Fractal dimension crossovers in turbulent passive scalar signals

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The fractal dimension $\delta_g^{(1)}$ of turbulent passive scalar signals is calculated from the fluid dynamical equation. $\delta_g^{(1)}$ depends on the scale. For small Prandtl (or Schmidt) number $Pr < 10^{-2}$ one gets two ranges, $\delta_g^{(1)} = 1$ for small scale r and $\delta_g^{(1)}$=5/3 for large r, both as expected. But for large $Pr > 1$ one gets a third, intermediate range in which the signal is extremely wrinkled and has $\delta_g^{(1)} = 2$. In that range the passive scalar structure function $D_\theta(r)$ has a plateau. We calculate the Pr-dependence of the crossovers. Comparison with a numerical reduced wave vector set calculation gives good agreement with our predictions.

As seen in recent experiments [1,2] the temperature power spectrum for thermally driven turbulence shows a strong dependence on the Prandtl number $Pr = \nu/\kappa$. ν is the viscosity of the fluid and κ the thermal conductivity (diffusivity) of the advected scalar. While for the helium cell ($Pr \approx 0.7$) there was a large scaling range [2], no universal scaling could be found for water ($Pr \approx 7$) [1].

We took these experiments as a motivation to examine the Pr dependence of self similarity features of a passive scalar field $\theta(\mathbf{x}, t)$. The passive scalar θ could be the temperature or a dye (then Pr is often denoted as Schmidt number Sc), convected by a turbulent, isotropic velocity field $\mathbf{u}(\mathbf{x}, t)$,

$$\partial_t \theta = -\mathbf{u} \cdot \nabla\theta + \kappa\nabla^2\theta + f_\theta. \tag{1}$$

$f_\theta(\mathbf{x}, t)$ is a forcing term replacing the boundary conditions.

Self similarity in turbulence is commonly characterized by the scaling exponents of power spectra or of structure functions. Alternatively, one may consider the fractal dimensions $\delta_g^{(d)}$ of the d-dimensional graphs of hydrodynamic fields. For the passive scalar $\theta(\mathbf{x}, t)$ the scaling exponents $\zeta_m^{(\theta)}$ of the structure functions are defined by

$$\langle|\theta(\mathbf{x} + \mathbf{r}, t) - \theta(\mathbf{x}, t)|^m\rangle \propto r^{\zeta_m^{(\theta)}}. \tag{2}$$

The scale dependence of the Hausdorff volume $H^{(d)}(G(B_r^{(d)}))$ of the graph $G(B_r^{(d)}) = \left\{(\mathbf{x}, \theta)|\mathbf{x} \in B_r^{(d)}, \theta = \theta(\mathbf{x})\right\}$ over a ball $B_r^{(d)}$ of radius r defines the fractal dimension $\delta_g^{(d)}$,

$$H^{(d)}(G(B_r^{(d)})) \propto r^{\delta_g^{(d)}}. \tag{3}$$

In particular, $\delta_g^{(3)}$ is the Hausdorff dimension of the passive scalar graph over a three dimensional ball $B_r^{(3)}$, and $\delta_g^{(1)} = \delta_g^{(3)} - 2$ [3] is the Hausdorff dimension of a turbulent signal in space (for fixed time) or – by the Taylor hypothesis [4] – in time for fixed position.

For the passive scalar the scaling exponent $\zeta_1^{(\theta)}$ of the structure function and the fractal dimension $\delta_g^{(d)}$ are connected by

$$\delta_g^{(d)} \le d + (1 - \zeta_1^{(\theta)}). \tag{4}$$

We suppose as in [3] that the inequality is in fact sharp.

In this paper we shall calculate the fractal dimension $\delta_g^{(3)}$ and thus via (4) also the scaling exponent $\zeta_1^{(\theta)}$ *from the dynamical equation* (1). The main tool of our calculation is the volume formula for the passive scalar graphs [5], which was introduced as a very useful tool into fluid dynamics by Constantin and Procaccia [6,3,7]. Its main advantage is that one can apply rigorous techniques with controlled approximations, based on the fluid dynamical equations. By extending the calculations of [3,8] we are able to go beyond an estimate of the exponents: we handle also the amplitudes. Doing so, we confirm the various scaling ranges addressed in [8] and calculate the Pr dependence of the crossovers. Comparison with experiments, simulations, and former theories is discussed.

