	0.79	0.79
θ	0.52	1.05	1.31	0.52	1.05	1.31

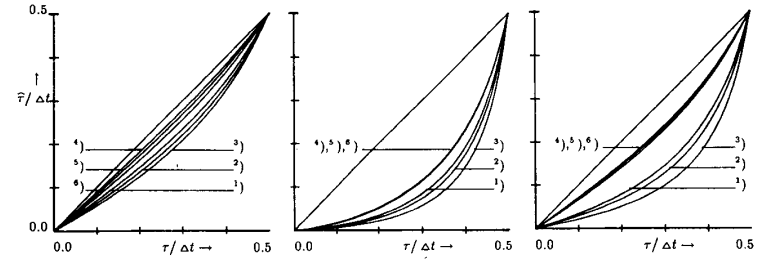

Fig. 3. The delay estimate $\hat{\tau}$ as a function of τ for the cross-covariance function (a) and the mutual information function (b) ($\lambda = 1.0$) and (c) ($\lambda = 0.5$). The delay estimates are calculated for the signal model of Fig. 1 with a second-order autoregressive filter with parameter ω and θ (see (3.4) and Table I).

Substitution of (3.5) into (2.3) provides us with the interpolated delay $\hat{\tau}$ as a function of τ. We studied six combinations of ω and θ (see Table I). Better parabolically shaped cross-covariance functions result in significantly smaller deviations of the delay estimate, though in Fig. 3(a) some clustering towards $\hat{\tau} = 0$ remains.

Similarly we calculate the interpolated delay for the mutual information function as correlator function:

$$c(t) = -\frac{1}{2}\log\left(1 - \left(\lambda \frac{e^{-\zeta\omega|t-\tau|}\sin\left(\theta + \omega \cdot |t-\tau| \cdot \sin\theta\right)}{\sin\theta}\right)^2\right). \tag{3.6}$$

The results are given in Fig. 3(b) and (c). Because of the sharper peaks of the mutual information function compared to the cross-covariance function, the deviations in Fig. 3(b) are larger than in Fig. 3(a). Deviations of the order of magnitude of $0.2\,\Delta t$ are to be expected.

IV. Discussion

The localization of a peak of the correlator function with an accuracy less than $1/2\,\Delta t$ requires knowledge about the behavior of the correlator function between the samples. If no *a priori* information about the peak form is available, an interpolation is used to reconstruct the peak. Several methods to interpolate sampled signals are mentioned in the literature, for a survey see [4, ch. 8]: reconstruction filters, polynomial interpolation [2], upsampling [5], and band-limited interpolation alias Whittaker's interpolation formula [3, p. 52]. Reconstruction filters, which are mainly used in hardware, have the disadvantage of modifying the signal at the sampling moments. The last two methods are essentially equivalent though they differ in the computation. When choosing an interpolation strategy several aspects are to be considered: computational efficiency, the bandwidth of the spectrum of the correlator function, the length of the interpolation interval, *a priori* knowledge, and the consequences of bias. Oversampling to avoid the interpolation results in inefficient computation. Band-limited interpolation is designed to reconstruct correlator functions with band-limited spectra. If the signals are band limited, the resulting correlator function is not necessarily band limited (e.g., mutual information

function). The correlator function is interpolated inside the interpolation interval, in which all samples have to be computed. The computation of samples of the mutual information function is quite expensive. The interpolation interval surrounding the peak can be extended if long-term dependencies between samples are assumed.

We can choose between polynomial interpolation and upsampling on a limited interval [5, sec. III]. This is the equivalent of fitting a polynomial, respectively, a sum of sine and cosine functions through a few samples assuming higher derivatives, respectively, frequency components above the Nyquist frequency are zero. The number of degrees of freedom of the function, which equals the number of samples used, remains arbitrary.

Boucher *et al.* [2] used the parabola fitting method successfully because their signal had only an effective bandwidth of $0.083/\Delta t$. Neither Holm [5] nor Cabot [6] present a performance analysis using upsampling. Chan *et al.* [7] studied a variant of Whittaker's interpolation formula for $\tau = 0.25$ and $0.50\ \Delta t$. They find, at least in some instances, a bias when $\tau = 0.25\ \Delta t$ as we did and, despite the symmetry, also a small bias when $\tau = 0.50\ \Delta t$. Because their signals are strictly band-limited, a detailed comparison of their results with ours is not possible.

V. Conclusions

Often the localization of the peak is considered to be the easiest part of the delay estimation procedure, in which hardly any mistake can be made. Therefore in the literature this problem is neglected. A more careful treatment of the peak localization based on better theoretical foundations is desirable.

Accurate interpolation based on local curve fitting requires *a priori* knowledge about the shape of the actual curve through the samples. If the fitted curve, in our case a parabola, does not correspond with the actual curve of the correlator function, interpolation errors can be made. These errors lead, as in our examples, to clustering of the delay estimates. To reduce the interpolation errors, we recommend transformation of the correlator function to a domain in which its form is more or less parabolic.

Another promising peak localization strategy is looking for global features of the correlator function instead of searching for the exact position of the maximum. For example, delay estimation by the expected value [8] or based on symmetry of the correlator function.

Without a reliable and theoretically acceptable peak localization strategy it is difficult to estimate time delays with an accuracy considerably better than half a sampling interval. Frequently time-delay estimation procedures and their peak localization are tested in special cases: for example, using a delay which is a multiple of Δt. A conscientious test should not be restricted to special cases only.

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Transform-Based Covariance Differencing Approach to the Array with Spatially Nonstationary Noise

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***Abstract*—Eigendecomposition-based methods developed to date required the noise covariance matrix to be known within a multiplicative constant. In this work, a transform-based covariance differencing approach is introduced which not only can be used to improve the resolution of the bearing estimation in the standard case, but can also be used to eliminate the effects of (unknown) unequal noise power on the array eigenstructure. The performance of this approach under different conditions is examined via simulations.**

I. Introduction

Two theoretically related covariance differencing techniques have recently been proposed which deal with completely unknown noise fields [1], [2]. The first one is based on the assumption that the noise field is invariant with respect to the array rotation [1]. The second method, which is based on a special transform covariance technique, assumes *a priori* knowledge about the structure of the noise covariance matrix (e.g., in bearing estimation it is assumed that the noise covariance matrix is symmetric Toeplitz).

In this work, the difference of two similarity transformations on the array covariance matrix is used so that the statistics of the noise process are preserved (as long as the noise process in the case of unknown unequal noise power remains spatially uncorrelated from sensor to sensor) while certain essential components of the signal are altered in a useful and predictable manner. This appraoch, in conjunction with the method of covariance differencing [2], can eliminate the effects of any uncorrelated noise in the bearing estimation process. The basic assumption here is that the noise covariance matrix is diagonal with unequal and unknown diagonal elements. This condition arises in situations where the additive white noise is not spatially stationary.

In this method, the resultant matrix, which is Hermitian, has a set of real eigenvalues in which the noise-related ones are all equal to zero, while the original array covariance would have unequal noise-related eigenvalues. Noise-related eigenvectors are all orthogonal to the modified signal space. This property is used to estimate bearing of incident wavefronts. In this methd, the transform-based covariance difference technique employed does not erase signals. Also the spectral function is modified so that it does not produce any spurious bearing estimates.

II. Formulation

Consider a uniformly spaced linear sensor array with L sensors. This assumption is not essential to the proposed method and is only used to simplify the formulation. The array covariance matrix R is

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