For convenience, we measure the passive scalar field in multiples of its rms, $\tilde\theta = \theta/\theta_{rms}$. We do not need to assume that upper bounds θ_{max} or u_{max} for the passive scalar or velocity field exist, loosening thus the assumptions made in [3]. We only need $\mathcal{L}_2$-integrability which anyhow is necessary for the existance of structure functions.

According to geometric measure theory [5,3] the Hausdorff volume $H^{(3)}(G(B_r^{(3)}))$ of the graph $r\tilde\theta(\mathbf{x})$ over the ball $B_r^{(3)} =: B_r$ reads

$$H^{(3)}(G(B_r)) = \int_{B_r} d^3x \sqrt{1 + r^2|\nabla^2\tilde\theta|^2}. \tag{5}$$

In one dimension eq.(5) is the well known formula for the length of the curve $r\tilde\theta(x)$. Dividing by $V^{(3)}(B_r) = 4\pi r^3/3$ and applying Cauchy-Schwarz's inequality we get

$$H^{(3)}(G(B_r))/V^{(3)}(B_r) \le \sqrt{1 + \frac{3}{4\pi r}\int_{B_r} d^3x|\nabla^2\tilde\theta|^2}. \tag{6}$$

Now the nice idea of [3] was to calculate $|\nabla\theta|^2$ from the heat transfer equation (1). In the stationary case it is

$$|\nabla\tilde\theta|^2 = \frac{1}{2\kappa}\left\{-\mathbf{u}\cdot\nabla + \kappa\nabla^2\right\}\tilde\theta^2 + \frac{f_\theta\tilde\theta}{\kappa\theta_{rms}}. \tag{7}$$

We insert the three terms of (7) in (6) and denote the resulting terms under the square root in (6) by I_1, I_2, and I_3, respectively. Directly from the definition of the thermal intensity dissipation rate ϵ_θ, we get $I_3 = \frac{\epsilon_\theta \eta^2}{\kappa \theta_{rms}^2} \left(\frac{r}{\eta}\right)^2$. Here $\eta = \nu^{3/4}/\epsilon^{1/4}$ is the Kolmogorov length and ϵ the energy dissipation rate [9]. I_3 can further be estimated by $I_3 \sim Pr Re^{-1/2}(r/\eta)^2$. Similarily, I_2 can be bound by $I_2 \leq \sqrt{3I_3} \sim Pr^{1/2} Re^{-1/4} r/\eta$.

For sufficiently large Re, I_2 and I_3 can be neglected for all relevant r, i.e., for r smaller than the outer length scale L. We are left with I_1. Applying Gauss' theorem, we get

$$I_1 = \frac{3}{8\pi\kappa r} \oint_{\partial B_r} \tilde{\theta}^2(\mathbf{x})\mathbf{u}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) dA(\mathbf{x}). \tag{8a}$$

$\mathbf{n}(\mathbf{x})$ is the unit vector normal to the sphere, directed inwards. The central idea of [3] now was to introduce velocity and scalar field *differences* in (8a) to connect the Hausdorff dimension of the passive scalar with the r-scaling exponents of velocity and scalar differences.

Here this is achieved in a slightly different way by adding a term $\propto \mathbf{u}(\mathbf{x_0})$ to the rhs of (8a), where $\mathbf{x}_0$ is the center of B_r. On average $\langle \mathbf{u}(\mathbf{x}_0) \rangle = 0$. Thus we are allowed to write

$$I_1 = \frac{3r}{2\kappa} \oint_{\partial B_r} \tilde{\theta}^2(\mathbf{x})(\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x_0})) \cdot \mathbf{n}(\mathbf{x}) \frac{dA(\mathbf{x})}{A_r}, \tag{8b}$$

where $A_r = 4\pi r^2$ is the surface of the sphere. We again apply Cauchy-Schwarz and get

$$I_1 \leq \frac{3r}{2\kappa} \sqrt{\left\langle \tilde{\theta}^4(\mathbf{x}) \right\rangle_{\partial B_r} \left\langle ((\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{x_0})) \cdot \mathbf{n}(\mathbf{x}))^2 \right\rangle_{\partial B_r}}. \tag{8c}$$

$\langle \ldots \rangle_{\partial B_r}$ denotes the averaging over the sphere. The first factor under the square root is the flatness of the scalar field, which is known to be 3 from experiment [9]. The second factor is the *longitudinal* velocity structure function $D_\parallel(r)$.

$D_\parallel(r)$ is related to the velocity structure function $D(r) = \langle |\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})|^2 \rangle$ via incompressibility [9],

$$D_\parallel(r) = r^{-3} \int_0^r \rho^2 D(\rho) d\rho. \tag{9}$$

If we measure lengths in multiples of $\eta = \nu^{3/4}/\epsilon^{1/4}$, and velocities in multiples of the Kolmogorov velocity $v_\eta = (\nu\epsilon)^{1/4}$ [9], $\tilde{D} = D/v_\eta^2$, $\tilde{r} = r/\eta$, we finally obtain

$$\frac{H^{(3)}(G(B_r))}{V^{(3)}(B_r)} = \text{const } r^{\delta_g^{(3)} - 3} \leq \sqrt{1 + \frac{3\sqrt{3}}{2} Pr \tilde{r} \sqrt{\tilde{D}_\parallel(\tilde{r})}}, \tag{10}$$

which is our main result. The inequalities arise from the Cauchy-Schwarz estimation. It is thus reasonable to assume that at least the r- and Pr-scaling behaviour is correctly given by (10). We thus have a controlled bound for the volume $H^{(3)}(G(B_r))$ of the passive scalar graphs, *including* the amplitudes.

Now from experiment it is known that the Batchelor interpolation formula is an excellent fit for the velocity structure function [9],

$$\tilde{D}(\tilde{r}) = \frac{\tilde{r}^2}{3(1 + a^2\tilde{r}^2)^{2/3}}, \qquad a^{-1} = 11.2. \tag{11}$$

We use the interpolation formula (11) to calculate the fractal dimension $\delta_g^{(3)}$ from (10) and (9) and get

$$\delta_g^{(3)} - 3 = \frac{d}{d \ln \tilde{r}} \ln \sqrt{1 + \frac{3\sqrt{3}}{2} Pr \tilde{r} \sqrt{\tilde{D}_\parallel(\tilde{r})}}. \tag{12}$$

The numerical result from (12) for $\delta_g^{(1)} = \delta_g^{(3)} - 2$ is given for several Pr numbers in Fig.1. In principle, four cases are possible which we now want to discuss.

For "small" Pr there are two ranges. If r is sufficiently small, we have $\delta_g^{(1)} = 1$ and from (4) $\zeta_1^{(\theta)} = 1$. On these small scales both the scalar and the velocity field are smooth, $\tilde{D}_\parallel(\tilde{r})$ is given by $\tilde{D}_\parallel(\tilde{r}) = \tilde{r}^2/15$. For increasing r a crossover occurs in $D_\parallel(r)$ and for $r > 14\eta$ the longitudinal structure function is given by $\tilde{D}_\parallel(\tilde{r}) = (3/11) b \tilde{r}^{\zeta_2^{(u)}}$ with [9] $b = a^{-4/3}/3 = 8.4$, $\zeta_2^{(u)}$ near $2/3$. The velocity field is now fractal but, as Pr is considerd to be small, no change can be observed in the fractal dimension $\delta_g^{(1)} = 1$ of the passive scalar field which stays to be smooth, as the 1 under the square root in (10) is still dominant. Physically this means that the diffusivity κ is so large that the passive scalar field is smooth even on turbulent scales. For sufficiently large scale,

$$r/\eta \geq \left(\frac{2}{3\sqrt{3}} \sqrt{\frac{11}{3b}} \right)^{3/4} Pr^{-3/4} = 0.36 Pr^{-3/4} \tag{13}$$

the second term in (10) becomes dominant. Then $\delta_g^{(1)} = 3/2 + \zeta_2^{(u)}/4$ and from (4) $\zeta_1^{(\theta)} = 1/2 - \zeta_2^{(u)}/4$. Without the at most tiny intermittency corrections we have $\delta_g^{(1)} = 5/3$ and $\zeta_1^{(\theta)} = 1/3$. Here both velocity and passive scalar field are fractal and scale alike. The classical Obukhov-Corrsin scaling theory [10] is recovered.

To observe the transition (13) we must have $0.36 Pr^{-3/4} > 14$, which implies the condition $Pr < 10^{-2}$. Thus "small" Pr are those with $Pr < Pr_l \approx 10^{-2}$. Furthermore, if Pr is even smaller than $(4Re)^{-1}$, the classical Obukhov-Corrsin scaling range can never be achieved, because $r < L$, i.e., the temperature signal is smooth on all scales.

For "large" Pr the situation is alike for sufficiently small and sufficiently large r. But for intermediate r the second term under the square root in (10) can become already dominant although still $r < 14\eta$, i.e., still in the viscous range of the velocity field, if only Pr is large enough. In this range $\delta_g^{(1)} = 2$ and consequently $\zeta_1^{(\theta)} = 0$. This means that the passive scalar signal is highly wrinkled although the velocity field is completely smooth on that scales. In that situation the dye or the heat is very efficiently mixed by the velocity field which is advected by the larger turbulent eddies. But since the diffusion is very slow (large Pr means small κ), concentration or temperature differences cannot be smeared out. Similar phenomena are obtained when non-turbulent, viscous fluids are mixed: fractal patterns develop [11].

From (10) we calculate that this intermediate range begins at

$$r/\eta \leq \sqrt{2\sqrt{5}/3} Pr^{-1/4} = 1.22 Pr^{-1/2}. \tag{14}$$

It ends at $r/\eta = 14$. Thus, to develop such an intermediate range of say, a decade, we must have $Pr > Pr_u \approx 1$. Such Pr we denote as "large".

Let us now have a look on the passive scalar structure function $D_\theta(r) = \langle|\theta(\mathbf{x}+\mathbf{r}) - \theta(\mathbf{x})|^2\rangle \propto r^{\zeta_2^{(\theta)}}$ Neglecting again possible intermittency corrections, we have $\zeta_2^{(\theta)} = 2\zeta_1^{(\theta)} = 4 - 2\delta_g^{(1)}$. For small r and large r we have $\zeta_2^{(\theta)} = 2$ and $\zeta_2^{(\theta)} = 2/3$, respectively, as for the velocity structure function. But, as derived above, for intermediate r in the case of large Pr we have a plateau in the passive scalar structure function, $\zeta_2^{(\theta)} = 0$, $D_\theta(r) = \text{const}$. The complete structure function can easily be reconstructed from the scale dependent scaling exponent $\zeta_2^{(\theta)}(r) = 2 - \delta_g^{(1)}(r)$. The result is shown is Fig.2.

The plateau has already been predicted by a mean field theory [12]. In that theory it is observed also for $Pr > Pr_u$ with a very similar Pr_u, see Fig.5 of [12]. The extension of the intermediate range here, $1.22 Pr^{-1/2} < r/\eta < 14$, is somewhat different from $5.49 Pr^{-1/4} < r/\eta < 15.6 Pr^{3/4}$ in [12], probably due to the mean field approximations in [12].

The theory of Batchelor [13,9] also predicts an intermediate range for $Pr > 1$. In that theory $D_\theta(r)$ depends only logarithmically on r in the intermediate range $r_1 < r < r_2$, but with $r_1 \propto Pr^{-1/2}$ and r_2 independent of Pr as in our theory.

For small $Pr < Pr_l$ all three theories predict the transition from $\zeta_1^{(\theta)} = 1$ to $\zeta_1^{(\theta)} = 1/3$ for a scale $\propto Pr^{-3/4}\eta$, see eq.(13), refs. [12], and [9].

Note that a plateau in the temperature structure function means a k^{-1}-behavior in the temperature spectrum. This k^{-1}-behavior was also postulated by Kraichnan [14].

Experiments and full numerical simulations for large Reynolds numbers Re *and* large Prandtl numbers Pr are very rare. There are some hints that there in fact is a plateau, see the collection of experimental data in [9]. To get independent confirmation of our predictions, we numerically solved eq. (1) together with the Navier-Stokes equation in a reduced wave vector set calculation for several Pr. For details concerning the method, see refs. [15,16]. The result is shown in Fig.3. For small wavevector p, i.e., large r, there is classical scaling, $\zeta_2^{(\theta)} = \zeta_2^{(u)} = 2/3$, for all scales. But for large $Pr > 1$ a plateau develops for medium scales as predicted by our theory. For small scales (large p) the spectra fall off exponentially, which reflects that the signal is smooth ($\delta_g^{(1)} = 1$) on these scales.

It becomes particularly evident from our figures, that $\delta_g^{(1)}$ and $\zeta_2^{(\theta)} = 4 - 2\delta_g^{(1)}$ should not be considered as *global* scaling exponent, but, instead, as *local* scaling exponents $\delta_g^{(1)}(r)$ and $\zeta_2^{(\theta)}(r)$, a concept which we have also introduced to examine the intermittency corrections in the Navier-Stokes dynamics [16]. If the r-ranges, where $\zeta_2^{(\theta)}(r)$ stays at a certain value, are small, it will be difficult to extract scaling exponents from experimental or simulated data.

Here this situation appears for $Pr > Pr_u \approx 1$. This might explain one aspect of the above mentioned Rayleigh-Benard experiments of Libchaber and coworkers where no global scaling exponent can be found for the "large" Pr number $Pr \approx 7$ [1]. Note that our theory can also be applied for an active scalar as for example the temperature field in Rayleigh-Benard convection. Eq.(10) remains valid for an active scalar, but, instead of the Batchelor interpolation (11), the velocity structure function of a *thermally* driven velocity field has to be used, which unfortunately is still not known reliably. If we assume Bolgiano-Obukhov scaling [17] for the velocity field, $D(r) \propto r^{6/5}$, we also get Bolgiano-Obukhov scaling for the temperature field, $D_\theta(r) \propto r^{2/5}$, if only r is sufficiently large.

The existance of *non-global* scaling ranges is also predicted for low Prandtl number Rayleigh-Benard convection, but only for very high Rayleigh numbers [18].

After having introduced *global* scaling as a paradigma in nonlinear dynamics, we might have to get used to *non-global* scaling ranges as the *normal* case, where universal exponents can hardly be derived from experiments or simulations, as the individual scaling ranges are too small.

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FIG. 1. Scale dependent fractal dimension $\delta_g^{(1)}$ of the passive scalar signal for several Pr.

FIG. 2. Temperature structure function $\tilde{D}_\theta(\tilde{r}) = D_\theta(\tilde{r})/(\epsilon_\theta \epsilon^{-1/2}\nu^{1/2})$. The plateau can easily be recognized for sufficiently large Pr. $\log \tilde{D}_\theta(\tilde{r})$ is obtained by numerical integration of $\zeta_2^{(u)}(\tilde{r})$ over $\log \tilde{r}$.

FIG. 3. Temperatur spectra from an approximate solution of the dynamical equations (by a reduced wave vector set method) for several Pr. The Reynolds number always is $Re = 2 \cdot 10^4$. The dashed-dotted line denotes the velocity spectrum of the advecting fluid. $k^{-\zeta}$-behavior in the calculation here with discrete wave vectors means $k^{-\zeta-1}$-behavior in a continuous wave vector calculation.

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